Sol^n's to Project-like Problems

§5.4#9 $\dfrac{dx}{dt} = 2y - x$, $\dfrac{dy}{dt} = e^x + y$ } this is a <u>system</u> of differential eq^n's we'll learn a method to solve these in the last part of this course (chapter 9). A sol^n would consist of two functions $x(t)$ and $y(t)$. The "<u>phase plane eq^n</u>" eliminates t as follows,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^x + y}{2y - x} \qquad \text{Eq}^n\,(2)$$

Need to solve Eq^n(2). Its not obvious how to sep. variables and I can't use the integrating factor method here since its not linear, lets hope its exact otherwise I'm out of standard tricks.

$$\underbrace{(2y - x)}_{N}dy - \underbrace{(e^x + y)}_{M}dx = 0 \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -1$$

its exact ☺.

A moment of thought yields $F = -xy - e^x + y^2$ this has $\partial F/\partial x = -y - e^x$ and $\partial F/\partial y = -x + 2y$. We learned that $F = k$ is a sol^n to Eq^n (2). These are called the <u>integral curves</u> for the system of DEq^n's.

$$\boxed{y^2 - e^x - xy = k}$$

<u>Remark</u>: for #7 I bet $e^{x+y} = e^x e^y$ is an useful observation.

$\boxed{\S 5.4 \# 11}$ Solve the phase plane eq^{n} and sketch some integral curves.

$$\left.\begin{aligned} \frac{dx}{dt} &= 2y \\ \frac{dy}{dt} &= 2x \end{aligned}\right\} \text{ system of ODEs} \rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2x}{2y} = \frac{x}{y}$$

Can sep. variables for this one, $\int y\,dy = \int x\,dx$

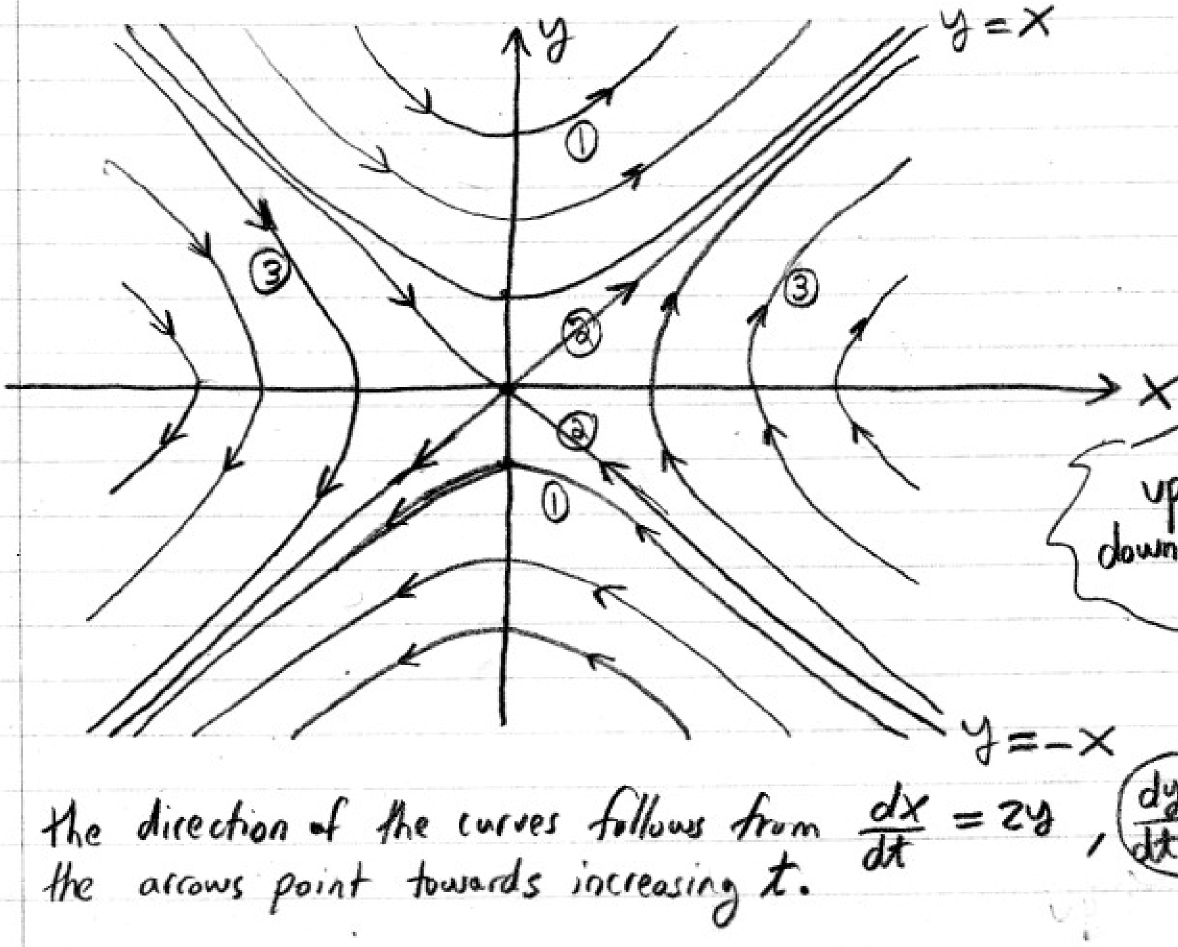
$$\tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + \tfrac{c}{2}$$

$$y^2 - x^2 = C$$

these are hyperbolas, and $y = \pm x$.

