LECTURE 3

• kinematics in one-dimension, definitions & calculus

Kinematics is the study of motion. Naturally this is a 3-dimensional question, however, we begin by examining the nature of 1-dimensional motion. We don't need vector notation in this context.

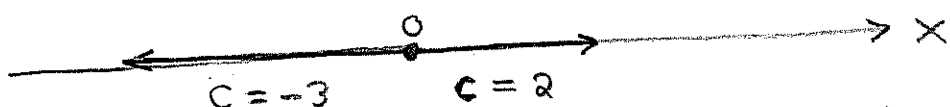
$\mathbb{R}^3$) $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$, $|\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}} = \sqrt{A_x^2+A_y^2+A_z^2} = A$

$\mathbb{R}^2$) $\vec{B} = B_x\hat{i} + B_y\hat{j}$, $|\vec{B}| = \sqrt{\vec{B}\cdot\vec{B}} = \sqrt{B_x^2+B_y^2} = B$

$\mathbb{R}^1$) $\vec{C} = C_x\hat{i}$, $|\vec{C}| = \sqrt{C_x^2} = \underbrace{|C_x|}_{\text{absolute value}}$ ⟵ in this case author does not say $|\vec{C}| = C$.

Writing $\vec{C} = C_x\hat{i}$ is not really needed if all we consider happens along the x-direction. Therefore, in the special case of one-dimensional motion we adopt the notation: $\hat{i} = 1$ thus $\vec{C} = C_x$. Usually we drop the $\rightarrow$ and write $C = C_x$. Which means we should not use $|\vec{C}| = C$ for one-dim. case since $|\vec{C}| \geq 0$ whereas C_x could be negative.

$\boxed{E1}$



Sign of one-dim. vector C indicates direction
(+) goes to right
(−) goes to left.

Remark: I probably could have said less here ☺
The pages to follow elucidate our 1-dim'l notation in detail.

Kinematics in one-dimension

Let x denote the position of a particle undergoing one dimensional motion. Here the term "particle" means we ignore the internal motions of the object in question. This amounts to an approximation since internal motions may effect the total motion, but for the examples we work in this part of the course it suffices to regard a object as if it were at a point.

E2

Car at A has $x = -5m$.
Car B has $x = 6m$.

If event A happened at $t = 2.0s$ then $x(2.0s) = -5m$.
If event B happened at $t = 1.0s$ then $x(1.0s) = 6m$.

Remark: the notation $x(t)$ means x at time t. This is function notation, not multiplication by t.

Def^{n}/ An event is a time and position.

E3 A is the event $(2.0s, -5m)$
B is the event $(1.0s, 6m)$ } using comments above of course.

Given two events we can calculate the average velocity as follows

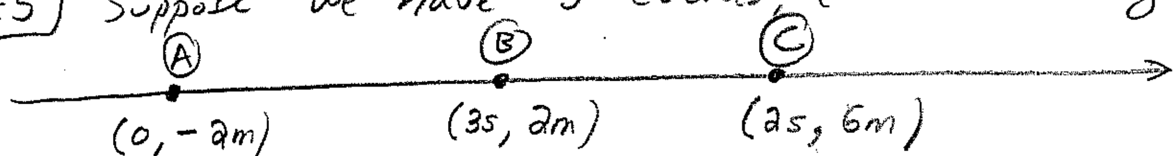
Def^n/ Given events (t_1, x_1), (t_2, x_2) with $t_1 < t_2$, average velocity is $V_{avg.} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$

E4 for our events A and B,

$$V_{avg} = \frac{-5m - 6m}{2.0s - 1.0s} = \frac{-11\,m}{1\,s} = \boxed{-11\,\frac{m}{s}}$$ (motion goes to left.)

- The dimensions for velocity are $\frac{L}{T} \longrightarrow \frac{m}{s}$ (in SI units)
  - Δx = the change in x
  - Δt = the change or duration of time.

