Remark: in $\boxed{E3}$ of $\boxed{\text{Lecture 25}}$ we observed that a given system of masses rigidly placed in relation to one another the moment of inertia is not simply a characteristic of the system. The location of the rotation axis is important to the value of I. Let's call such a system a "rigid body".

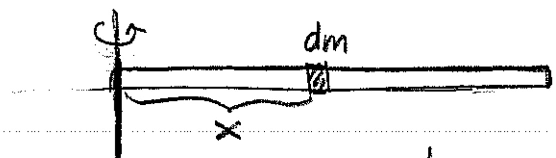
Continuous Mass Distributions

Suppose $\rho = \frac{dm}{dV}$ and let dm rotate about some axis a radial distance r at ω over the whole object S. Calculate,

$$KE_{TOTAL} = \int_S \frac{1}{2}(dm)\,v^2 = \int_S \frac{1}{2}(dm)\,r^2\omega^2 = \frac{1}{2}\underbrace{\left(\int r^2 dm\right)}_{I}\omega^2$$

Hence $KE_S = \frac{1}{2} I\omega^2$ where I is defined by an integral (could be line, area or volume depending on the dimension of S)

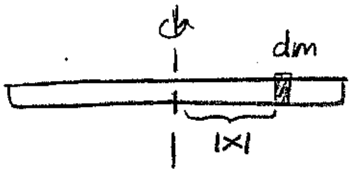
E1 Calculate I for rod of mass M of length L rotating about one end. Assume uniform density λ



$$dm = \lambda\,dx = \frac{M}{L}\,dx$$

$$I = \int_0^L x^2 \frac{M}{L}\,dx = \frac{1}{3}\frac{M}{L}x^3\Big|_0^L = \boxed{\frac{1}{3}ML^2}$$

$\boxed{E2}$ Rotate uniform rod of mass M around center, length of rod is L



$$dm = \lambda dx = \frac{M}{L} dx$$

$$dI = r^2 dm = |x|^2 \frac{M}{L} dx = x^2 \frac{M}{L} dx$$

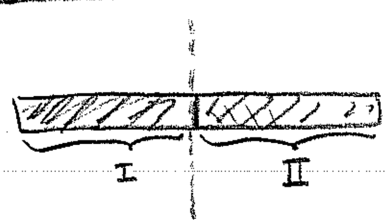
$$I = \int_{-L/2}^{L/2} \frac{M}{L} x^2 dx = \frac{M}{L} \frac{x^3}{3} \Big|_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left(\frac{L^3}{8} - \left(\frac{-L^3}{8}\right)\right)$$

$$= \boxed{\frac{1}{12} ML^2} = I_{\text{rod rotated about central point.}}$$

Notice $I_{endpt.} = \frac{1}{3} ML^2 > I_{center\ point}$. Again same mass dist. can give differing moments.

$\boxed{E3}$ Because I is defined by an integration we have additivity for I of a system. In other words, if we can break the problem into pieces we can attack a given problem in parts. Try $\boxed{E2}$ again but use $\boxed{E1}$ to find I



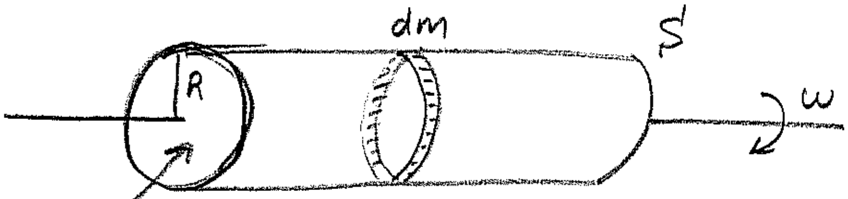
(I) $\frac{M}{2}$ length $\frac{L}{2}$ rotated around endpt.

