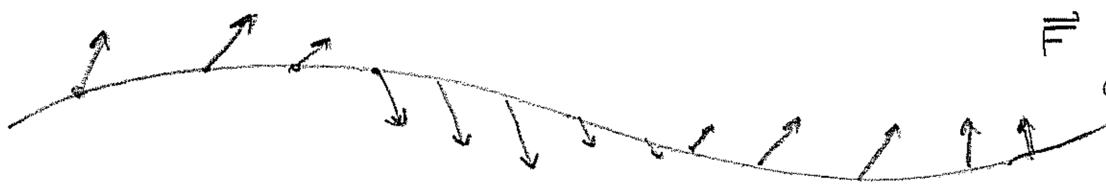
LECTURE 15

- The material in this "lecture" extends beyond the talk I gave in class. We define work, kinetic energy, potential energy & conservative forces and discuss the Work-Energy Th$^{\underline{m}}$ and Conservation of Energy. The mathematical backdrop involves some calculus $\underline{III}$ topics and I will supplement the text by offering numerous mathematical examples in this document. The lectures that follow this are mainly to add texture and depth to the framework we create here.

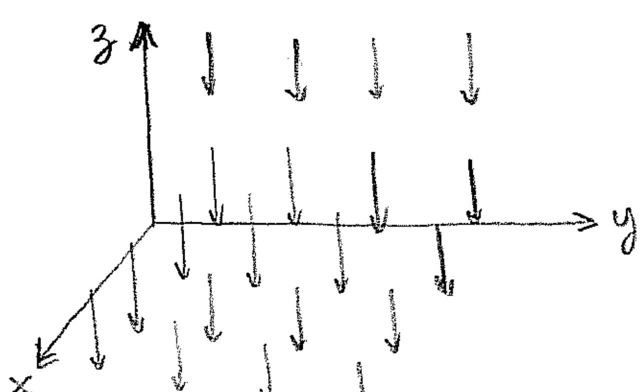
<u>Remark</u>: we've thought a lot about forces applied to particular points. The next step is to think about how a force is applied to a whole series of points along some curve (or more generally over some space)



$\vec{F}$ along a curve C

- This might be the force of a roller-coaster on a person and C could be the path the coaster travels.

Other forces, like gravity, are due to a <u>field</u> which fills space and is responsible for the communication of the gravitational force



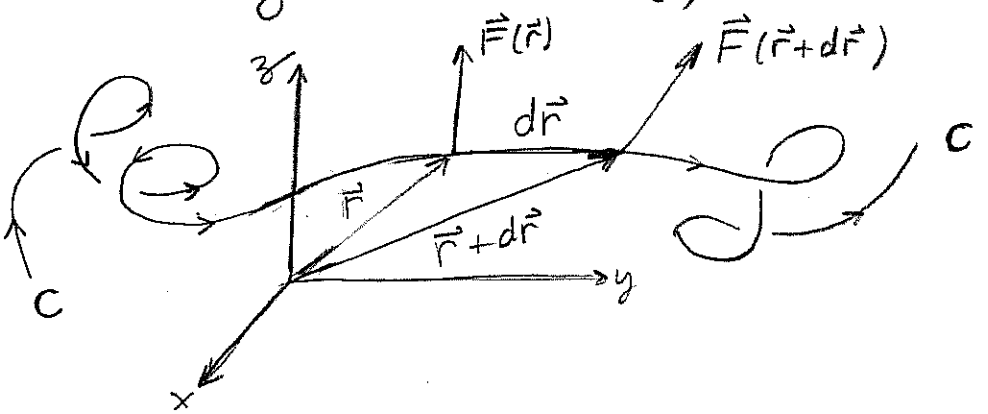
$$\underbrace{\vec{a} = -g\hat{k}}_{\text{gravitational field fills space, gives}}$$

$\vec{F} = -mg\hat{k}$ when m is found.