E5 Suppose we have 3 events (insert story here)



$t_A = 0s$ $\qquad$ $t_B = 3s$ $\qquad$ $t_C = 2s$

$x_A = -2m$ $\qquad$ $x_B = 2m$ $\qquad$ $x_C = 6m$

We can calculate several average velocities,

$$V_{avg}(A,B) = \frac{x_B - x_A}{t_B - t_A} = \frac{2m - (-2m)}{3s - 0s} = \frac{4}{3}\,\frac{m}{s} \cong \underline{1.333\,\frac{m}{s}}$$

$$V_{avg}(B,C) = \frac{x_B - x_C}{t_B - t_C} = \frac{2m - 6m}{3s - 2s} = \underline{-4\,\frac{m}{s}}$$

Notice that $V_{avg.}(B,C) < 0$ means the particle travelled left whereas $V_{avg}(A,B) > 0$ indicated motion to the right.

I'll define <u>distance</u> <u>travelled</u> carefully with calculus, but for now I hope we can all agree that in $\boxed{\text{E5}}$ the distance travelled from event A $\longrightarrow$ C is 8m then from C $\longrightarrow$ B we travel 4m. Thus the net-distance travelled from event A $\longrightarrow$ B is $8m + 4m = \boxed{12m}$

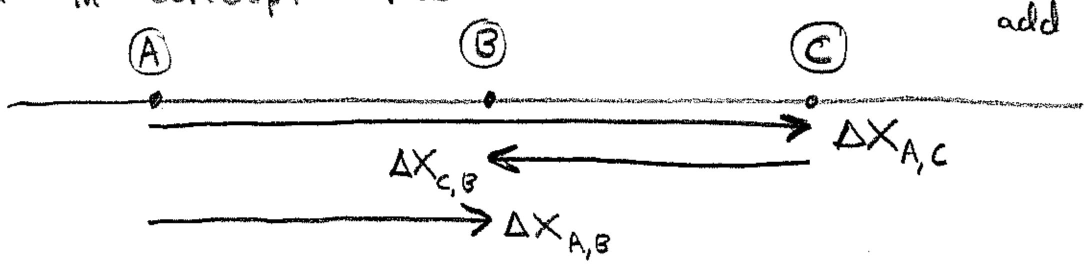
> **Def^n/** <u>Displacement</u> between two events (t_A, x_A), (t_B, x_B) is $\Delta x_{AB} = x_B - x_A$

(I do not insist $t_A < t_B$)
(the displacement between two positions is independent of time in principle.)

$\boxed{\text{E6}}$ Considering once more the events in $\boxed{\text{E5}}$,

$$\Delta x_{AB} = x_B - x_A = 4m \neq 12m = \text{distance travelled.}$$
$$\Delta x_{BA} = x_A - x_B = -4m$$

Displacement is a <u>vector</u> it has direction whereas distance is a <u>scalar</u> which is necessarily positive and in concept has no direction. Note that displacements add like vectors,



Notice $\underbrace{\Delta x_{A,C} + \Delta x_{C,B}}_{} = \Delta x_{A,B}$.

net-displacement is vector-sum of all steps in the motion.

Instantaneous Velocity

Average velocity is a sloppy concept since we need two times. In contrast,

Def^n/ The velocity of a particle with position $x(t)$ is

$$V(t) = \frac{dx}{dt} = \lim_{\Delta t \to 0}\left[\frac{x(t+\Delta t) - x(t)}{\Delta t}\right]$$

This velocity is defined at one time t, or an instant of time, so we call it instantaneous. However, this def^n is primary so we just write V (not $V_{instantaneous}$) as by default velocity is defined with calculus.