(II) likewise, $I_I = I_{II}$ by symmetry!

$$I_I = \frac{1}{3} M_I L_I^2 = \frac{1}{3}\left(\frac{M}{2}\right)\left(\frac{L}{2}\right)^2 \quad \text{by } \boxed{E1}$$

$$\therefore \quad \underline{I_{TOTAL} = 2 I_I = \frac{1}{12} ML^2} \ \checkmark$$

E4 cylindrical shell of mass M at R from rotation axis, find moment of inertia



hollow

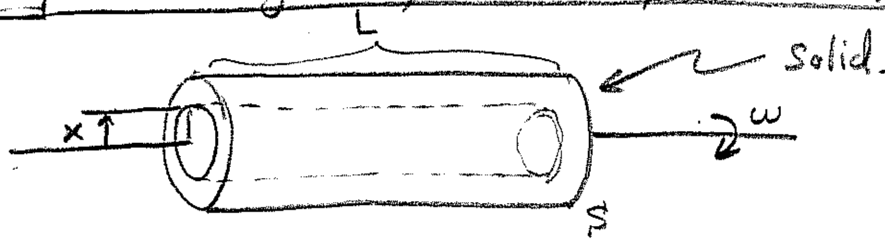
$$dI = R^2 dm$$ cylindrical shell.

$$I = \int_S R^2 dm = R^2 \int_S dm = \boxed{MR^2}$$

because all the mass is at same distance R from axis of rot.

it's like a point particle of mass M rotating at R.

E5 Solid cylinder, mass M, radius R, find I



Use **E4** to set-up the integral over S. Parse the object into series of cylindrical shells.

$$dI = x^2 dM = x^2 \rho \, dV$$

$$= x^2 \rho \, (2\pi x L dx)$$

$$= x^2 \left(\frac{M}{\cancel{\pi} R^2 \cancel{L}}\right) 2\cancel{\pi} x \cancel{L} dx$$

$$= \frac{2Mx^3 dx}{R^2}$$

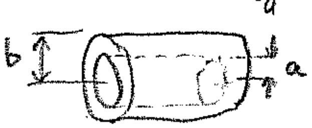
Solid cylinder.

$$I = \int_0^R \frac{2Mx^3 dx}{R^2} = \frac{2MR^4}{4R^2} = \boxed{\frac{1}{2}MR^2}$$

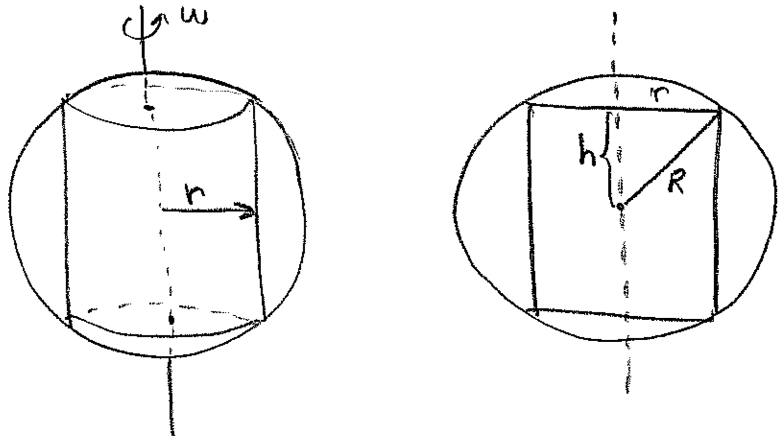
E6 for M distributed over $a \leq x \leq b$ (picture as above)

$$I = \int_a^b x^2 \underbrace{\left(\frac{M}{\pi(b^2-a^2)\cancel{L}}\right)}_{\text{Mass/volume}} 2\cancel{\pi} x \cancel{L} dx = \left(\frac{2M}{b^2-a^2}\right)\left(\frac{b^4}{4} - \frac{a^4}{4}\right)$$

$$= \frac{1}{2}M \frac{1}{\cancel{b^2-a^2}} \cancel{(b^2-a^2)}(b^2+a^2) = \boxed{\frac{1}{2}M(a^2+b^2)}$$



$\boxed{E7}$ Find I for sphere of radius R rotated about a diameter



$$h = \sqrt{R^2 - r^2}$$

$$dV = \underbrace{(2\pi r)(2h)}_{\text{area of shell}}\underbrace{dr}_{\text{thickness}} \rightsquigarrow dm = 4\rho\pi r\sqrt{R^2-r^2}\,dr$$

all at radius r from rotation axis, $dI = r^2\,dm$.

