Your solutions should be neat, correct and complete. Same instructions as Mission 1 apply here.

**Recommended Homework from Textbook:** problems:

5.2, 5.3, 5.7, 5.14, 5.24, 5.31, 5.32, 5.34, 5.40, 5.42, 5.63, 5.72, 5.77, 5.80, 5.103, 5.114.

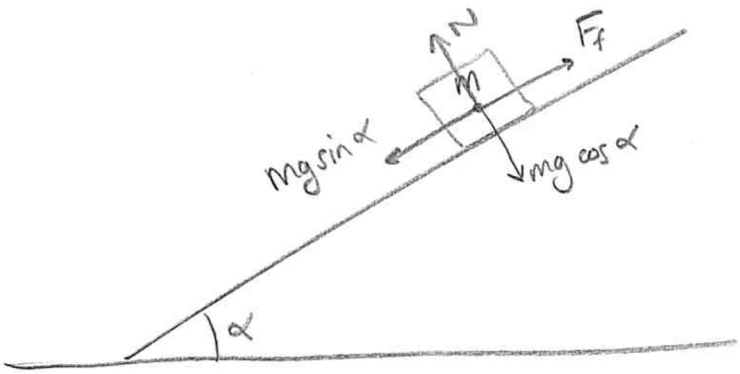
I also reccommend you work on understanding whatever details of lecture seem mysterious at first.

**Required Reading 3** [1pt] Your signature below indicates you have read:

**(a.)** I read Lectures 9, 10, 11, 13 and 15 by Cook as announced in Blackboard: ________________.

**(b.)** I read Chapters 5 and 6 of the required text: ________________.

**Problem 21** [3pts] Problem 5.36 (inclined plane)




$\mu_k = 0.25$

$\mu_s = 0.35$

(a.) $0 = F_f - mg\sin\alpha$

$0 = N - mg\cos\alpha \rightarrow N = mg\cos\alpha \rightarrow F_f = \mu_s mg\cos\alpha$

Thus, for max $\alpha$,

$0 = \mu_s \cancel{mg}\cos\alpha - \cancel{mg}\sin\alpha$

$\mu_s = \tan\alpha \Rightarrow \alpha = \tan^{-1}(\mu_s) = \boxed{19.29° = \alpha_{max}}$

(b.) $ma = \mu_k mg\cos\alpha - mg\sin\alpha$

$a = g(\mu_k\cos\alpha - \sin\alpha) = (9.8\ m/s^2)(0.23596 - 0.33035)$

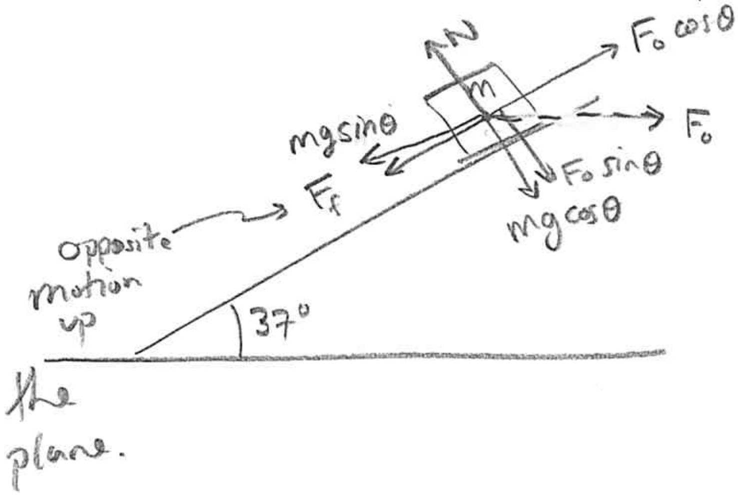
$\Rightarrow \boxed{a = -0.925\ m/s^2}$

(aka. $0.925\ m/s^2$ down the plane)