E7 Suppose $x(t) = V_o t - \frac{1}{2}gt^2$ where V_o, g are constants. Find V (and V_{avg} for $t_1 < t_2$).

$$V = \frac{dx}{dt} = \frac{d}{dt}\left[V_o t - \frac{1}{2}gt^2\right] = \underline{V_o - gt}.$$

omitted t-dependence here

put back here.

$$\therefore \boxed{V(t) = V_o - gt}$$

In contrast,

$$V_{avg}(t_1,t_2) = \frac{\Delta x}{\Delta t} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

$$= \frac{\left(V_o t_2 - \frac{1}{2}gt_2^2\right) - \left(V_o t_1 - \frac{1}{2}gt_1^2\right)}{t_2 - t_1}$$

$$= \boxed{\frac{V_o(t_2 - t_1) - \frac{1}{2}g(t_2^2 - t_1^2)}{t_2 - t_1} = V_{avg}}$$

- this simplifies, can you see how?

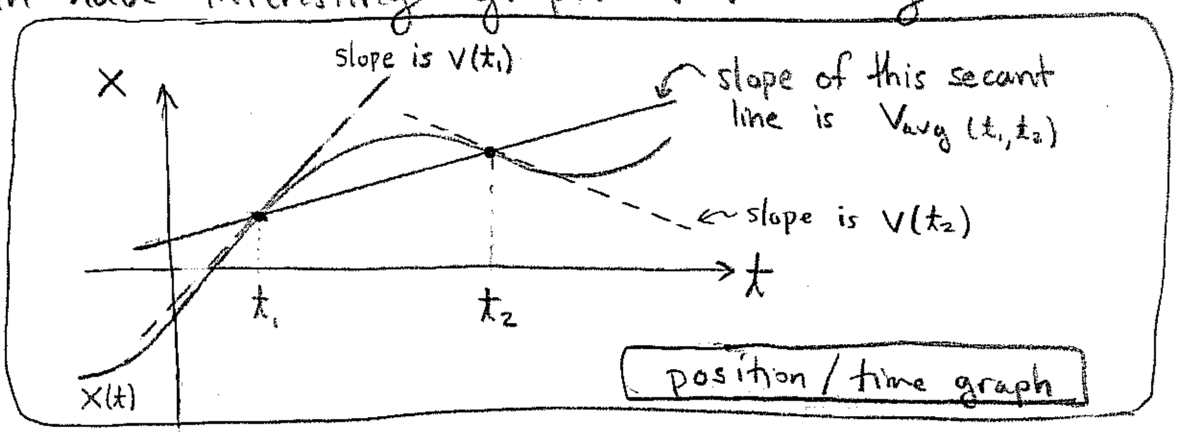
Def$^{\underline{n}}$/ Speed is the magnitude of the velocity; $\frac{ds}{dt} = |V(t)|$

E8 for example **E7** we find $\frac{ds}{dt} = |V_o - gt|$

absolute value bars

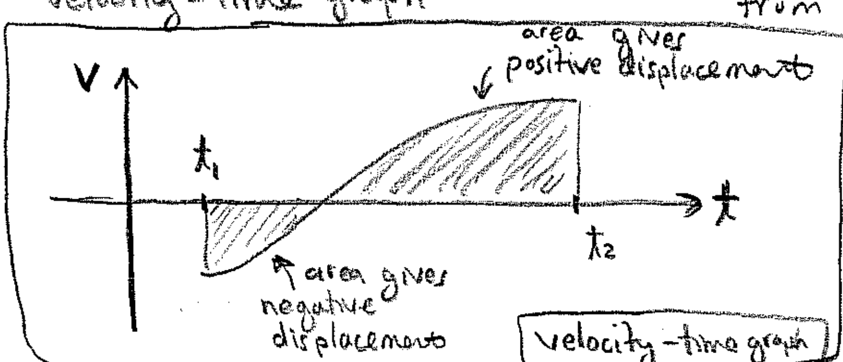
or, $\frac{ds}{dt} = \sqrt{(V_o - gt)^2}$
if you want an formula.

