§4.6 #3 p. 312

Find two positive numbers (m, n) whose product is 100 $(mn = 100)$ and whose sum $s = m + n$ is a minimum.

$mn = 100 \Rightarrow n = \frac{100}{m}$ so $s = m + \frac{100}{m}$ (substituted for n)

Now we find minimum,

$$s' = 1 - \frac{100}{m^2} = 0 \Rightarrow m^2 = 100 \Rightarrow m = \pm 10 \Rightarrow \underline{m = 10}$$

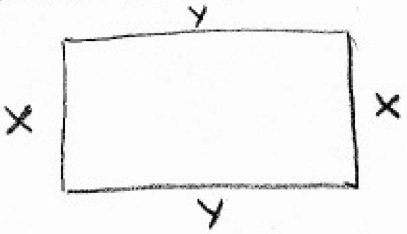
So our only critical number is $m = 10$

$$s'' = \frac{200}{m^3} \quad \text{so} \quad s''(10) = \frac{200}{1000} > 0 \quad \therefore \quad s(10) \text{ is a minimum by } 2^{nd} \text{ Der. Test.}$$

Thus the numbers are $\boxed{10 \text{ and } 10}$ $\boxed{\text{E7}}$ AGAIN ☺

§4.6 #5 p. 313

Find dimensions of rectangle with perimeter of 100 with maximum area. Draw a picture make up your variables:



$P = 2x + 2y = 100$

$y = 50 - x$ solving for y

Now the area $A = xy = x(50 - x) = 50x - x^2$, now the calculus, lets find $A(x)$'s maximum,

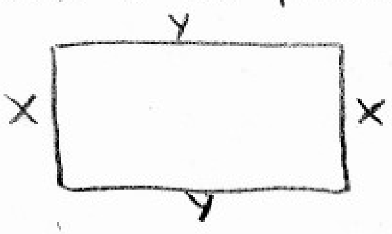
$$A'(x) = 50 - 2x$$
$$A''(x) = -2$$

So $A'(x) = 0 = 50 - 2x \Rightarrow \underline{x = 25}$ is the only critical #.

And $A''(25) = -2 < 0 \therefore$ By 2^{nd} Derivative $A(25)$ is maximum area so the dimensions are $\boxed{x = 25}$ and $\boxed{y = 50 - 25 = 25}$, it's a <u>square</u>.

§4.6 #6 p. 313

Similar to last problem, except area is fixed & perimeter varies,



$A = xy = 1000$

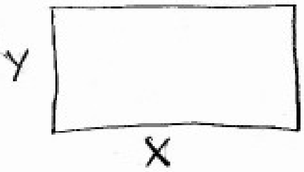
$y = \frac{1000}{x}$

$$P = 2x + 2y = 2x + \frac{2000}{x} \Rightarrow P'(x) = 2 - \frac{2000}{x}$$

So then $P'(x) = 0 = 2 - \frac{2000}{x} \Rightarrow \underline{x = 1000 \text{ is critical \#}}$

$P''(x) = \frac{2000}{x^2}$ so $P''(1000) = \frac{2000}{(1000)^2} > 0$, Thus $x = 1000$, $y = 1$ gives minimum perimeter

§4.6#11 p.313 (a.) Show that all the rectangles with a given area, the one with the smallest perimeter is a square. A is some fixed #.



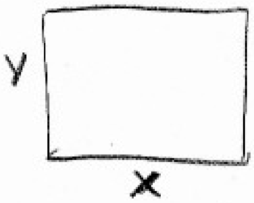
$A = xy \Rightarrow y = A/x$

$P = 2x + 2y = 2x + \frac{2A}{x}$

$P'(x) = 2 - \frac{2A}{x^2} = 0 \Rightarrow 2 = \frac{2A}{x^2} \Rightarrow x = \pm\sqrt{A} \Rightarrow \underline{x = \sqrt{A}}$

$P''(x) = \frac{4A}{x^3} \Rightarrow P''(\sqrt{A}) = \frac{4A}{(\sqrt{A})^3} > 0$, Hence by 2^{nd} Derivative test the min. perimeter is obtained for $x = \sqrt{A}$, $y = \frac{A}{\sqrt{A}} = \sqrt{A}$ is a square.