The curves fall into 3 categories

1. $C > 0$ then $y = 0$ not possible $\Rightarrow$ open up/down.
2. $C = 0$ $y^2 - x^2 = 0$ a.k.a. $y^2 = x^2 \Rightarrow y = \pm x$.
3. $C < 0$ then $x = 0$ not possible $\Rightarrow$ open left/right



up if $x > 0$
down if $x < 0$

the direction of the curves follows from $\frac{dx}{dt} = 2y$, $\frac{dy}{dt} = 2x$
the arrows point towards increasing t.

$\boxed{\S 5.4 \# 15}$ Find critical points then use technology to sketch the direction field in the phase plane and describe the stability of the critical points.

$$\frac{dx}{dt} = 2x + y + 3$$
$$\frac{dy}{dt} = -3x - 2y - 4$$

By def^{n} a critical point has both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ zero.

$$\left.\begin{aligned} 0 &= 2x + y + 3 \\ 0 &= -3x - 2y - 4 \end{aligned}\right\}$$ 2 eq^{n}'s 2 unknowns, no problem, I'll use substition.

$$y = -2x - 3$$
$$0 = -3x - 2[-2x-3] - 4$$
$$0 = -3x + 4x + 6 - 4 \Rightarrow \boxed{x = -2}$$
$$y = -2(-2) - 3 = 4 - 3 = \boxed{1 = y}$$

The algebra reveals there is just one critical point $\boxed{(-2, 1)}$

- I do want you to use Maple or something to sketch the direction field for these problems. (# 17)
- I have a DIRECTION FIELD MAPLE SHEET POSTED IN one of the old ma 241 pages I think.
- Look at the sol^{n}'s at the end of the text. we can see $(-2,1)$ is an unstable equilbrium point because the tangents point towards it from one direction and away from it in another. This is characteristic of a Saddle Point

(see figure 5.12 for other cases possible)

$\boxed{\S 5.4 \# 20}$ Convert given 2^{nd} order ODE $y'' + y = 0$ into a system of 1^{st} order ODEs (the first of which is merely a definition $v = y' = dy/dt$)

$$y'' + y = 0$$

$$\frac{d}{dt}(y') + y = 0 \longrightarrow \frac{dv}{dt} + y = 0$$

The phase eq^{n} in the yv-plane is $\frac{dy}{dv}$

$$\frac{dy}{dv} = \frac{dy/dt}{dv/dt} = \frac{v}{-y}$$

$$\therefore \; -y\,dy = v\,dv$$

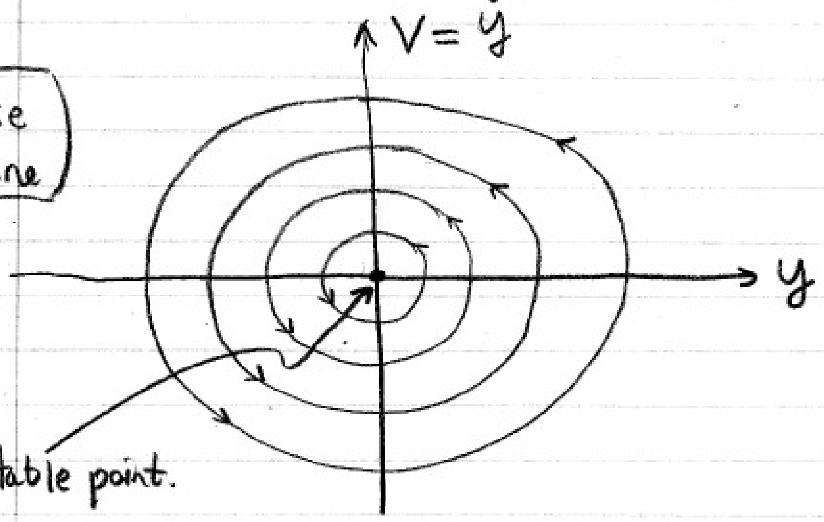
$$-\frac{y^2}{2} = \frac{v^2}{2} - \frac{k^2}{2}$$: I choose my arbitrary constants.

$$v^2 + y^2 = k^2$$: isn't that convenient circles!

