Lecture 36:

- Simple Harmonic Motion (SHM) and energy associated with SHM. To begin we discuss a bit of mathematics that frames this Lecture and the next.

Problem: Solve $Ay'' + By' + Cy = 0$ where A, B, C are real constants and $A \neq 0$

Sol^{n}: we can show there are 3 types of sol^{n}'s

I.) If $B^2 - 4AC > 0$ then let λ_1, λ_2 be sol^{n}'s to $A\lambda^2 + B\lambda + C = 0$ then

$$\boxed{y = c_1 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t}}$$

Where c_1, c_2 are arbitrary constants which allow us to fit given initial conditions.

II.) If $B^2 - 4AC = 0$ then let $\lambda = -B/2A$ and

$$\boxed{y = c_1 e^{-\lambda t} + c_2 t e^{-\lambda t}}$$

where again c_1, c_2 are arbitrary constants to fit initial conditions

---

III.) If $B^2 - 4AC < 0$ then let $\alpha = \frac{-B}{2A}$ and let $\beta = \frac{\sqrt{4AC - B^2}}{2A}$ to write sol^{n}

$$\boxed{y = c_1 e^{-\alpha t} \cos\beta t + c_2 e^{-\alpha t} \sin\beta t}$$

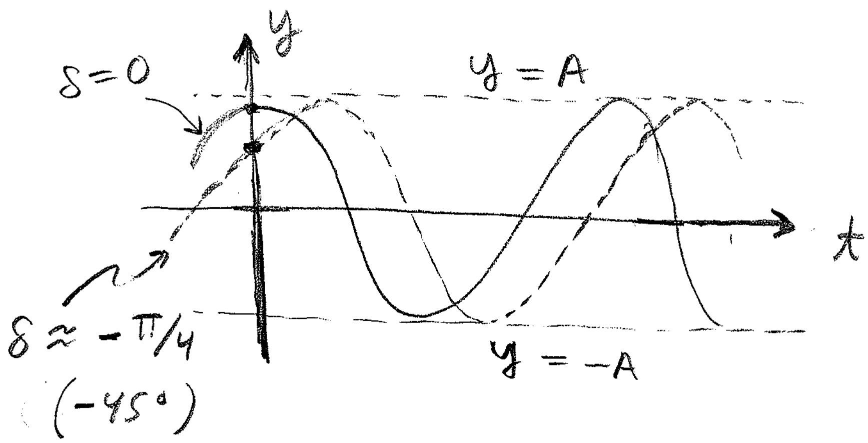
Cases II and III of most importance and in case III we often have $\alpha = 0$ so $\boxed{y = c_1 \cos\beta t + c_2 \sin\beta t}$ (SHM)

Further Mathematics on Case III

The case $\lambda = \alpha \pm i\beta$ generally yields sol^{n} $y = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t$. However when $\alpha = 0$ this simplifies to a pure oscillatory sol^{n} of the form $y = c_1\cos\beta t + c_2\sin\beta t$. We can just as well express this by a general formula using the <u>amplitude</u> (A) and the phase (δ)

$$\boxed{y = A\cos(\beta t + \delta)}$$

Usually we let $\beta = \omega$ in these sort of formulas but I wanted to connect to the I, II, III formulas.



$y(0) = A\cos(\delta)$

$y(t) = A\cos(\beta t + \delta)$

shifts graph left for $\delta > 0$. right for $\delta < 0$.

E1 Suppose $y'' + \omega^2 y = 0$ for some real constant ω,

$$\lambda^2 + \omega^2 = 0 \Rightarrow \lambda = \pm\sqrt{-\omega^2} = \pm i\omega$$

Thus $y(t) = A\cos(\omega t + \delta)$

E2 $y'' + 9y = 0$

$\longrightarrow y(t) = A\cos(3t + \delta)$

<u>Remark</u>: the DEq^{n} tells us the angular freq. of the sol^{n}.