(c.) $V_f^2 = V_0^2 + 2a\Delta S$

$V_f = \sqrt{2(0.925\ m/s^2)(5m)} = \boxed{3.04\ m/s = V_f}$

**Problem 22** [3pts] Problem 5.66 (inclined plane)




$m = 6.0\,kg$

$\mu_k = 0.3$

$F_0 = ?$ for $a = 4.2\,m/s^2$

$$\perp: \quad 0 = N - mg\cos\theta - F_0\sin\theta \quad \therefore \quad N = mg\cos\theta + F_0\sin\theta$$

$$\parallel: \quad ma = F_0\cos\theta - mg\sin\theta - \mu_k N$$

$$ma = F_0\cos\theta - mg\sin\theta - 0.3(mg\cos\theta + F_0\sin\theta)$$

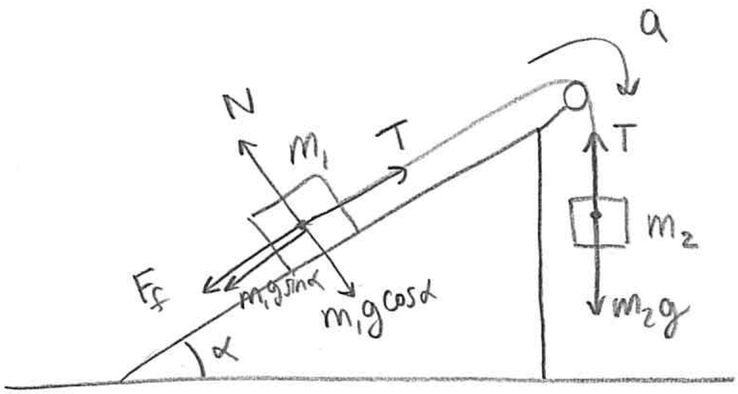
$$ma + mg\sin\theta + 0.3mg\cos\theta = F_0(\cos\theta - 0.3\sin\theta)$$

$$F_0 = \frac{m[a + g(\sin\theta + 0.3\cos\theta)]}{\cos\theta - 0.3\sin\theta}$$

$$= \frac{(6.0\,kg)[4.2\,m/s^2 + 9.8\,m/s^2(\sin 37° + 0.3\cos 37°)]}{\cos(37°) - 0.3\sin(37°)}$$

$$\cong \boxed{121\,N}$$

**Problem 23** [3pts] Problem 5.68 (inclined plane)




$m_1 = 20.0\,kg$

$\alpha = 53.1°$

$\mu_k = 0.4$

$m_2 = ?$

$\Delta y = 12.0\,m$ for $\Delta t = 3s$ given $V_0 = 0\,m/s$ at $t = 0$.