We consider some mass m which travels along a path $C = \{\vec{r}(t) \mid a \leq t \leq b\}$. Suppose the force $\vec{F}$ varies along C such that at each point $\vec{r}(t)$ the force $\vec{F}(\vec{r}(t))$ is applied. Given all this we <u>define the work done by $\vec{F}$</u>

$$\boxed{W = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \left(\vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\right) dt}$$

Perhaps you learned in some earlier course that work = force × distance. We can still think that way infinitesimally,



<u>Idea</u>: $\vec{F}(\vec{r}) \approx \vec{F}(\vec{r}+d\vec{r})$ since $\|d\vec{r}\| \approx 0$ thus $dW = \vec{F}(\vec{r}) \cdot d\vec{r}$. We need the dot-product to insure we select the part of $\vec{F}$ in the $d\vec{r}$ direction. Finally we multiply by one,

$$dW = \underbrace{\vec{F}(\vec{r}) \cdot d\vec{r}}_{\text{an idea}} = \underbrace{\left(\vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\right) dt}_{\text{a calculable quantity.}}$$

Then integrate.
Perhaps this motivates the definition for you. My next example shows how we recover easy case

E1 Suppose a constant force $\vec{F}(\vec{r}(t)) = \vec{F}_o$ acts on a mass m which travels along the straight-line path
$$t \longmapsto \vec{r}(t) = \vec{r}_o + t\vec{V}_o$$
Find the work done during $t_1 \leq t \leq t_2$

$$W = \int \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$

note that $\frac{d}{dt}(r_o + t\vec{V}_o) = \vec{V}_o$

$$= \int_{t_1}^{t_2} (\vec{F}_o \cdot \vec{V}_o)\, dt$$

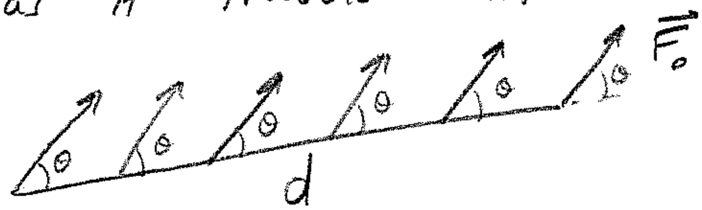
$$= \vec{F}_o \cdot \vec{V}_o \int_{t_1}^{t_2} dt$$

$$= \vec{F}_o \cdot \vec{V}_o (t_2 - t_1)$$

$$= \underline{\vec{F}_o \cdot \Delta\vec{r}}.$$

constant velocity $\Rightarrow$ displacement $\Delta\vec{r} = \vec{V}_o \Delta t$.

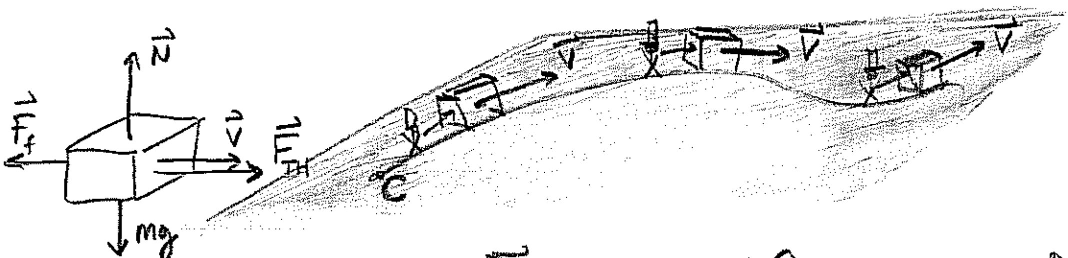
Perhaps you saw $W = F_o d \cos\theta$ to calculate the work done by constant force F_o on a mass as it travels distance d where



Remark: note that $\vec{F}_o$ need not be $\vec{F}_{net}$! If multiple forces are at work then each force may either speed or slow the motion. We can show the net-work is additive. If $\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n$ then $W_{net} = W_1 + W_2 + \cdots + W_n$ where I mean to define $W_j = \int_C \vec{F}_j \cdot d\vec{r}$ and $W_{net} = \int_C \vec{F}_{net} \cdot d\vec{r}$.

$\boxed{E2}$ Suppose $\overset{\text{⊓}}{X}$ pushes a box along a rough, horizontal surface. If the box has mass m and $F_f = \mu_k N$ then what is the work done by friction if Mr. Tophat pushes horizontally along a path of length L?