$$I = \int_0^R 4\rho\pi r^3\sqrt{R^2-r^2}\,dr$$

: Let $u = R^2 - r^2$
then $du = -2r\,dr \to r\,dr = \frac{-du}{2}$
and $u(0) = R^2$, $u(R) = 0$
note $r^2 = R^2 - u$

$$= \int_{R^2}^0 4\rho\pi r^2\sqrt{u}\left(\frac{-du}{2}\right)$$

$$= \int_0^{R^2} 2\rho\pi\,(R^2 - u)\sqrt{u}\,du$$

$$= 2\left[\frac{M}{\frac{4}{3}\pi R^3}\right]\pi\left(\frac{2}{3}R^2u^{3/2} - \frac{2}{5}u^{5/2}\right)\Bigg|_0^{R^2}$$

$$= \frac{3M}{2R^3}\left(\frac{2}{3}R^5 - \frac{2}{5}R^5\right)$$

$$\frac{2}{3} - \frac{2}{5} = \frac{10}{15} - \frac{6}{15} = \frac{4}{15}$$

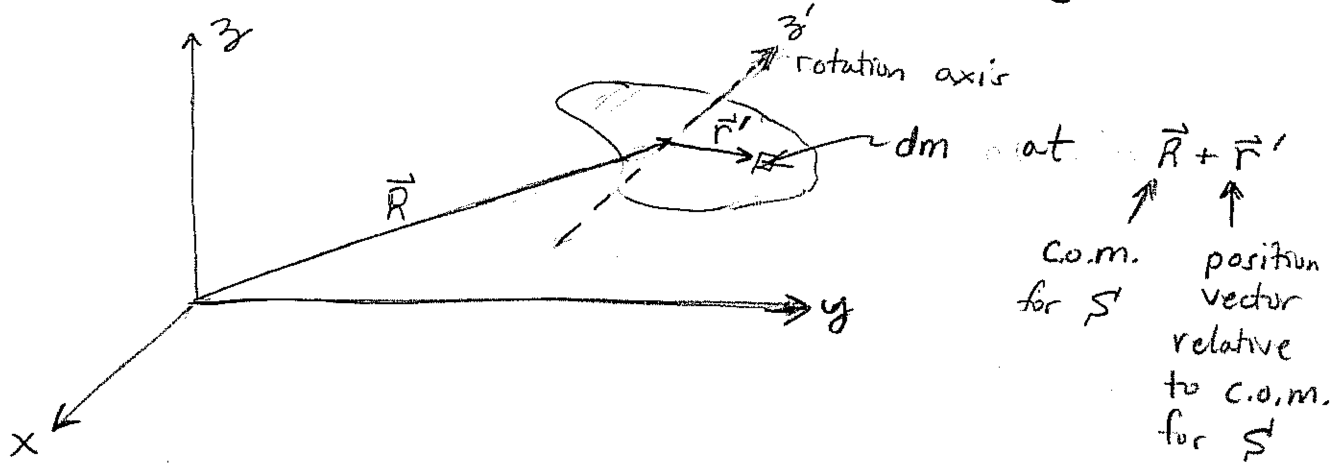
$$\frac{4}{15}\cdot\frac{3}{2} = \frac{2}{5}$$

$$= \boxed{\frac{2}{5}MR^2 = I_{sphere}}$$

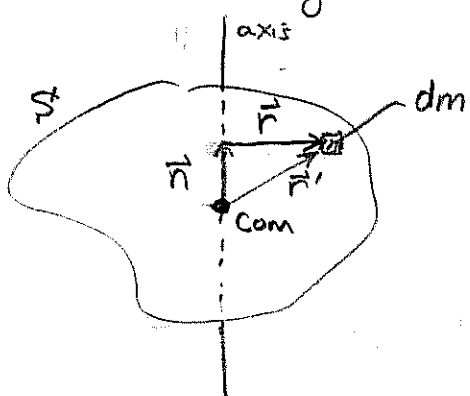
Well folks, I've done enough, see Table 9.1 for more. I will provide 9.1 for test 3, but I may ask you to derive one of those (as I have just done in $\boxed{E1}$ through $\boxed{E7}$).

(Generalization to moving system)

Most ideas from the linear context have mirrors in our current context. We now consider a system S of particles which form a rigid body.