For $m_1$

$\perp$ $0 = N - m_1 g \cos\alpha \Rightarrow N = m_1 g \cos\alpha$

$\parallel$ $m_1 a = T - m_1 g \sin\alpha - 0.4\, m_1 g \cos\alpha$ *

For $m_2$ $\quad m_2 a = m_2 g - T$ **

add * & ** to obtain

$$(m_1 + m_2) a = m_2 g - m_1 g \sin\alpha - 0.4\, m_1 g \cos\alpha$$

$$m_2 (a - g) = -m_1 a - m_1 g \sin\alpha - 0.4\, m_1 g \cos\alpha$$

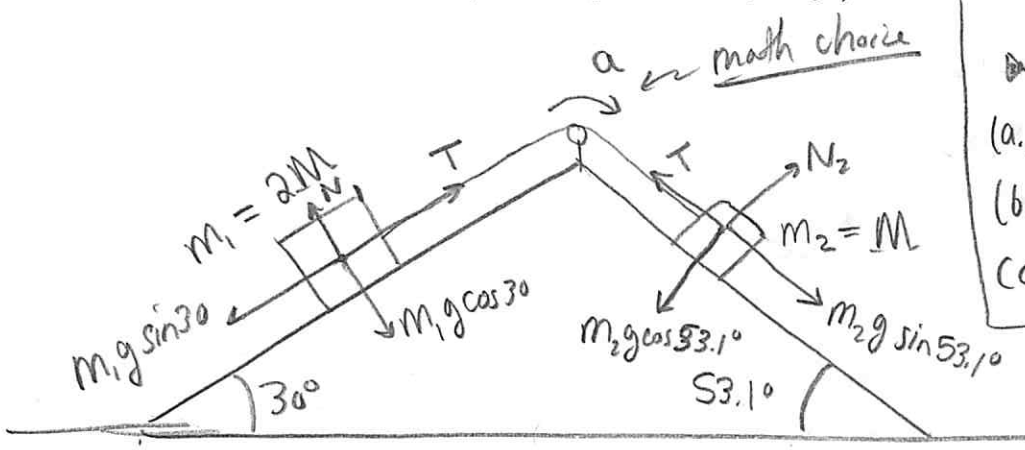
$$m_2 = \frac{m_1 \left(a + g(\sin\alpha + 0.4 \cos\alpha)\right)}{g - a} \quad ☆$$

Notice, $\Delta y = \frac{1}{2} a (\Delta t)^2 \Rightarrow a = \frac{2(12.0\,m)}{9.0\,s^2} = 2.667\,m/s^2$

Plug the #'s into ☆ to obtain

$$\boxed{m_2 \cong 36.0\,kg}$$

**Problem 24** [3pts] Problem 5.92 (inclined plane with pulleys)




- no friction

(a.) direction of motion?

(b.) $a = ?$

(c.) tension $T = ?$

$M = 50\,\text{kg}$

$$m_1 a = T - m_1 g \sin 30°$$
$$m_2 a = m_2 g \sin(53.1°) - T$$

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$$(m_1 + m_2)\, a = [m_2 \sin(53.1°) - m_1 \sin(30°)]\, g$$

$m_1 + m_2 = 2M + M = 3M$

$$a = \frac{1}{3M}\left[M \sin(53.1°) - 2M \underbrace{\sin 30°}_{1/2}\right] g$$

$$\Rightarrow a = \frac{1}{3}(\sin(53.1°) - 1)\, g < 0 \Rightarrow \boxed{\text{(a.) motion is to the left}}$$

Moreover,

$$a = \frac{1}{3}(-0.2003)(9.8\,\text{m/s}^2) \Rightarrow \boxed{a = -0.6544\,\text{m/s}^2}$$

Thus,

$$T = m_1 a + m_1 g \sin 30°$$
$$= (100\,\text{kg})\left(-0.6544\,\text{m/s}^2 + (9.8\,\text{m/s}^2)\left(\tfrac{1}{2}\right)\right)$$
$$= \boxed{424.6\,\text{N}}$$

**Problem 25** [3pts] Problem 5.108 (v-dependent force)

$$F_R = -k\sqrt{V}$$

(a.) Find $V(t)$, $X(t)$ given $V(0) = V_0$, $\underbrace{X(0) = 0}_{\text{defining it.}}$

(b.) $V(t_f) = 0$ for what $t_f$?

(c.) $X(t_f) = ?$

$$m\frac{dV}{dt} = -k\sqrt{V}$$

$$\frac{dV}{\sqrt{V}} = \frac{-k}{m}dt$$

$$\int_{V_0}^{V_f} V^{-1/2}\, dV = -\int_0^{t_f} \beta\, dt$$

$$2\sqrt{V_f} - 2\sqrt{V_0} = -\beta t_f$$

$$\Rightarrow \sqrt{V} = \sqrt{V_0} - \frac{\beta t}{2}$$

$$\boxed{V(t) = \left(\sqrt{V_0} - \frac{kt}{2m}\right)^2} = V_0 - \frac{\sqrt{V_0}\,kt}{m} + \frac{k^2t^2}{4m^2}.$$

$$V(t) = \frac{dx}{dt} \Rightarrow \boxed{x(t) = V_0 t - \frac{1}{2}\frac{\sqrt{V_0}\,k}{m}t^2 + \frac{1}{12}\frac{k^2}{m^2}t^3}$$