Let $t \longmapsto \vec{r}(t)$ denote the path traveled. At each point we can picture $\vec{V}(t) = \frac{d\vec{r}}{dt}$. It points in the direction of the motion



At each point $\vec{F}_f = -\mu N \hat{V}$ where $N = |\vec{N}|$ and $\hat{V} = \frac{1}{V}\vec{V}$ and $\vec{V} = \frac{d\vec{r}}{dt}$. Note $\vec{N} = mg\,\hat{k}$ since we assume $\vec{F}_{TH}$ is horizontally applied. Thus,

$$\vec{F}_f = -\mu mg\,\hat{V}$$

Calculate the work done by $\vec{F}_f$,

$$W = \int_C \vec{F}_f \cdot d\vec{r} = \int_a^b \left(-\mu mg\,\hat{V}\right)\cdot \frac{d\vec{r}}{dt}\,dt \quad \left(\text{where } \vec{r}([a,b]) = C\right)$$

$$= -\mu mg \int_a^b \frac{1}{V}\,\vec{V}\cdot\vec{V}\,dt \quad : \text{note } \vec{V}\cdot\vec{V} = V^2$$

$$= -\mu mg \int_a^b V\,dt$$

$$= -\mu mg \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= -\mu mg\,L$$

<u>arclength</u> given to be L

Thus, the work is negative. Much like $\boxed{E1}$ the work is still of form force × distance.

The interesting thing about $\boxed{E2}$ is that even though the $\vec{F}_f$ is <u>not</u> constant (notice it's always changing direction) it is constant magnitude. Moreover, the direction lines up with $\frac{d\vec{r}}{dt}$ in the same manner for all points along C. The $\vec{F}_f$ is always antiparalell to $\frac{d\vec{r}}{dt}$. The net-result mimichs the simplest case: $W = (\text{force})(\text{distance})$.

<u>Remark</u>: the work done by $\vec{F}$ along C on a mass m somehow measures the ability of $\vec{F}$ to "move" m. We can be more preicice,

$\underline{\text{Th}^{\underline{m}}}$ <u>(Work-Energy)</u>. Define $K = \frac{1}{2}m\vec{V}\cdot\vec{V}$ to be the <u>Kinetic</u> <u>Energy</u> of a mass m with velocity $\vec{V}$. If $\vec{F}_{net}$ is the net-force and C goes from $\vec{r}(a)$ to $\vec{r}(b)$,

$$W_{net} = \int_C \vec{F}\cdot d\vec{r} = K_b - K_a$$

<u>Proof</u>: Note $\frac{dK}{dt} = \frac{1}{2}m\frac{d}{dt}(\vec{V}\cdot\vec{V}) = \frac{1}{2}m\left[\frac{d\vec{V}}{dt}\cdot\vec{V} + \vec{V}\cdot\frac{d\vec{V}}{dt}\right]$ (product rule)

It follows that $\frac{dK}{dt} = \left(m\frac{d\vec{V}}{dt}\right)\cdot\vec{V}$. Calculate,

$$\begin{aligned} W_{net} = \int_C \vec{F}_{net}\cdot d\vec{r} &= \int_a^b \left(\vec{F}_{net}\cdot\frac{d\vec{r}}{dt}\right)dt \\ &= \int_a^b \left(m\frac{d\vec{V}}{dt}\cdot\vec{V}\right)dt \quad : \text{Newton's } 2^{nd} \text{ Law holds along C.} \\ &= \int_a^b \frac{dK}{dt}\,dt \\ &= K(\vec{r}(b)) - K(\vec{r}(a)) \\ &= K_b - K_a . // \end{aligned}$$

Observation (I): If $\vec{F}_{net} = 0$ along a path C then the speed of the mass in question must be constant. This is a simple consequence of the Work-Energy Th$^{\underline{m}}$,

$$W_{net} = \int_C \vec{F}_{net} \cdot d\vec{r} = \int_C \vec{0} \cdot d\vec{r} = 0$$

Hence $0 = K_b - K_a \Rightarrow K_a = K_b$

$$\Rightarrow \tfrac{1}{2} m V_a^2 = \tfrac{1}{2} m V_b^2$$

$$\Rightarrow V_a^2 = V_b^2$$

$$\Rightarrow \underline{V_a = V_b}.$$

Observation (II): If $\vec{F}_{net} \perp \vec{V}$ then we get constant speed motion. Again this follows from the Work-Energy Th$^{\underline{m}}$, Note $\vec{F}_{net} \perp \vec{V}$ means that $\vec{F}_{net} \cdot \vec{V} = 0$,

$$W_{net} = \int_C \vec{F}_{net} \cdot d\vec{r} = \int_a^b \big(\underbrace{\cancel{\vec{F}_{net} \cdot \vec{V}}}^{0}\big)\, dt = 0.$$

For example, circular motion at constant speed.