(b.) Show that of all the rectangles with a given fixed perimeter P the one with greatest area is a square



$P = 2x + 2y \Rightarrow y = \frac{1}{2}P - x$

$A = xy = x(\frac{1}{2}P - x) = \frac{1}{2}Px - x^2$

So then $A'(x) = \frac{1}{2}P - 2x$ and $A''(x) = -2$. Critical numbers are again found by $A'(c) = 0 = \frac{1}{2}P - 2c \Rightarrow \underline{c = P/4 \text{ critical } \#}$ and $A''(P/4) = -2 < 0$, Hence $x = P/4$ gives max. area. And $y = \frac{1}{2}P - \frac{P}{4} = P/4$ so we have a square with side $P/4$.

§4.6 #13 p.313 Find point on line $y = 4x + 7$ closest to origin $(0,0)$ { Just like in class example ! }

$d^2 = (x-0)^2 + (y-0)^2 = x^2 + (4x+7)^2$ ← distance from origin to point (x,y) on line squared. we minimize d^2 to find minimum d. Call $d^2(x) = f(x)$

$$f'(x) = 2x + 2(4x+7)\cdot 4 = 34x + 56$$

$$f''(x) = 34$$

So $f'(c) = 0 = 34c + 56 \Rightarrow c = \frac{-56}{34}$ is critical #, and $f''(\frac{-56}{34}) = 34 > 0$ thus $f(-56/34)$ is the minimum distance squared.

$\sqrt{f(-56/34)} = \sqrt{(\frac{-56}{34})^2 + (4(\frac{-56}{34}) + 7)^2} =$ well that's not it asked for, is it?

We just had to find the point which is $x = \frac{-56}{34} = \frac{-28}{17}$

$y = 4(\frac{-28}{17}) + 7 = \frac{-112 + 119}{17} = \frac{7}{17}$ the point is $\boxed{(\frac{-28}{17}, \frac{7}{17})}$

§4.6 #14 p. 313

This is really close to the in-class example. $x+y^2=0 \Rightarrow x=-y^2$

$$d^2 = (x-0)^2 + (y+3)^2 = x^2 + (y+3)^2 = y^4 + (y+3)^2 \equiv f(y)$$

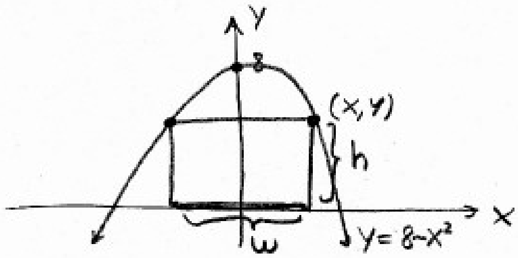
$$f'(y) = 4y^3 + 2(y+3)$$

$$f'(y) = 0 = 4y^3 + 2y + 6$$

Notice that $4y^3+2y+6=0$ when $y=-1$. (Don't worry I wouldn't ask you to solve a cubic w/o some help, this is hwk you have calculators, you could graph $f'(y)$ to see where it crosses the axis to guess $y=-1$, in general there's no easy way to factor a cubic ☹.) Anyway $f''(y) = 12y^2+2 \Rightarrow f''(-1) = 14$ so its a min. by 2nd derivative test, moreover the point is $\boxed{(-1,-1)}$

§4.6 #16 p. 313

Find dimensions of rectangle of largest area that has base on the x-axis and its other two vertices above x-axis on the parabola $y = 8-x^2$.



$w = 2x$

$h = y = 8-x^2$ (since corner is on graph)

$A = wh = 2x(8-x^2) = 16x-2x^3$

(note we can consider $x>0$ by picture)

$$A(x) = 16x - 2x^3$$

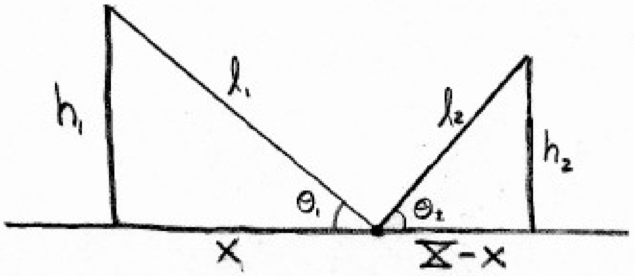
$$A'(x) = 16-6x^2 \Rightarrow A'(x) = 0 = 16-6x^2 \Rightarrow 6x^2 = 16 \Rightarrow x = \pm\sqrt{\tfrac{8}{3}}$$

$$A''(x) = -12x$$

Then $A''\left(\sqrt{\tfrac{8}{3}}\right) = -12\sqrt{\tfrac{8}{3}} < 0 \Rightarrow$ we find max. area for $x = \sqrt{\tfrac{8}{3}}$.

Bonus Point: Prove Snell's Law, see #33 p. 315.

