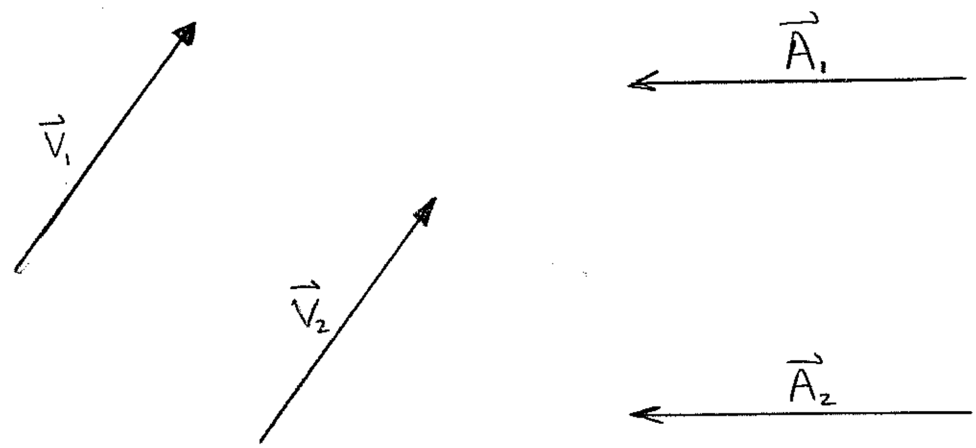
LECTURE 2

• what is a vector?

A vector in physics is a mathematical object which has both a <u>magnitude</u> (or <u>length</u>) and a <u>direction</u>. We can visualize vectors as directed line segments.



Usually, we allow these arrows to move about the space while maintaining their direction. This means $\vec{V}_1 = \vec{V}_2$ and $\vec{A}_1 = \vec{A}_2$.
(if you want to distinguish these, more power to you, and that's worthwhile, just not to Physics 231)

DIMENSIONS: (in the sense of units, not spatial extension)

The "length" of a vector need not represent an actual physical length. Often our vectors are force-vectors and the <u>units</u> of the magnitude will actually be ML/T^2.
Vectors <u>represent</u> physical variables.

HOW TO DESCRIBE VECTORS:

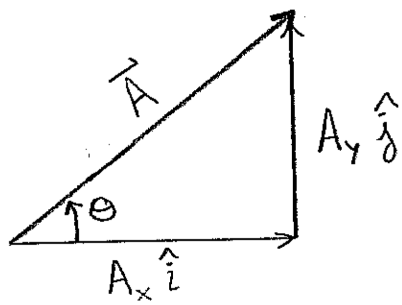
- In the two-dimensional case

(i.) graphically.

(ii.) magnitude and standard angle (or some other angle)

(iii) Cartesian components.

We use all three ideas and we must be able to freely convert between all.



$$\vec{A} = \underbrace{A_x \hat{i}}_{\substack{x\text{-vector} \\ \text{component} \\ \text{of } \vec{A}}} + \underbrace{A_y \hat{j}}_{\substack{y\text{-vector} \\ \text{component} \\ \text{of } \vec{A}}}$$

The magnitude of $\vec{A}$ is denoted by A and clearly

$$\boxed{A = \sqrt{A_x^2 + A_y^2}}$$

We call A_x, A_y the <u>scalar</u> <u>components</u> of $\vec{A}$ since they are just numbers (or functions). The standard angle is (up to a $2\pi k$ degeneracy) defined by the equations

$$\boxed{\begin{aligned} A_x &= A\cos\theta \\ A_y &= A\sin\theta \end{aligned}}$$

Given A and θ it's easy to find A_x & A_y. However, given A_x and A_y we have to be a bit cautious in how we select θ.

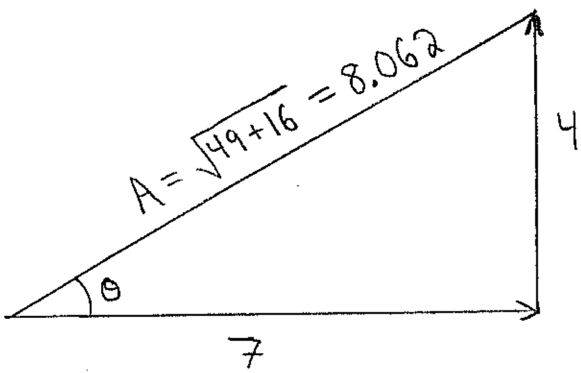
Remark: $\vec{A} = \hat{i}$ has $A = \sqrt{1^2+0^2} = 1$ and $\theta = 0$
$\vec{B} = \hat{j}$ has $B = \sqrt{0^2+1^2} = 1$ and $\theta = 90°$