Observation (III): If the speed is constant then the forces must either balance to zero or possibly only act nontrivially in directions $\perp$ to the direction of the motion. For example, MR. TOPHAT pushing at $|\vec{F}_{TH}| = F_f$ would result in no net-gain of speed in $\boxed{E2}$. In that case $\vec{F}_{TH} = -\vec{F}_f$ thus

$$W_{TH} = \int_C \vec{F}_{TH} \cdot d\vec{r} = -\int \vec{F}_f \cdot d\vec{r} = -W_{friction}$$

Hence $W_{net} = W_{TH} + W_{friction} = 0.$

(If $|F_{TH}| > F_f$ then we speed-up the box)

Further Examples: (forgive my sins of dimensional incorrectness)

Technically $\int_C \vec{F} \cdot d\vec{r}$ is a line-integral. These are covered by calculus III in additional depth. Basically we just need some parametrization $t \longmapsto \vec{r}(t)$ for the oriented curve $C = \vec{r}([a,b])$ and we can calculate.

E3 Suppose $\vec{F} = x\hat{i} + y\hat{j}$ and C is parametrized by $\vec{r}(t) = 3\cos t\,\hat{i} + 2\sin t\,\hat{j}$ for $0 \le t \le \pi$. Find W.

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi} \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\,dt$$

$$= \int_0^{\pi} \left(3\cos t\,\hat{i} + 2\sin t\,\hat{j}\right) \cdot \left(-3\sin t\,\hat{i} + 2\cos t\,\hat{j}\right) dt$$

$$= \int_0^{\pi} \left(-9\cos t\sin t + 4\sin t\cos t\right) dt$$

$$= \int_0^{\pi} (5\cos t)(-\sin t\,dt) \longrightarrow \begin{cases} u = \cos t \\ du = -\sin t\,dt \\ u(0) = \cos(0) = 1 \\ u(\pi) = \cos(\pi) = -1 \end{cases}$$

$$= \int_1^{-1} 5u\,du$$

$$= \frac{5u^2}{2}\Big|_1^{-1}$$

$$= \boxed{0}$$

E4 Suppose $x = t$, $y = \sin t$, $z = \cos t$. If $\vec{F} = -x\hat{i}$ then find $W_{\vec{F}}$ for $0 \le t \le 1$.

Note $\vec{r}(t) = t\hat{i} + \sin t\,\hat{j} + \cos t\,\hat{k}$ so $\frac{d\vec{r}}{dt} = \hat{i} + \cos t\,\hat{j} - \sin t\,\hat{k}$

$$W_{\vec{F}} = \int_0^1 \left(-t\hat{i}\right) \cdot \left(\hat{i} + \cos t\,\hat{j} - \sin t\,\hat{k}\right) dt$$

$$= \int_0^1 -t\,dt$$

$$= \frac{-t^2}{2}\Big|_0^1 = \boxed{\frac{-1}{2}}$$

Note: motion not in $\vec{F}$-direction didn't contribute to $W_{\vec{F}}$

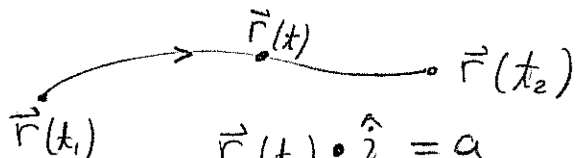
If you want to see more examples of line integrals then you might look in my calculus III notes or perhaps Stewart. I've tried to do a few examples to show you the reach of the work-definition. Sophistication aside, we mostly work with simple cases in this course. Typically either our force is constant, or we have a force which varies in magnitude but not direction. Usually we don't need the full force of the line integral, but I want you to appreciate that is because we deal with <u>special</u> <u>cases</u>

E5 Suppose a force $\vec{F} = f(x)\hat{i}$ acts on a path from $x = a$ to $x = b$. Find $W_{\vec{F}}$

<u>We calculate,</u>

$$W_{\vec{F}} = \int_C \vec{F} \cdot d\vec{r} = \int \left(f(x)\hat{i}\right) \cdot \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right) dt$$

$$= \int_{t_1}^{t_2} f(x(t))\frac{dx}{dt}\, dt$$



$$\underbrace{\begin{matrix} \vec{r}(t_1) \cdot \hat{i} = a \\ \vec{r}(t_2) \cdot \hat{i} = b \end{matrix}}_{\text{get it ?}}$$

$$= \int_a^b f(x)\, dx$$

This is what I teach in calculus II.