Continuing with general mathematical analysis of $y'' + w^2 y = 0$,

$$y = A\cos(wt + \delta)$$

is bounded between $y = -A$ and $y = A$. The period T is the smallest value T such that $y(t+T) = y(t)$ $\forall t$. (this defines periodic motion of which $y = A\cos(wt+\delta)$ is a rather famous case). Cosine cycles as its arguments go from 0 to 2π thus $\Rightarrow$

$$(\omega T + \delta) - \delta = 2\pi$$

$$\Rightarrow \omega T = 2\pi \qquad \therefore \; \omega = \frac{2\pi}{T}$$

$$\text{or} \quad \boxed{T = \frac{2\pi}{\omega}}$$

The frequency $f = 1/T$ has $\omega = 2\pi f$.

$\boxed{E3}$ Consider $y'' + 7y = 0$ (in meters & seconds)

$$w^2 = 7 \Rightarrow w = \sqrt{7}\ \tfrac{rad}{s} = 2.646\ \tfrac{rad}{s}$$

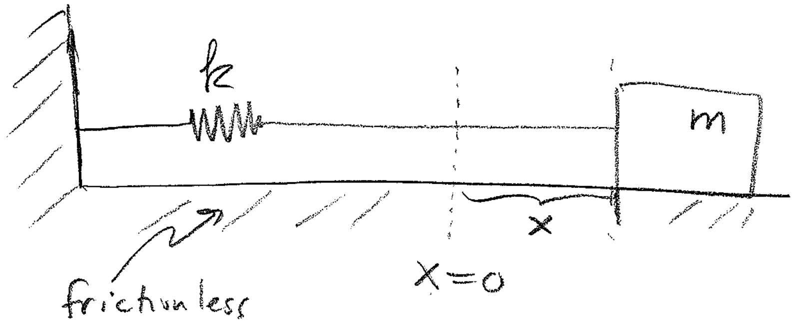
$$\Rightarrow T = \frac{2\pi}{\sqrt{7}}\ s = \underline{2.375\ s}.$$

$$\Rightarrow f = \frac{\sqrt{7}}{2\pi}\ Hz = \underline{0.421\ Hz}.$$

"Hertz" is $\frac{1}{s}$ and is the natural unit for frequency.

Spring Mass System and SHM

- we now apply the math to a common physical situation (even though this is not directly seen, many systems are essentially the same as a spring/mass system w/o friction, generically we call this <u>Simple Harmonic Motion</u>)



Hooke's Law

$$F = -kx$$

Newton's 2nd Law $\quad m\dfrac{d^2x}{dt^2} = -kx$

$$x'' + \frac{k}{m}x = 0$$

$$\Rightarrow \lambda^2 + \frac{k}{m} = 0 \quad \Rightarrow \quad \lambda = \pm\sqrt{-\frac{k}{m}} = \pm i\omega$$

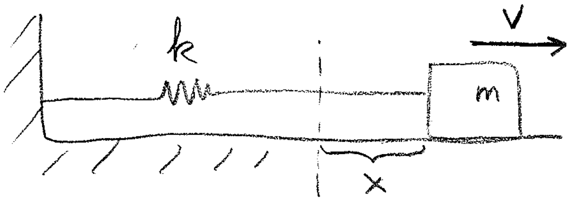
where we define $\omega = \sqrt{\dfrac{k}{m}}$

Therefore,

$$\boxed{\begin{aligned} x(t) &= A\cos(\omega t + \delta) \\ \omega &= \sqrt{\frac{k}{m}} \\ T &= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \\ f &= \frac{1}{2\pi}\sqrt{\frac{k}{m}} \end{aligned}}$$

Analyzing SHM for k, m system (from 4)

What is the energy of the spring/mass system as function of time.



$$E = \underbrace{\frac{1}{2}kx^2}_{\text{PE of spring}} + \underbrace{\frac{1}{2}mv^2}_{\text{KE of mass}}$$