Remark: A_x & A_y are the Cartesian components of $\vec{A}$ if $\vec{A}$ has the form $\vec{A} = A_x\hat{i} + A_y\hat{j}$. It is at times convenient to use other notation,

$$\vec{A} = A_x\hat{i} + A_y\hat{j} = \langle A_x, A_y \rangle = A_x\hat{x} + A_y\hat{y}$$

I'll use $\hat{i}, \hat{j}$ in this cource so my notation matches Tipler.

E1 Given $A_x = 7$ and $A_y = 4$ find the magnitude and standard angle for $\vec{A} = A_x\hat{i} + A_y\hat{j}$

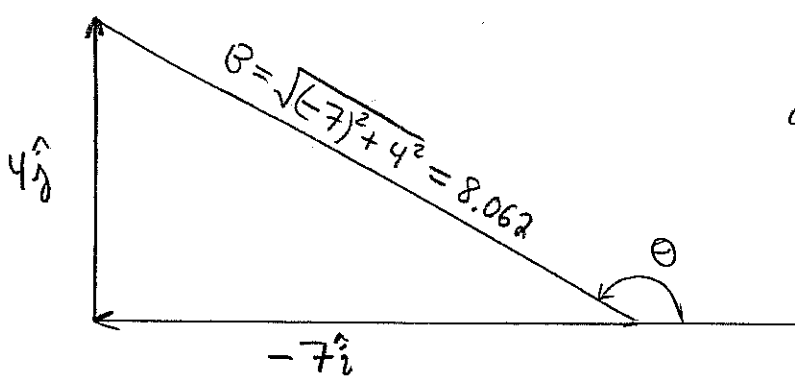


$$\tan\theta = \frac{A_y}{A_x} \quad \text{since} \quad \frac{A_y}{A_x} = \frac{A\sin\theta}{A\cos\theta} = \tan\theta$$

$$\Rightarrow \theta = \tan^{-1}\left[\frac{4}{7}\right] = \underbrace{\boxed{29.74^\circ = \theta}}$$

notice this angle is logical since $A_x, A_y > 0$ we ought to have θ in quad I where $0 < \theta < 90^\circ$

E2 Suppose $\vec{B} = -7\hat{i} + 4\hat{j}$ find B and θ for $\vec{B}$



notice $\tan\theta = \frac{-4}{7}$

and $\underbrace{\tan^{-1}(-4/7) = -29.74^\circ}$

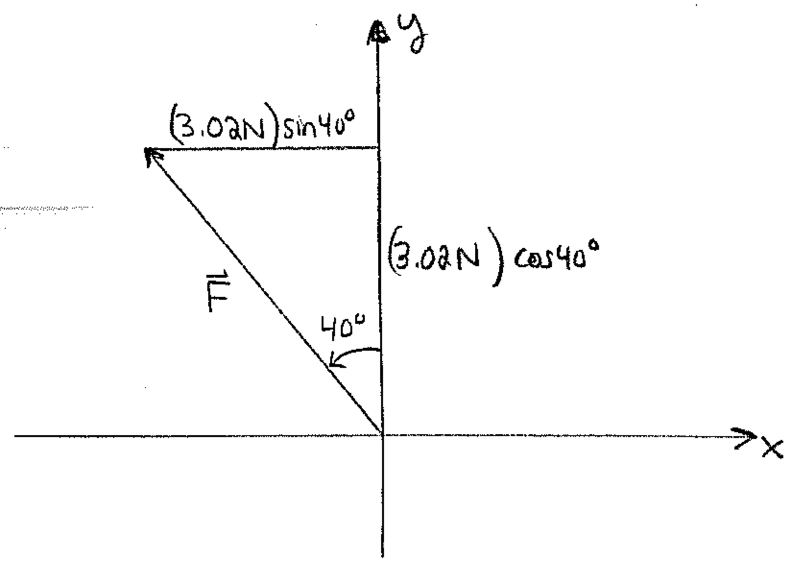
not θ for $\vec{B}$ since -29.74° puts $\vec{B}$ in quad. IV

geometrically it is clear $\theta = 180^\circ - 29.74^\circ$

$\therefore \boxed{\theta = 150.26^\circ}$

Remark: to find standard angle a good strategy is to find $\tan^{-1}\left[|A_y|/|A_x|\right]$ then simply add or subtract angle to obtain correct θ. My algorithm for this as a student was to draw a picture and think.