§4.6 #34 p. 315 Two poles of unequal heights are secured by a very taught rope as pictured below. Show the shortest length of such a rope occurs when $\theta_1 = \theta_2$



h_1 and h_2 are fixed but arbitrairy. Also
$$x_1 + x_2 = X$$
which is also a constant.
$x_1, x_2, \theta_1, \theta_2, l_1, l_2$ all can vary as the point where the rope is attached to ground is adjusted.

$l = l_1 + l_2$: we want to minimize.

Let us endeavor to express l as a function of x. Notice

$$l_1 = \sqrt{x^2 + h_1^2}$$
$$l_2 = \sqrt{(X - x)^2 + h_2^2}$$

Now we can write l as a function of x & differentiate,

$$\frac{dl}{dx} = \frac{2x}{2\sqrt{x^2+h_1^2}} - \frac{2(X-x)}{2\sqrt{(X-x)^2+h_2^2}}$$

$$= \frac{x\sqrt{(X-x)^2+h_2^2} - (X-x)\sqrt{x^2+h_1^2}}{\sqrt{x^2+h_1^2}\sqrt{(X-x)^2+h_2^2}} = 0 \quad \left(\text{for critical \#s}\right)$$

Equivalently we may solve,

$$x\sqrt{(X-x)^2+h_2^2} = (X-x)\sqrt{x^2+h_1^2}$$
$$x^2\left((X-x)^2+h_2^2\right) = (X-x)^2(x^2+h_1^2)$$
$$x^2\left[X^2 - 2Xx + x^2 + h_2^2\right] = \left[X^2 - 2Xx + x^2\right](x^2+h_1^2)$$
$$\cancel{x^2X^2} - \cancel{2Xx^3} + \cancel{x^4} + x^2h_2^2 = \cancel{X^2x^2} - \cancel{2Xx^3} + \cancel{x^4} + X^2h_1^2 - 2Xxh_1^2 + x^2h_1^2$$
$$\boxed{x^2(h_2^2 - h_1^2) + x(2Xh_1^2) - X^2h_1^2 = 0}$$

hey, its just a quadratic eq$^{\underline{n}}$! ∴ we can solve it.

§4.6 #34
p. 315
Continued

We just found $\frac{dl}{dx} = 0$ when

$$x^2(h_2^2 - h_1^2) + x(2\Sigma h_1^2) - \Sigma^2 h_1^2 = 0$$

This is a quad. eq^n with sol^n's,

$$x = \frac{-2\Sigma h_1^2 \pm \sqrt{4\Sigma^2 h_1^4 + 4(h_2^2 - h_1^2)\Sigma^2 h_1^2}}{2(h_2^2 - h_1^2)}$$

$$= \frac{-\Sigma h_1^2 \pm \sqrt{\cancel{\Sigma^2 h_1^4} + h_2^2 h_1^2 \Sigma^2 - \cancel{\Sigma^2 h_1^4}}}{h_2^2 - h_1^2}$$

$$= \frac{-\Sigma h_1^2 \pm \Sigma h_1 h_2}{h_2^2 - h_1^2}$$

$$= \Sigma\left(\frac{h_1(h_1 \pm h_2)}{(h_2 + h_1)(h_2 - h_1)}\right)$$

$$= \frac{-\Sigma h_1}{h_2 - h_1} \quad \text{OR} \quad \boxed{\frac{\Sigma h_1}{h_1 + h_2}}$$

Notice that $\Sigma - x = \Sigma - \frac{\Sigma h_1}{h_1 + h_2} = \Sigma\left(\frac{h_1 + h_2 - h_1}{h_1 + h_2}\right) = \frac{h_2 \Sigma}{h_1 + h_2}$

Lets relate θ_1 & θ_2 given these critical lengths of x.

$$\tan\theta_1 = \frac{h_1}{x} = \frac{h_1}{\left(\frac{\Sigma h_1}{h_1 + h_2}\right)} = \frac{h_1 + h_2}{\Sigma}$$

$$\tan\theta_2 = \frac{h_2}{\Sigma - x} = \frac{h_2}{\left(\frac{h_2 \Sigma}{h_1 + h_2}\right)} = \frac{h_1 + h_2}{\Sigma}$$

Thus $\tan\theta_1 = \tan\theta_2 \Rightarrow \boxed{\theta_1 = \theta_2}$

Now we assumed that $\frac{dl}{dx} = 0$ gave the minimum length, we ought to prove that via the 1st or 2nd derivative tests.

<u>Bonus</u>: Rework this problem & <u>prove</u> this is indeed the minimum length.