To find work done by $f(x)$ for $a \leq x \leq b$ just add-up $dW = f(x)dx$ to get total $W = \int_a^b f(x)dx$.

If you think about the examples, even the meager set we've seen thus far, you might notice that the details of the path matter not. For $\boxed{E5}$, I only gave info about the x-coordinates for the endpoints. Moreover for other examples, it seems apparent that if we introduce additional motions $\perp$ to $\vec{F}$ then we again fail to change $W_{\vec{F}}$. On the other hand, if you study work done by $\vec{F} = (-y\hat{i} + x\hat{j})\frac{N}{m}$ then you'll find that different paths between the same pair of endpoints give different $W_{\vec{F}}$. Or think about $\vec{F}_f$ and $\boxed{E2}$, the W_{F_f} depends on the details of the path (in particular the arclength L). All of this leads us to think about the concept of a <u>conservative</u> <u>force</u>

<u>Concept</u>: the work done by a conservative force can be reversed. In other words, if we reverse the path in question then we'll either recover the energy we lost or lose the energy we gained. Somehow the work done depends only on the endpoints. Where does the energy go? It's stored in the <u>potential</u> energy U.

(Before I discuss U, I'll take a math detour to discuss $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \nabla$ etc...)

Partial Differentiation

The concept of partial differentiation applies to functions of several independent variables. Often we deal with functions of x, y or x, y, z. In this context,

$$\frac{\partial}{\partial x}\left(f(x,y)\right)\bigg|_{(a,b)} \stackrel{\text{def}^{\underline{n}}}{=} \frac{d}{dx}\Big[f(x,b)\Big]\bigg|_{x=a} \stackrel{\text{def}^{\underline{n}}}{=} \frac{\partial f}{\partial x}(a,b)$$

$$\frac{\partial}{\partial y}\left(f(x,y)\right)\bigg|_{(a,b)} \stackrel{\text{def}^{\underline{n}}}{=} \frac{d}{dy}\Big[f(a,y)\Big]\bigg|_{y=b} \stackrel{\text{def}^{\underline{n}}}{=} \frac{\partial f}{\partial y}(a,b)$$

$\frac{\partial f}{\partial x}$ is read "partial f, partial x" the notation "∂" means about the same as d for calc. I. In Math 231 (calc. III) we'll explain more, but for here let's just study the mechanics of the calculation,

Key Formulas:

$$\left.\begin{matrix} \frac{\partial x}{\partial x} = 1, & \frac{\partial x}{\partial y} = 0, & \frac{\partial x}{\partial z} = 0 \\ \frac{\partial y}{\partial x} = 0, & \frac{\partial y}{\partial y} = 1, & \frac{\partial y}{\partial z} = 0 \\ \frac{\partial z}{\partial x} = 0, & \frac{\partial z}{\partial y} = 0, & \frac{\partial z}{\partial z} = 1 \end{matrix}\right\}$$

in-short, $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$

0 if $i \neq j$
1 if $i = j$

Beyond this, just differentiate like we did already in calculus I.

$\boxed{E6}$

$$\frac{\partial}{\partial x}(x^2yz) = 2xyz$$

$$\frac{\partial}{\partial y}(x^2yz) = x^2z$$

$$\frac{\partial}{\partial z}(x^2yz) = x^2y$$

$$\frac{\partial}{\partial x}\left(e^{x^2y+z}\right) = e^{x^2y+z}\frac{\partial}{\partial x}(x^2y+z) = \underline{2xy\, e^{x^2y+z}}.$$

$$\underbrace{\frac{\partial}{\partial x_i}(x_j^3) = 3x_j^2\frac{\partial x_j}{\partial x_i} = 3x_j^2\,\delta_{ij}}_{\text{ask if interested ☺}}$$