The average and instantaneous velocity are usually distinct. Both have interesting graphical meaning



Observe that $V(t_1) \neq V_{avg}(t_1, t_2) \neq V(t_2)$. Since we've all had calculus I we know FTC hence,

$$\underbrace{\int_{t_1}^{t_2} V(t)\, dt}_{\text{signed-area under velocity-time graph}} = \int_{t_1}^{t_2} \frac{dx}{dt}\, dt = \underbrace{x(t_2) - x(t_1)}_{\text{displacement from } x(t_1) \text{ to } x(t_2)}$$



Def^n/ Let $x(t)$ be the position of a particle then the distance travelled from time $t_1 \to t_2$ is given by the integration below:

$$S_{t_1 \to t_2} = \Delta S = \int_{t_1}^{t_2} |V(t)|\, dt$$

We can also define distance travelled from time t_1 to time t as $s(t)$ where

$$s(t) = \int_{t_1}^{t} |V(\tau)|\, d\tau$$

dummy variable of integration needed since t is argument of $s(t)$.

Notice that

$$\frac{ds}{dt} = \frac{d}{dt}\int_{t_1}^{t} |V(\tau)|\, d\tau = |V(t)|$$

The speed is the instantaneous change in the distance travelled. Speed and distance travelled are both non-negative.

E9 Let $x(t) = A\cos(\omega t)$ (Eq^n of a spring which oscillates with amplitude A and angular freq. ω. Find displacement and distance travelled for complete cycle.

$$V(t) = \frac{dx}{dt} = -A\omega \sin(\omega t)$$

$$\Rightarrow \frac{ds}{dt} = A\omega|\sin \omega t|$$

We need $0 \le \omega t \le 2\pi$ to get a whole oscillation.

$\Rightarrow$ use $t_1 = 0$ and $t_2 = \frac{2\pi}{\omega}$.

$$\Delta x_{1,2} = \int_0^{\frac{2\pi}{\omega}} -A\omega \sin\omega t\, dt = 0 \quad \text{vs.} \quad \Delta S_{1,2} = \int_0^{\frac{2\pi}{\omega}} A\omega|\sin \omega t|\, dt = \boxed{4A}$$

(WORK IT OUT!)

ACCELERATION AND AVERAGE ACCELERATION

Average acceleration is defined for a pair of events (t_1, x_1), (t_2, x_2) where $V(t_1)$ and $V(t_2)$ are the respective velocities at t_1 & t_2, $(t_1 < t_2)$

$$a_{avg}(t_1, t_2) = \frac{\Delta V}{\Delta t} = \frac{V(t_2) - V(t_1)}{t_2 - t_1}.$$

This quantity is sometimes useful for crude-estimations of processes we don't have the ability to calculate the instantaneous acceleration.

Def^n/ $a(t) = \frac{dV}{dt} = \frac{d^2X}{dt^2}$ for a particle with position $X(t)$, $a(t)$ is the <u>acceleration</u>

This means $a(t)$ is the instantaneous rate of change for the velocity.

E10 Consider $x(t) = V_0 t - \frac{1}{2} g t^2$ from **E7**. Compare the average and instantaneous accelerations.

Note $V = \frac{dx}{dt} = V_0 - gt$ thus $a = \frac{dV}{dt} = -g$.

This particle has <u>constant</u> acceleration of $-g$.

The average acceleration for $[t_1, t_2]$ is

$$\begin{aligned} a_{avg} = \frac{\Delta V}{\Delta t} &= \frac{V(t_2) - V(t_1)}{t_2 - t_1} \\ &= \frac{(V_0 - gt_2) - (V_0 - gt_1)}{t_2 - t_1} \\ &= \frac{-g(t_2 - t_1)}{t_2 - t_1} \\ &= -g. \end{aligned}$$

In the case of <u>constant</u> acceleration we find $a_{avg} = a$.
This is also true for constant velocity; $V_{avg} = V$.