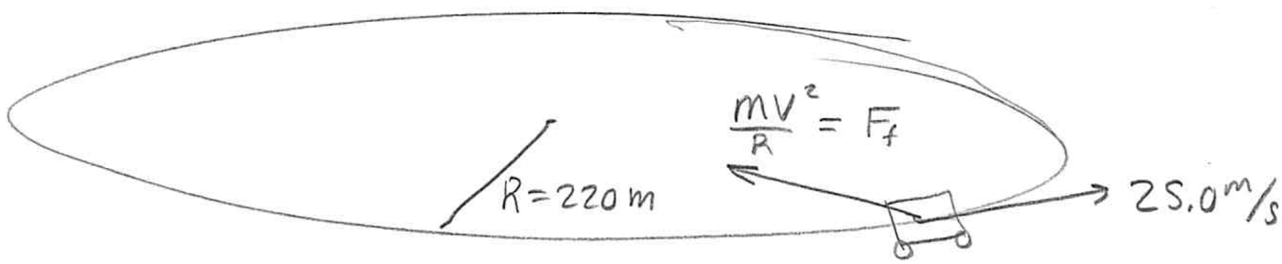
(b.) $V(t) = 0 \Rightarrow \sqrt{V_0} = \dfrac{kt}{2m} \Rightarrow \boxed{t_f = \dfrac{2m\sqrt{V_0}}{k}}$

(c.) $X(t_f) - X(0) = \Delta X = \cancel{V_0\left(\frac{2m\sqrt{V_0}}{k}\right) - \frac{1}{2}\frac{k\sqrt{V_0}}{m}\left(\frac{4m^2V_0}{k}\right)} + \dfrac{k^2}{12m^2}\left(\dfrac{8m^3V_0}{k^3}\right)^{3/2}$

$$\Rightarrow \boxed{\Delta X = \frac{2mV_0^{3/2}}{3k}}$$

$$\frac{8}{12} = \frac{2}{3}.$$

**Problem 26** [3pts] Problem 5.44 (flat race track with friction)




(a.) $\frac{mv^2}{R} = \mu_s N = \mu mg$ (min. case)

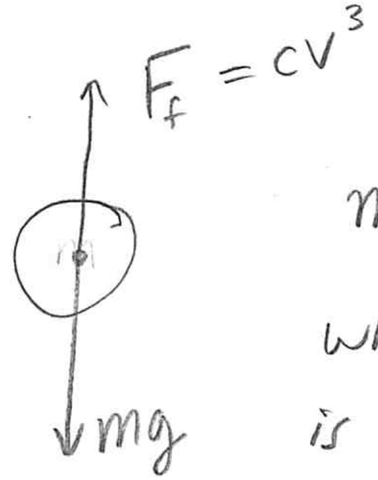
$$\mu = \frac{v^2}{gR} = \frac{(25.0\,m/s)^2}{(9.8\,m/s^2)(220.0\,m)} = \boxed{0.29}$$

(b.) $\mu_{ice} = \frac{0.29}{3} = 0.097$

$$\frac{mV_{max}^2}{R} = \mu_{ice}\,mg \longrightarrow V_{max} = \sqrt{\mu_{ice}\,g R}$$

$$\boxed{V_{max} = 14.43\ m/s}$$

**Problem 27** [3pts] A velocity-dependent friction force of $F_f = cv^3$ acts on a falling mass. Find the terminal velocity and draw a free-body diagram to illustrate the direction of the gravitational and friction forces. What are the dimensions of the constant $c$?




$$ma = mg - cv^3$$

when terminal velocity is attained we see $a = 0$

hence $0 = mg - cv^3$

$$\Rightarrow v^3 = \frac{mg}{c}$$

$$\Rightarrow \boxed{v = \sqrt[3]{\frac{mg}{c}}}$$

$$N = [cv^3] = [c]\left(\frac{m}{s}\right)^3 \Rightarrow [c] = \frac{Ns^3}{m^3} = \frac{kg\,m}{s^2}\,\frac{s^3}{m^3}$$

also good. $\boxed{[c] = \frac{kg\,s}{m^2}}$.