$\boxed{E7}$ Suppose $U(x,y) = \tan^{-1}(x-y)$

$$\frac{\partial U}{\partial x} = \frac{1}{1+(x-y)^2}\frac{\partial}{\partial x}(x-y) = \frac{1}{1+(x-y)^2}$$

$$\frac{\partial U}{\partial y} = \frac{1}{1+(x-y)^2}\frac{\partial}{\partial y}(x-y) = \frac{-1}{1+(x-y)^2}$$

$\boxed{E8}$ Suppose $r = \sqrt{x^2+y^2+z^2}$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial x}(x^2+\cancelto{0}{y^2}+\cancelto{0}{z^2}) = \frac{2x}{2r} = \frac{x}{r}.$$

$$\frac{\partial r}{\partial y} = \frac{1}{2\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial y}(\cancelto{0}{x^2}+y^2+\cancelto{0}{z^2}) = \frac{2y}{2r} = \frac{y}{r}.$$

$$\frac{\partial r}{\partial z} = \frac{1}{2\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial z}(\cancelto{0}{x^2}+\cancelto{0}{y^2}+z^2) = \frac{2z}{2r} = \frac{z}{r}.$$

$\boxed{E9}$ Let $w = r^n$ where $r = \sqrt{x^2+y^2+z^2}$

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(r^n) = nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-1}\frac{x}{r} = nxr^{n-2}.$$

Likewise, by symmetry, $\frac{\partial w}{\partial y} = nyr^{n-2}$, $\frac{\partial w}{\partial z} = nzr^{n-2}$

$\boxed{E10}$ Let $w = \ln|x+y^2|$

$$\frac{\partial w}{\partial x} = \frac{1}{|x+y|}\frac{\partial}{\partial x}(x+y^2) = \frac{1}{|x+y|}$$

$$\frac{\partial w}{\partial y} = \frac{1}{|x+y|}\frac{\partial}{\partial y}(x+y^2) = \frac{2y}{|x+y|}$$

$\boxed{E11}$ Let $U = -1/r$

$$\frac{\partial U}{\partial x} = \frac{1}{r^2}\frac{\partial r}{\partial x} = \frac{1}{r^2}\frac{x}{r} = \frac{x}{r^3} \Rightarrow \overbrace{\frac{\partial U}{\partial y} = \frac{y}{r^3},\ \frac{\partial U}{\partial z} = \frac{z}{r^3}}^{\text{by symmetry.}}$$

Remark: the examples I gave in lecture were probably better. But, the $\boxed{E7} \longrightarrow \boxed{E11}$ calculations are likely more interesting. If you want to see more either ask, browse, or perhaps take a look at my calculus III resources. I have oodles posted. My notes go well beyond Stewart and contain many useful tid-bits for vector calculus in non-Cartesian coordinates... ok, I digress, sorry.

The gradient of a scalar function

If $f = f(x,y)$ then $\nabla f = \hat{i}\dfrac{\partial f}{\partial x} + \hat{j}\dfrac{\partial f}{\partial y}$

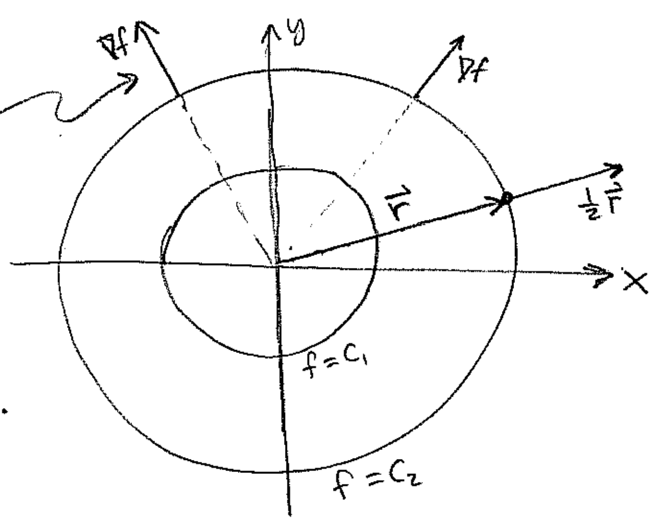
If $f = f(x,y,z)$ then $\nabla f = \hat{i}\dfrac{\partial f}{\partial x} + \hat{j}\dfrac{\partial f}{\partial y} + \hat{k}\dfrac{\partial f}{\partial z}$

The operator $\nabla = \hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}$ converts scalar functions into vector fields. The geometry is simple enough. ∇f points in direction of maximal increase for f.