Let's break $\vec{r}'$ down into a radial vector $\vec{r}$ and a vector $\vec{n}$ along the axis. As S rotates about



the axis only the $\vec{r}$-vector changes, the $\vec{n}$-vector is fixed under the rotation.

$$\vec{V}_{rel} = \frac{d\vec{r}\,'}{dt} = \frac{d}{dt}\left(\vec{n} + \vec{r}\right) = \frac{d\vec{r}}{dt}$$

The KE of dm is $\frac{1}{2}(dm)\left\|\frac{d}{dt}\left(\vec{R} + \vec{r}\,'\right)\right\|^2$. Let's see if we can simplify this using $\vec{V}_{cm} = \frac{d\vec{R}}{dt}$ and $\vec{V}_{rel} = \frac{d\vec{r}\,'}{dt}$,

$$\left\| \frac{d\vec{R}}{dt} + \frac{d\vec{r}\,'}{dt} \right\|^2 = \left(\frac{d\vec{R}}{dt} + \frac{d\vec{r}}{dt}\right)\cdot\left(\frac{d\vec{R}}{dt} + \frac{d\vec{r}}{dt}\right)$$

$$= \left(\vec{V}_{cm} + \vec{V}_{rel}\right)\cdot\left(\vec{V}_{cm} + \vec{V}_{rel}\right)$$

$$= \vec{V}_{cm}\cdot\vec{V}_{cm} + 2\vec{V}_{rel}\cdot\vec{V}_{cm} + \vec{V}_{rel}\cdot\vec{V}_{rel}$$

Thus the KE associated with dm is

$$dK = \frac{1}{2}dm\left(V_{cm}^2 + V_{rel}^2 + 2\vec{V}_{rel}\cdot\vec{V}_{cm}\right)$$

Continuing, add up all the dK's for S' by integrating over the rigid body S',

$$K = \int_{S'} dK = \int_{S} \frac{1}{2} dm \left(V_{cm}^2 + V_{rel}^2 + 2\vec{V}_{rel} \cdot \vec{V}_{cm} \right)$$

$$= \underbrace{\frac{1}{2} V_{cm}^2 \int_{S} dm}_{KE_{cm}} + \underbrace{\frac{1}{2} \int_{S'} V_{rel}^2 \, dm}_{KE_{rotational}} + \underbrace{\vec{V}_{cm} \cdot \int_{S'} \vec{V}_{rel} \, dm}_{\text{zero by Lemma ③}}$$

Let me break down the integrals further and introduce a new concept to help quantify $KE_{rotational}$ for common objects.

$$KE_{cm} = \frac{1}{2} V_{cm}^2 \int_{S} dm = \frac{1}{2} M V_{cm}^2 \quad \text{where } M = \text{total mass of } S'.$$

Hopefully, not too surprising. We dealt with com motion in some previous work. The $KE_{rotational}$ is new. Let ω = angular velocity of S' and let r = distance from axis to dm then $V_{rel} = r\omega$ and we have

$$KE_{rotational} = \frac{1}{2} \int_{S} V_{rel}^2 \, dm$$

$$= \frac{1}{2} \int_{S'} r^2 \omega^2 dm$$

note, ω constant over S' since S' is rigid body.

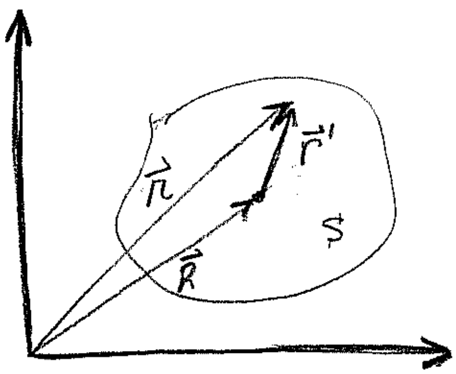
$$= \left(\frac{1}{2} \int_{S'} r^2 dm \right) \omega^2$$

$\rho = \frac{dm}{dV}$ = mass density

$$= \frac{1}{2} \underbrace{\left[\int_{S} \rho r^2 dV \right]}_{\substack{\text{Moment of Inertia for } S' \\ \text{with respect to axis discussed.} \\ \text{Usually denoted "} I \text{"}}} \omega^2 = \frac{1}{2} I \omega^2$$

③ Lemma: Claim $\vec{V}_{cm} \bullet \int_S \vec{V}_{rel}\, dm$

"Proof": $\displaystyle \int_S \vec{V}_{rel}'\, dm = \int_S \frac{d\vec{r}'}{dt} dm = \frac{d}{dt}\int_S \vec{r}'\, dm =$



$$\vec{r} = \vec{R} + \vec{r}'$$

$\vec{r}$: position vector of dm relative to fixed origin of an inertial frame

$\vec{R}$: position of com

$\vec{r}'$: position vector of dM relative to $\vec{R}$.