**Problem 28** [3pts] We define **gradients** $\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}}$ for $f = f(x, y)$ or, for three dimensional cases, $g = g(x, y, z)$, we define $\nabla g = \frac{\partial g}{\partial x}\hat{\mathbf{x}} + \frac{\partial g}{\partial y}\hat{\mathbf{y}} + \frac{\partial g}{\partial g}\hat{\mathbf{z}}$. Geometrically, $f(x, y) = k$ has a solution set which is a curve, sometimes called the level-$k$ curve. Geometrically, $g(x, y, z) = k$ has a solution set which is a surface, sometimes called the level $k$-surface. Calculate the gradient vector field of each function below.

**(a.)** $f(x, y) = x^2 + 2y^2$,

**(b.)** $f(x, y) = e^{xy}$,

**(c.)** $g(x, y, z) = ax + by + cz$ where $a, b, c$ are constants,

**(d.)** $g(x, y, z) = (x^2 + y^2 + z^2)^n$ where $n \in \mathbb{Z}$.

(a.) $\nabla f = \left\langle \frac{\partial}{\partial x}(x^2+2y^2), \frac{\partial}{\partial y}(x^2+2y^2) \right\rangle = \boxed{\langle 2x, 4y \rangle}.$

(b.) $\nabla f = \left\langle \frac{\partial}{\partial x}(e^{xy}), \frac{\partial}{\partial y}(e^{xy}) \right\rangle$

$= \left\langle e^{xy}\frac{\partial}{\partial x}(xy), e^{xy}\frac{\partial}{\partial y}(xy) \right\rangle$ : chain rule

$= \boxed{\langle y e^{xy}, x e^{xy} \rangle}.$

$= e^{xy}\langle y, x \rangle$ ⇐ nice formula.

(c.) $\nabla g = \left\langle \frac{\partial}{\partial x}(ax+by+cz), \frac{\partial}{\partial y}(ax+by+cz), \frac{\partial}{\partial z}(ax+by+cz) \right\rangle$

$= \boxed{\langle a, b, c \rangle}.$

(d.) $\nabla g = \nabla (x^2+y^2+z^2)^n$

YEP. IT DOES THIS. ☺

$= n(x^2+y^2+z^2)^{n-1}\,\nabla(x^2+y^2+z^2)$

$= n(x^2+y^2+z^2)^{n-1}\langle 2x, 2y, 2z \rangle$

$= \boxed{2n(x^2+y^2+z^2)^{n-1}\langle x, y, z \rangle}$

**Problem 29** [3pts] Let $C_1$ be the helix parametrized by $\vec{\mathbf{r}}_1(t) = \langle \cos t, \sin t, t\rangle$ for $t \in [0, 2\pi]$. Also, let $C_2$ be the line from $(1,0,0)$ to $(1,0,2\pi)$ parametrized by $\vec{\mathbf{r}}_2(t) = \langle 1, 0, t\rangle$. Both parametrizations have $0 \le t \le 2\pi$.

(a.) if $\vec{\mathbf{F}}_1 = (x+y)\hat{\mathbf{x}} + (y+z)\hat{\mathbf{y}} + (z+x)\hat{\mathbf{z}}\rangle$ then calculate $\int_{C_1} \vec{\mathbf{F}}_1 \bullet d\vec{\mathbf{r}}$ and $\int_{C_2} \vec{\mathbf{F}}_1 \bullet d\vec{\mathbf{r}}$

(b.) if $\vec{\mathbf{F}}_2 = x^3\hat{\mathbf{x}} + y^2\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ then calculate $\int_{C_1} \vec{\mathbf{F}}_1 \bullet d\vec{\mathbf{r}}$ and $\int_{C_2} \vec{\mathbf{F}}_1 \bullet d\vec{\mathbf{r}}$