E3 Suppose that $\vec{F}$ has a magnitude of 3.02 N in a direction 40° west of north. Find the Cartesian components of $\vec{F}$



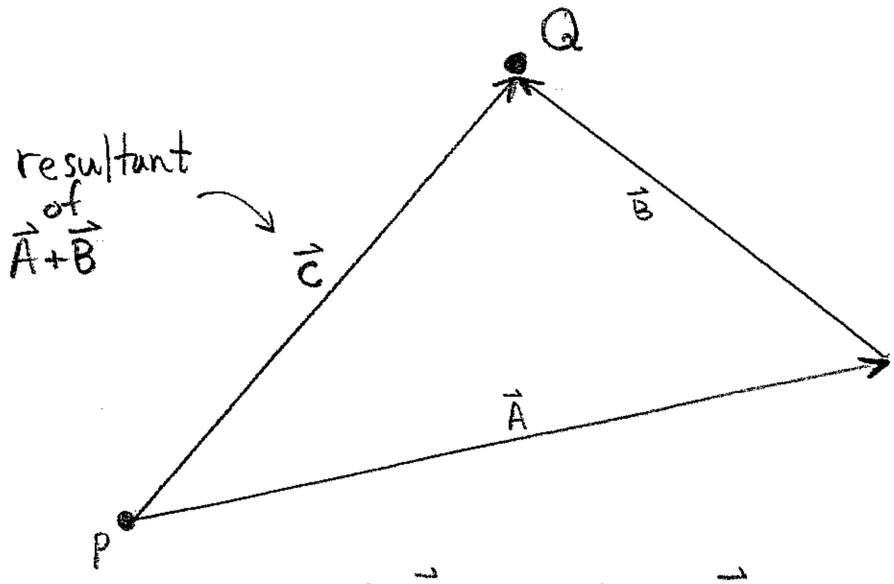
(+) y is North
(+) x is East
(convention used unless otherwise stipulated)

$$|F_x| = (3.02\,N)\sin 40° = 1.941\,N$$
$$|F_y| = (3.02\,N)\cos 40° = 2.313\,N$$
$$\therefore \boxed{\vec{F} = -(1.941\,N)\hat{i} + (2.313\,N)\hat{j}}$$

Question: writing the hat on i does seem like a pain, and do I really need the $\longrightarrow$ on $\vec{F}$? Answer: yes you do. $\vec{F}$ and F are not the same object usually. $\hat{i}$ and i are not the same. Write what is correct not what is convenient. Sorry for the sermon here, just trying to squash some bad habits before they mature ...

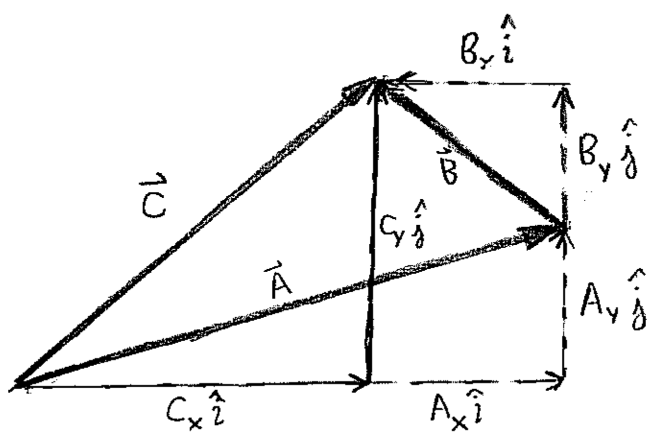
Addition of Vectors & Scalar Multiples

Graphically we add vectors tip-to-tail.
Suppose $\vec{A}$ and $\vec{B}$ are added to give $\vec{C}$, it works like so



start at P, travel $\vec{A}$ and then $\vec{B}$ we get to Q.
start at P, travel $\vec{C}$ and we get to Q.
Thus $\vec{A}+\vec{B}$ and $\vec{C}$ give the same magnitude and direction.