Continuing,

$$\int_S \vec{V}_{rel}\, dm = \frac{d}{dt}\int_S (\vec{r} - \vec{R})\, dm$$

$$= \frac{d}{dt}\left[\int_S \vec{r}\, dm - \vec{R}\int_S dm\right]$$

why can I pull $\vec{R}$ out of the $\int_S$?

$$= \frac{d}{dt}\left[M\vec{R} - \vec{R}M\right]$$

$$= 0.$$

Where I have recalled $\vec{R} = \frac{1}{M}\int_S \vec{r}\, dm$ is the continuous formulation of the com vector for S and $\int_S dm = M$ = total mass of S.

Summary: for a rigid body S with total mass M and moment of inertia I the total mechanical energy is

$$\boxed{K = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} I\omega^2}$$

V_{cm}: velocity of com

ω: angular velocity of body with respect to a given axis.

Remark: the past 3 pages are elegantly argued in Tipler for the finite case in his §8.2. Technically this is from §9.6 on rolling objects, but I've added it here because I think it fits in this Lecture. It will help illuminate the utility of the formula

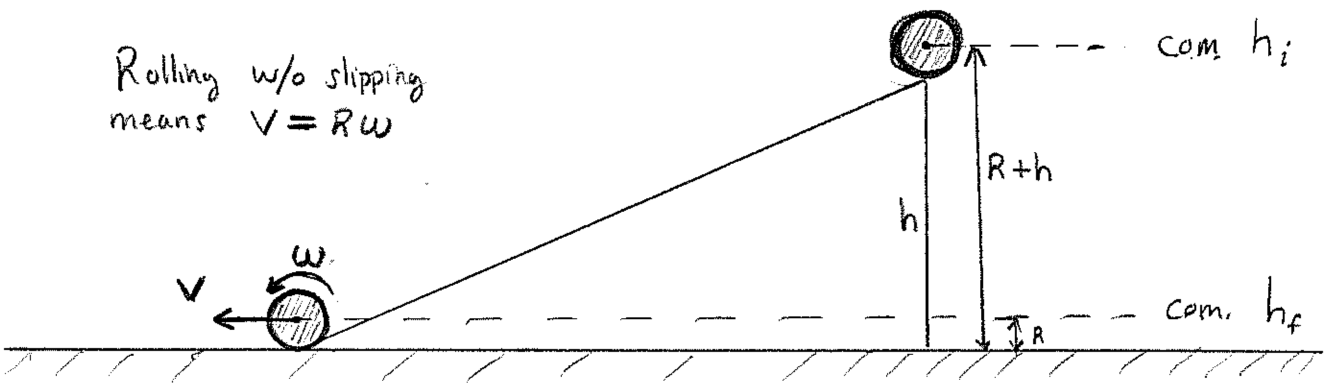
$$K_{TOTAL} = \underbrace{K_{TRANSLATIONAL}}_{\frac{1}{2}MV_{cm}^2} + K_{ROTATIONAL}$$

w.r.t. the com axis. It's $\frac{1}{2}I_{cm}\omega^2$

E8 Suppose a cylinder with mass M and radius R rolls without slipping down an inclined plane of height h. How fast does M roll past the base of the plane?

Rolling w/o slipping is a conservative process, no energy lost as a result of the rolling. On the other hand, energy is converted from $PE = Mg(h+R)$ to KE as the cylinder rolls down the plane

Rolling w/o slipping means $V = R\omega$



$$E_i = E_f$$

$$Mg(R+h) = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 + MgR$$

: where $I = \frac{1}{2}MR^2$ since we have a cylinder rotating around its axis.

Notice the MgR terms vanish and often as I work problems such as this I simply anticipate this by shifting $PE = 0$ to the height $h = R$. Continuing,

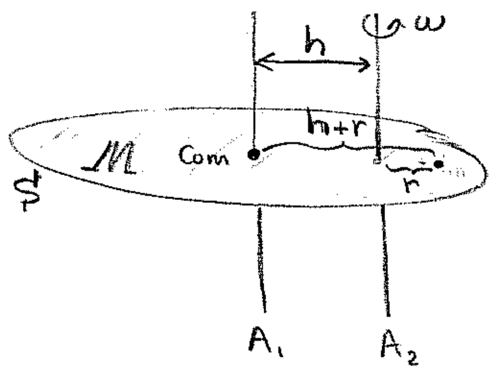
$$\cancel{M}gh = \frac{1}{2}\cancel{M}v^2 + \frac{1}{4}\cancel{M}R^2\left(\frac{V}{R}\right)^2 = \frac{3}{4}\cancel{M}v^2$$

$$\therefore \boxed{V = \sqrt{\frac{4gh}{3}}}$$

try working this example for other reasonable shapes from Table 9.1.