$\boxed{E12}$ $f(x,y) = \frac{1}{4}(x^2+y^2)$ ← level curves of f are circles.

$$\nabla f = \hat{i}\frac{\partial}{\partial x}\left(\frac{x^2+y^2}{4}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{x^2+y^2}{4}\right) = \hat{i}\left(\frac{x}{2}\right) + \hat{j}\left(\frac{y}{2}\right)$$
$$= \frac{1}{2}(x\hat{i} + y\hat{j})$$
$$= \frac{1}{2}\vec{r}$$

I just plotted a couple samples of ∇f.



the ∇f is a vector field it gives the vector $(\nabla f)(a,b)$ at each point (a,b). (I can draw all of them!)

Let me summarize the situation in E12:
the gradient vector field $\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y}$
supplies the vector $\nabla f\big|_{(a,b)}$ at the point (a,b).
This vector $\nabla f\big|_{(a,b)}$ points in the direction <u>normal</u>
to the level curve $f(x,y) = c$ where $f(a,b) = c$.
This plays a big role in physics 232, but we'll
probably not need the finer points here for now.

<u>BIG IDEA</u>: $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}$ gives us a
way to get vectors from scalars; $\underset{\text{Scalar field}}{U} \longrightarrow \underbrace{-\nabla U = \vec{F}}_{\text{Vector field}}$

Now, only special vector fields can be obtained
in this way. <u>We define</u> $\vec{F}$ to be <u>conservative</u>
iff we can find a <u>potential function</u> U such
that $\vec{F} = -\nabla U$ on the total domain for $\vec{F}$.
This ties together with our comments on 9 due
to the following excellent Th$^{\underline{m}}$ (proved in 231 or 332)

> Th$^{\underline{m}}$ (FTC for line integrals!). Suppose C is a curve from a to b in $\mathbb{R}^n$ then:
> $$\int_{C_{a\to b}} (\nabla U)\cdot d\vec{r} = U(b) - U(a)$$

This says that $\int_{C_{a\to b}} \vec{F}\cdot d\vec{r} = -\int (\nabla U)\cdot dr = -\big(U(b) - U(a)\big)$
$$= \underbrace{U(a) - U(b)}_{\text{independent of the particular path taken by } C\,!}$$

$\boxed{E13}$ Note $U(x,y) = U_o \tan^{-1}(x-y)$ has (by $\boxed{E7}$)
$\nabla U = \frac{U_o}{1+(x-y)^2}(\hat{i} - \hat{j})$. We can calculate
the work done by $\vec{F} = -\nabla U$ along curve from $a \to b$ as

$$W_{a\to b} = \int_{C_{a\to b}} \vec{F} \bullet d\vec{r} = -\int_{C_{a\to b}} \nabla U \bullet d\vec{r} = -U(b) + U(a).$$

If the point $a = (a_1, a_2)$ and $b = (b_1, b_2)$ then

$$W_{a\to b} = -U_o \tan^{-1}(b_1 - b_2) + U_o \tan^{-1}(a_1 - a_2)$$

Remark: I would give the constant U_o units of energy to maintain proper dimensions!

$\boxed{E14}$ Consider $\boxed{E1}$ from our current view. Can we find $U(x,y,z)$ such that $\vec{F} = \vec{F}_o = F_{ox}\hat{i} + F_{oy}\hat{j} + F_{oz}\hat{k} = -\nabla U$?

We need to solve $F_{ox}\hat{i} + F_{oy}\hat{j} + F_{oz}\hat{k} = -\frac{\partial U}{\partial x}\hat{i} - \frac{\partial U}{\partial y}\hat{j} - \frac{\partial U}{\partial z}\hat{k}$
which means we need three things

$$F_{ox} = -\frac{\partial U}{\partial x}, \quad F_{oy} = -\frac{\partial U}{\partial y}, \quad F_{oz} = -\frac{\partial U}{\partial z}$$