E4 If $\vec{A} = A_x\hat{i} + A_y\hat{j}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j}$ then
$$\vec{C} = \vec{A}+\vec{B} = (A_x+B_x)\hat{i} + (A_y+B_y)\hat{j}.$$

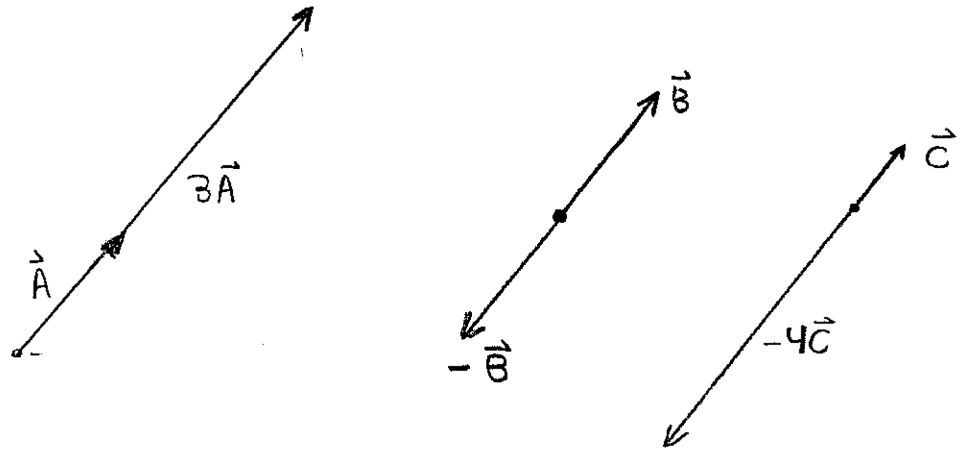


$B_x < 0$, $A_x > 0$

$C_x = A_x + B_x$ automatically accounts for directions of $\vec{A}$, $\vec{B}$ as we'd like

(draw your own picture if you don't like mine.
We can always <u>break-down</u> vectors into Cartesian component this is how we add.

E5 Scalar multiplying is a way to change the length of the vector and/or to reverse direction. For examples:

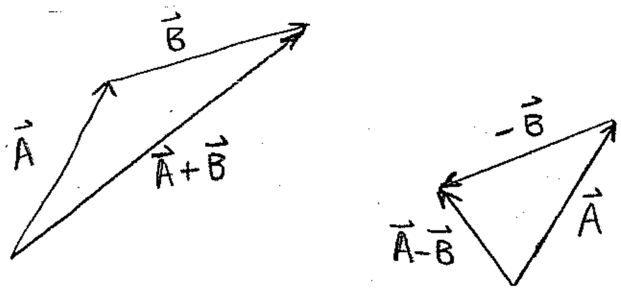


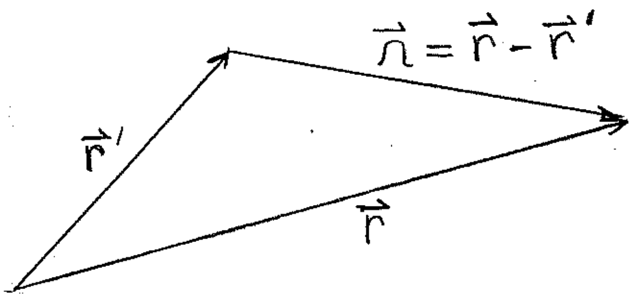
Not too suprisingly I hope, if $\vec{B} = B_x\hat{i} + B_y\hat{j}$ then

$$-\vec{B} = -B_x\hat{i} - B_y\hat{j}$$

E6 Vector subtraction is addition of reversed vector;

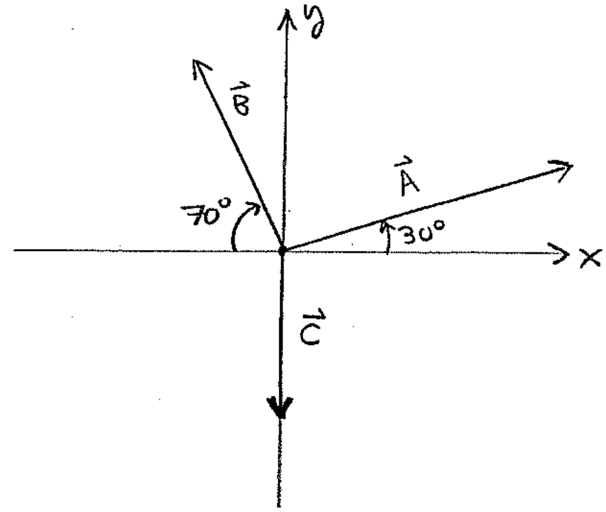
$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B}).$$