Remark: an experiment similar to $\boxed{E8}$ may have helped Galileo to shed his Aristotelian ideals as they failed to exhibit themselves in experiment. Heavy and light spheres rolled with the same acceleration, an acceleration based on geometry and the height of the incline, not on the raw size or what we now understand as mass.

THE PARALLEL-AXIS THEOREM: NEW MOMENTS FROM OLD



$$I_{A_2} = I_{cm} + Mh^2$$

moment of inertia w.r.t. parallel axis A_2

moment of inertia for some axis A_1 through the com

h is distance between A_1 & A_2 which is meaningful since $A_1 \parallel A_2$.

Proof: Suppose S rotates around A_2 with ω then $KE = \frac{1}{2} I_{A_2}\omega^2$ but on the other hand, using $K = \frac{1}{2}MV_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$ we find $K = \frac{1}{2}M(h\omega)^2 + \frac{1}{2}I_{cm}\omega^2$ as $V_{cm} = h\omega$ (think!)

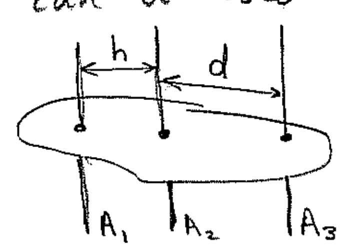
$$\therefore \quad \frac{1}{2}Mh^2\omega^2 + \frac{1}{2}I_{cm}\omega^2 = \frac{1}{2}I_{A_2}\omega^2$$

$$\frac{1}{2}(Mh^2 + I_{cm})\omega^2 = \frac{1}{2}I_{A_2}\omega^2$$

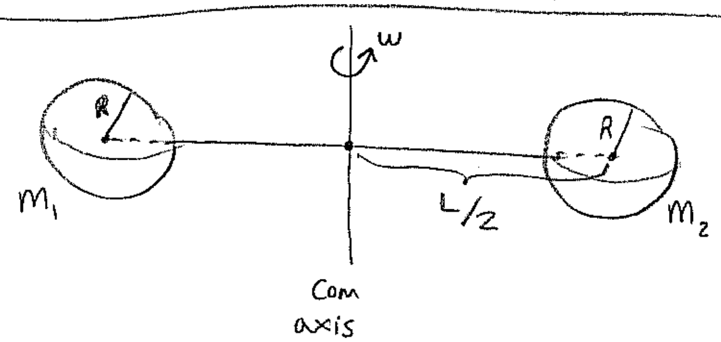
This holds for arbitrary ω $\therefore$ $\boxed{I_{A_2} = I_{cm} + Mh^2}$ //

Remark: if we throw in third axis A_3 a distance d from A_1 then $I_{A_3} = I_{cm} + Md^2$ and $I_{A_2} = I_{cm} + Mh^2$ can be used to eliminate $I_{cm} = I_{A_3} - Md^2 = I_{A_2} - Mh^2$

$$\therefore \quad I_{A_3} = I_{A_2} + M(d^2 - h^2)$$



$\boxed{E9}$ Find I_{cm} for barbell with length L (see picture below) and weights of radius R and mass M. Suppose they're spherical.



$$I_{cm} = I_{m_1} + I_{m_2} = 2I_{m_2} \quad \text{by symmetry.}$$
$$= 2\left[\tfrac{2}{5}MR^2 + M\left(\tfrac{L}{2}\right)^2\right]$$
$$= \underline{\tfrac{4}{5}MR^2 + \tfrac{1}{2}ML^2}.$$

Note that as $R \longrightarrow 0$ we recover $\boxed{E3}$ part A of LECTURE 25.

Assignment: go, work through Examples 9-6 and 9-7 of §9.3. These are great examples to help you ingest the meaning of "I". I think of it this way, an object with $I \neq 0$ and $\omega \neq 0$ has extra energy stored in rotational motion. In effect, it's another source of inertia (resistance to change in motion). It takes more energy to roll a ball without slipping to a speed v than it does to just throw it w/o using energy to pump-up the KE_{rot}.

CONFESSION: our treatment thus far is naive. For most objects we need the inertia tensor to properly understand the 3D motion which may be superpositions of several rotational motions. We'd need eigenvectors and eigenvalues and if you wish see my Math 321 notes for more. (or ask, I have books ☺)