But, we assume in $\boxed{E1}$ that $\vec{F}_o$ constant hence these are easy to solve; $\boxed{U(x,y,z) = -xF_{ox} - yF_{oy} - zF_{oz}}$.
If you prefer, we can write $U(x,y,z) = -\vec{r} \bullet \vec{F}_o$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Now calculate W from $\vec{r}_1$ to $\vec{r}_2$

$$\underbrace{W_{\vec{r}(t_1)\to\vec{r}(t_2)}}_{C_{12}} = \int_{C_{12}} \vec{F}_o \bullet d\vec{r} = -\int_{C_{12}} \nabla U \bullet d\vec{r} = -U(\vec{r}_2) + U(\vec{r}_1)$$
$$= \vec{r}_2 \bullet \vec{F}_o - \vec{r}_1 \bullet \vec{F}_o$$
$$= (\vec{r}_2 - \vec{r}_1) \bullet \vec{F}_o = \Delta\vec{r} \bullet \vec{F}_o$$

(Contrast this with $\boxed{E1}$, two ways to calculate same ↗)

If we try to find a potential function U for the $\vec{F}_f$ from $\boxed{E2}$ we should realize from the outset it's impossible. Why? Because the work done by friction is very much path dependent whereas forces with $\vec{F} = -\nabla U$ do work which depends only on the endpoints of the path considered. We return to the work-energy Th$^{\underline{m}}$ from 5.

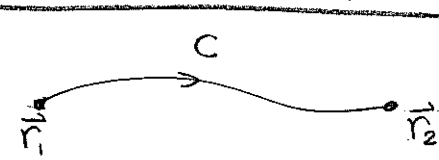
Th$^{\underline{m}}$/ (Conservation of Energy for $\vec{F}_{net} = -\nabla U$) Suppose $\vec{F}_{net} = -\nabla U$ and consider a path from point $\vec{r}_1$ to $\vec{r}_2$ where the respective velocities are likewise labeled $\vec{V}_1$ and $\vec{V}_2$,

$$U(\vec{r}_1) + K(\vec{V}_1) = U(\vec{r}_2) + K(\vec{V}_2)$$

Or defining $E = U + K$ we have $\frac{dE}{dt} = 0$ along the equations of motion where $\vec{F}_{net} = m\vec{a}$

Proof:

$$W_{net} = \int_C \vec{F}_{net} \cdot d\vec{r}$$



$$= -\int_C \nabla U \cdot d\vec{r}$$

$$= -U(\vec{r}_2) + U(\vec{r}_1)$$

But, by the work-energy Th$^{\underline{m}}$, $W_{net} = K(\vec{V}_2) - K(\vec{V}_1)$

Hence, $K(\vec{V}_2) - K(\vec{V}_1) = -U(\vec{r}_2) + U(\vec{r}_1)$ or

$$K(\vec{V}_2) + U(\vec{r}_2) = K(\vec{V}_1) + U(\vec{r}_1). \quad //$$

Remark: I prefer to write K_2, U_2, K_1, U_1 for applied problems. Moreover, I will use $K = KE$ or $U = PE$ to emphasize meaning later.

$\boxed{E15}$ Suppose $\vec{F} = \frac{-GmM}{r^3}\vec{r}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

then by $\boxed{E8}$, $\boxed{E9}$ and $\boxed{E11}$ we might not be surprised to try $U = -\frac{GmM}{r}$ for a potential energy function for $\vec{F}$. Let's see if it works, here $r = \sqrt{x^2+y^2+z^2}$

$$\nabla U = \nabla\left(-\frac{GmM}{r}\right)$$

$$= -GmM\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$= -GmM\left(-\hat{i}\frac{1}{r^2}\frac{\partial r}{\partial x} - \hat{j}\frac{1}{r^2}\frac{\partial r}{\partial y} - \hat{k}\frac{1}{r^2}\frac{\partial z}{\partial r}\right)$$

$$= -GmM\left(-\hat{i}\frac{x}{r^3} - \hat{j}\frac{y}{r^3} - \hat{k}\frac{z}{r^3}\right)$$

$$= \frac{GmM}{r^3}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{GmM}{r^3}\vec{r} \qquad \therefore \quad \vec{F} = -\nabla U$$

and we have shown that $\vec{F} = -\frac{GmM}{r^3}\vec{r}$ is conservative.

Phew: we'll discuss $\vec{F} = -\frac{GmM}{r^2}\hat{r}$ for a few lectures later this semester. This is gravity as an inverse-square law. We just proved this is a conservative force law. Imagine if gravity wasn't conservative maybe we'd spiral into the Sun.... poof.

(of course, we have a Reason not to worry about such inadequecies in physical models!)