Critical points: have $\frac{dy}{dt} = 0$ & $\frac{dv}{dt} = 0$,

$$\left.\begin{aligned} y' &= v = 0 \\ v' &= -y = 0 \end{aligned}\right\} \longrightarrow$$ origin $(0,0)$ is the only critical point.

Phase Plane



physically, this $y'' + y = 0$ is the eq^{n} for a frictionless spring. If $y = 0$ & $\dot{y} = 0$ the spring stays put. If $y \neq 0$ OR $\dot{y} \neq 0$ the spring moves in periodic motion which correspond to circles in phase plane.

§12.4 is on "Energy Methods", essentially these are tricks to help guide analysis of the phase plane. Physically, studying the energy tells you alot about what is physically possible for a given system.

$\boxed{\text{§12.4 \# 2}}$ $\dfrac{d^2x}{dt^2} + \cos(x) = 0$

find the potential energy function $G(x)$ & the energy function $E(x,v)$ for the given eq^{n}. From pg. 767 applied to our problem $g(x) = \cos(x)$

$$G(x) = \int \cos(x)\,dx + C$$
$$\boxed{G(x) = \sin(x) + C}$$

Then $E = v^2/2 + G(x) = v^2/2 + \sin(x) + C$
the text wants $E(0,0) = 0 \Rightarrow 0 + \sin(0) + C = 0$
thus $C = 0$ then $\boxed{E = v^2/2 + \sin(x)}$

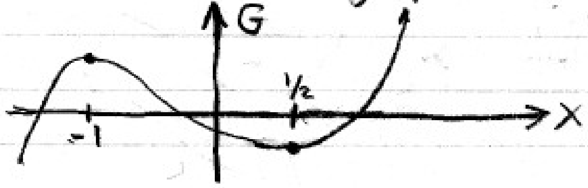
$\boxed{\text{§12.4 \# 9}}$ $\dfrac{d^2x}{dt^2} + 2x^2 + x - 1 = 0 \Rightarrow G(x) = \int (2x^2 + x - 1)\,dx$
$$\underline{G(x) = \tfrac{2}{3}x^3 + \tfrac{1}{2}x^2 - x + C}$$

The phase plane (xv) where
$v = dx/dt$ so $dv/dt + 2x^2 + x - 1 = 0$.

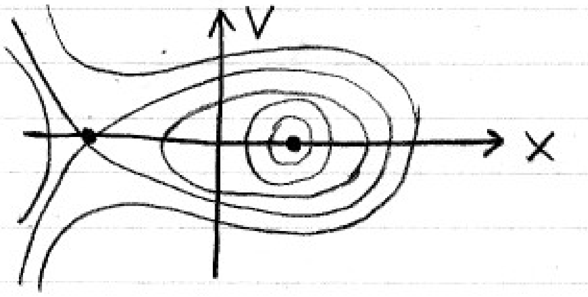
- Read the section to get the idea of what $G(x)$ has to do with the phase plane plot.

Notice $G'(x) = 2x^2 + x - 1 = 0 \Rightarrow (2x-1)(x+1) = 0$
and $G''(x) = 4x + 1$ $\quad x = 1/2,\ x = -1$ critical pts. of G.
$G''(1/2) = 3$ thus $G(1/2)$ is loc. min
$G''(-1) = -3$ thus $G(-1)$ is loc. max.

I did this calculus to graph $G(x)$ carefully.



$\Rightarrow$ critical points in phase plane are $(-1,0)$ and $(1/2, 0)$.



- see pages 768–770 to see logic behind $G(x) \Rightarrow$ phase plane plot.

$\boxed{\S 12.4 \# 9}$ Comment: the energy method gave us a rough plot of the integral curves in the phase plane, however we never had to solve the phase eq^n directly. Instead analyzing $G(x)$ told us everything. Physically the idea is that $G(x)$ is like the potential energy and different trajectories correspond to different kinetic energies. Depending on the amount of kinetic energy we may find localized motion (possibly periodic) or unbounded motion (like $V > V_{escape} \Rightarrow$ fly off into space)

$\boxed{\S 12.4 \# 7 \& 11}$ these are conservative, there is no friction term if we think of it as a spring. This means that $E = \text{constant} = V^2/2 + G(x)$ on each trajectory. So level curves of the energy function give the integral curves in the phase plane. (I'll let you work out the details, well I'll tell you this,

$$\int \frac{x}{x-2}\,dx = \int \left(\frac{x-2}{x-2} + \frac{2}{x-2}\right)dx = x + 2\ln|x-2| + C$$

$\boxed{\S 12.4 \# 13}$ $\displaystyle \frac{d^2x}{dt^2} + \boxed{\frac{dx}{dt}} + x - x^3 = 0$ ← friction term. See pg. 772-773 on how to do this one.

Remark: I'd like for you to see if you can use §12.4 ideas to help navigate project SE. I'm not saying it'll help each part, just keep §12.4 in mind.