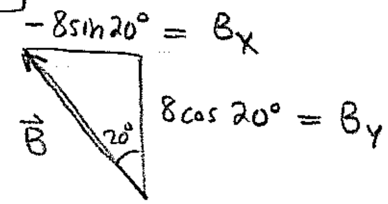
(Note that $\vec{r}\,' + \vec{\mathscr{r}} = \vec{r}$ by vector tip-tail addition so we must conclude $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$)

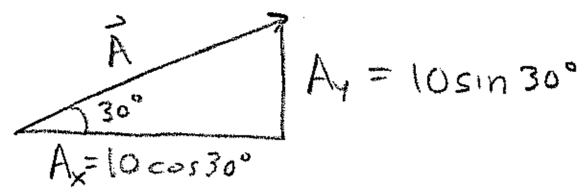
E7 Adding more than two vectors is done by resolving vectors into their Cartesian components. Suppose we wish to add $\vec{A}$, $\vec{B}$, $\vec{C}$ pictured below:



$A = 10$
$B = 8$
$C = 5$

Sol^n: Notice $\vec{C} = -5\hat{y}$ by picture hence $C_x = 0$, $C_y = -5$.



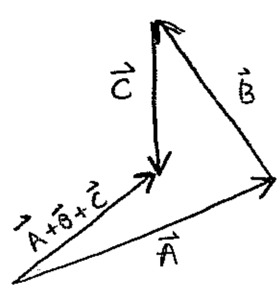


Then make a table,

	x	y
$\vec{A}$	8.660	5
$\vec{B}$	−2.736	7.518
$\vec{C}$	0	−5
$\vec{A}+\vec{B}+\vec{C}$	5.924	7.518

$$\therefore \boxed{\vec{A}+\vec{B}+\vec{C} = 5.924\,\hat{i} + 7.518\,\hat{j}}$$

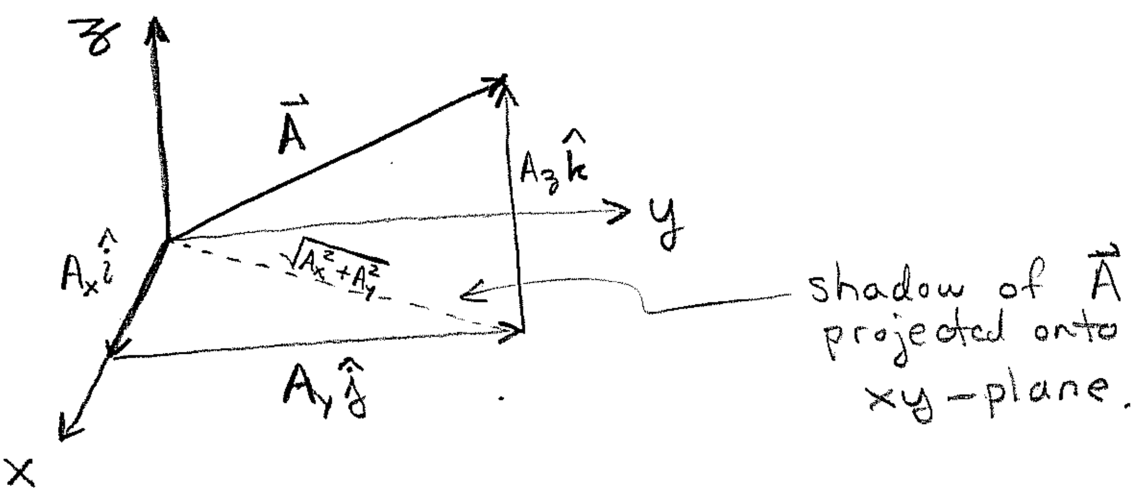
$$|\vec{A}+\vec{B}+\vec{C}| = \underline{9.572} \quad \text{at} \quad \theta = \tan^{-1}\left[\frac{7.518}{5.924}\right] = \boxed{51.76°}$$

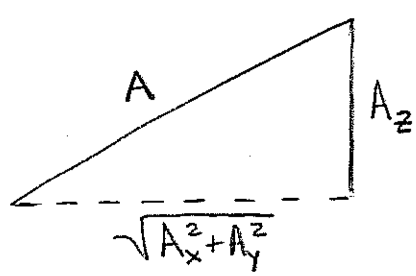


<u>Remark:</u> we have considered two-dimensional vectors in $\boxed{E1} \longrightarrow \boxed{E7}$. Often our problems are covered by this approach. However, the natural world is actually <u>3-dimensional</u> hence we should say a bit more:

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \leftarrow \text{3-dimensional vector.}$$

We have scalar components A_x, A_y, A_z. We can visualize the vector sum of the x, y, z component vectors of $\vec{A}$ as follows:





$\longrightarrow$ $\boxed{A = \sqrt{A_x^2 + A_y^2 + A_z^2}}$

magnitude for 3-vector.