E11 Consider $x(t) = A\cos(\omega t)$ as in **E9**. Compare the acceleration and average acceleration.

We find $v = -A\omega\sin(\omega t)$ since A, ω are assumed constant hence $\frac{dv}{dt} = -A\omega^2\cos(\omega t)$. Thus $\boxed{a(t) = -A\omega^2\cos(\omega t)}$

Interesting to note $a(t) = -\omega^2 x(t)$ thus $\frac{d^2x}{dt^2} = -\omega^2 x(t)$

Our position function is the sol$^{\underline{n}}$ of the diff. eq$^{\underline{n}}$.

Average acceleration, look at $[t_1, t_2]$,

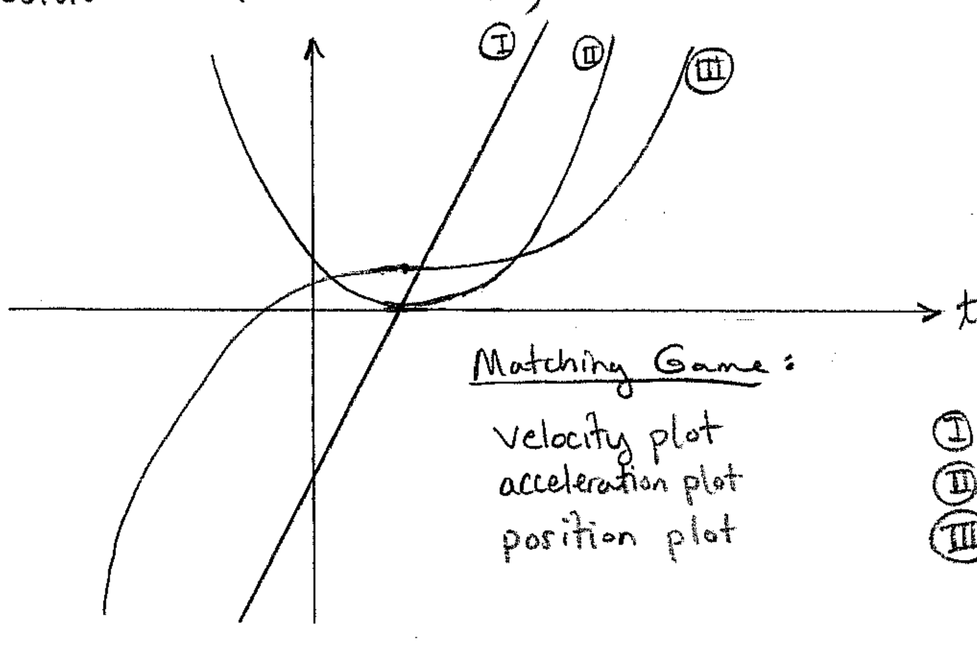
$$a_{avg} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

$$= \frac{-A\omega\sin(\omega t_2) + A\omega\sin(\omega t_1)}{t_2 - t_1} \neq a(t) \text{, usually.}$$

for given A, t_1 and ω we could likely find t_2 such that $a(t_2) = a_{avg}[t_1, t_2]$. but we cannot have $a(t) = a_{avg}$ for all t here.

Graphical Question

If we draw plots of $x(t)$, $v(t)$, $a(t)$ on the same plot then we can appreciate the interrelation of position, velocity and acceleration a little better,



Matching Game:

Velocity plot	I
acceleration plot	II
position plot	III

Remarks: when plotting v, x, a on same graph the magnitudes of v, x, a are rescaled as to make simultaneous viewing possible. Technically we plot $x, k_1 v, k_2 a$

Remark: up to this point my focus on kinematics has been primarily mathematical. In my view these terms are essentially basic and warrant no more primary justification. That said, what makes x, v, a and t physical is their utility to describe and analyze real-world problems. Once paired with Newton's Laws we have physics which is an experimentally verifiable science. Philosophy aside, to the word problems we go ...