$\vec{F}_2$ $\quad$ $\vec{F}_2$

$$\int_{C_1} \vec{F}_1 \cdot d\vec{r} = \int_{C_1} \langle x+y,\ y+z,\ z+x\rangle \cdot d\vec{r} \qquad C_1: \begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}$$

$$= \int_0^{2\pi} \langle \cos t + \sin t,\ \sin t + t,\ t + \cos t\rangle \cdot \langle -\sin t, \cos t, 1\rangle\, dt$$

$$= \int_0^{2\pi} \left(\cancel{-\sin t\cos t} - \sin^2 t + \cancel{\sin t \cos t} + t\cos t + t + \cos t\right) dt$$

$$= \int_0^{2\pi} \left(-\frac{1}{2} + \frac{1}{2}\cancel{\cos(2t)}^{\,0} + t\cos t + t + \cancel{\cos t}^{\,0}\right) dt$$

$$= -\frac{1}{2}(2\pi) + \frac{(2\pi)^2}{2} + \left(\cancel{t\sin t}^{\,0} - \cancel{\cos t}^{\,0}\right)\Big|_0^{2\pi}$$

$$= -\pi + 4\pi^2/2 +$$

$$= \boxed{2\pi^2 - \pi}.$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_2} (x+y)\cancel{dx}^{\,0} + (y+z)\cancel{dy}^{\,0} + (z+x)dz \ : \ C_2: \begin{cases} x = 1, & dx = 0 \\ y = 0, & dy = 0 \\ z = t, & dz = dt \end{cases}$$

$$= \int_0^{2\pi} (t+1)\, dt$$

$$= \frac{1}{2}(2\pi)^2 + 2\pi$$

$$= \boxed{2\pi^2 + 2\pi}.$$

(b.) $\int_{C_1} \vec{F}_2 \cdot d\vec{r} = \int_{C_1} x^3 dx + y^2 dy + z dz$ $\quad C_1 : \begin{cases} x = \cos t & dx = -\sin t\, dt \\ y = \sin t & dy = \cos t\, dt \\ z = t & dz = dt \end{cases}$

$$= \int_0^{2\pi} \left[ \cancel{\cos^3 t (-\sin t) dt}^{\,0} + \cancel{(\sin^2 t)(\cos t\, dt)}^{\,0} + t\, dt \right]$$

$$= \frac{1}{2}(2\pi)^2$$

$$= \boxed{2\pi^2}.$$

$$\int_{C_2} \vec{F}_2 \cdot d\vec{r} = \int_{C_2} x^3 \cancel{dx}^{\,0} + y^2 \cancel{dy}^{\,0} + z dz \quad C_2 : \begin{cases} x = 1 ; & dx = 0 \\ y = 0 ; & dy = 0 \\ z = t ; & dz = dt \end{cases}$$

$$= \int_0^{2\pi} t\, dt$$

$$= \frac{1}{2}(2\pi)^2$$

$$= \boxed{2\pi^2}$$

$\boxed{\text{PROBLEM 30}}$ Note $\int_{C_1} \vec{F}_1 \cdot d\vec{r} \neq \int_{C_2} \vec{F}_1 \cdot d\vec{r}$ despite $C_1$ & $C_2$ both going from $(1,0,0)$ to $(1,0,2\pi)$. Thus $\vec{F}_1$ is <u>not</u> path-independent $\Rightarrow \vec{F}_1 \neq -\nabla U_1$.

However, there is hope for $\vec{F}_2$, a moments thought reveals

$$U(x,y,z) = -\frac{1}{4}x^4 - \frac{1}{3}y^3 - \frac{1}{2}z^2$$

We can easily verify $-\nabla U = \vec{F}_2$.

$$W = -\Delta U = U(1,0,0) - U(0,0,2\pi)$$

$$W = -\frac{1}{4} + \frac{1}{2}(2\pi)^2 = \boxed{2\pi^2 - 1/4}$$