$\boxed{E8}$ $\vec{A} = 3\hat{i} + 2\hat{j} - \hat{k}$

$$A = \sqrt{9+4+1} = \sqrt{14}.$$

<u>Observation</u>: one angle does not specify the direction in $\mathbb{R}^3$. We need something new.

Unit Vectors

Def^{n}/ If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \neq 0$ then we <u>define</u> $\hat{A}$ to be the following vector:

$$\hat{A} = \frac{1}{A}\vec{A} = \frac{1}{\sqrt{A_x^2+A_y^2+A_z^2}}\left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)$$

<u>Observation</u>: any nonzero vector $\vec{A}$ can be expressed as:

$$\vec{A} = \underset{\text{magnitude}}{A}\ \underset{\text{direction}}{\hat{A}}$$

The unit-vector specifies the direction of the vector.

$\boxed{E9}$ Let $\vec{A} = 3\hat{i} - 4\hat{k}$ then $A = \sqrt{9+16} = 5$. thus $\hat{A} = \frac{1}{5}(3\hat{i} - 4\hat{k})$. Moreover,

$$\vec{A} = \underset{\text{magnitude}}{5}\ \underbrace{\left[\frac{1}{5}(3\hat{i} - 4\hat{k})\right]}_{\text{direction.}}$$

$\boxed{E10}$ Vectors have units generally speaking, however the unit-vector is dimensionless. For example

$$\vec{B} = \left(3\tfrac{m}{s}\right)\hat{i} + \left(4\tfrac{m}{s}\right)\hat{j} - \left(2\tfrac{m}{s}\right)\hat{k}$$

$$\Rightarrow B = \sqrt{9\frac{m^2}{s^2} + 16\frac{m^2}{s^2} + 4\frac{m^2}{s^2}} = \left(\sqrt{29}\right)\frac{m}{s}$$

The magnitude has units of m/s whereas,

$$\hat{B} = \left(\frac{1}{\sqrt{29}\ \frac{m}{s}}\right)\vec{B} = \underbrace{\frac{3}{\sqrt{29}}\hat{i} + \frac{4}{\sqrt{29}}\hat{j} - \frac{2}{\sqrt{29}}\hat{k}}_{\text{unitless. (dimensionless)}}$$

DOT-PRODUCTS

Finally, I mention that

$$\boxed{\vec{A} \bullet \vec{B} = AB\cos\theta = A_x B_x + A_y B_y + A_z B_z}$$

This fantastically useful formula allows us to recast many of our def^{n}'s in this lecture in a compact form:

$$A = |\vec{A}| = \sqrt{\vec{A} \bullet \vec{A}} \quad \text{OR} \quad \vec{A} \bullet \vec{A} = A^2$$

You can verify that

$$\vec{A} \bullet (\vec{B} + \vec{C}) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}$$

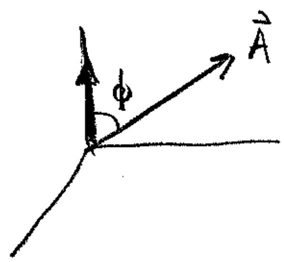
$$\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$$

$$\vec{A} \bullet (c\vec{B}) = c(\vec{A} \bullet \vec{B})$$

These are useful.

Pragmatically, we can use this to calculate angles between 3-vectors!

E11 What is angle between $\vec{A} = \hat{i} + \hat{j} + \hat{k}$ and the z-axis. (same as $\hat{k}$).



$$\vec{A} \bullet \hat{k} = A|\hat{k}|\cos\phi \quad , \quad |\hat{k}| = 1$$

$$\therefore \cos\phi = \frac{\vec{A} \bullet \hat{k}}{A} = \frac{1}{\sqrt{3}}$$

$$\therefore \phi = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = \boxed{54.74^\circ}$$