Recall once more for $\omega = \sqrt{k/m}$

$$x = \underline{A\cos(\omega t + \delta)}.$$

$$\Rightarrow \frac{dx}{dt} = v = \underline{-A\omega\sin(\omega t + \delta)}.$$

Let $\theta = \omega t + \delta$ for convenience, calculate,

$$\begin{aligned}
E &= \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \\
&= \frac{1}{2}kA^2\cos^2\theta + \frac{1}{2}m(-A\omega\sin\theta)^2 \\
&= \frac{1}{2}kA^2\cos^2\theta + \frac{1}{2}m\omega^2A^2\sin^2\theta \qquad \text{but } \omega^2 = \frac{k}{m} \text{ thus,} \\
&= \frac{1}{2}kA^2\cos^2\theta + \frac{1}{2}kA^2\sin^2\theta \\
&= \frac{1}{2}kA^2(\cos^2\theta + \sin^2\theta) \\
&= \frac{1}{2}kA^2 \qquad \text{(independent of time!)}
\end{aligned}$$

$$\boxed{E = \frac{1}{2}kA^2}$$

Of course, as $k = m\omega^2$ we can also derive

$$\boxed{E = \frac{1}{2}mA^2\omega^2 = \frac{1}{2}mV_{max}^2} \qquad (V_{max} = A\omega)$$

Continuing, why $V_{max} = A\omega$,

$$V(t) = -A\omega \sin(\omega t + \delta) \quad (\text{from } 5)$$

It follows $\underline{V_{max} = A\omega}$ since $|\sin(\omega t + \delta)|$ is at most 1.

Phase Plane Analysis (Poincare Plane : position vs. velocity)

We begin with $m\ddot{x} + kx = 0$ for spring mass system. Multiply by $\dot{x}$ to obtain

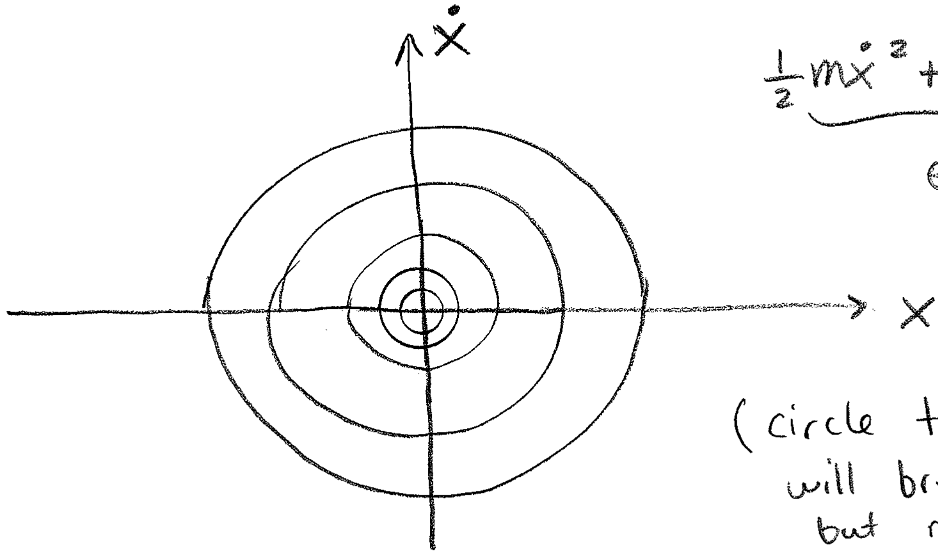
$$\underline{m\dot{x}\ddot{x} + kx\dot{x} = 0} \;\star$$

Notice $\frac{d}{dt}(\dot{x}^2) = 2\dot{x}\ddot{x}$ and $x\dot{x} = x\frac{dx}{dt} = \frac{1}{2}\frac{d}{dt}(x^2)$

Hence, $\star$ is actually the mathematical statement of energy conservation!

$$\frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right] = 0$$

Thus $E(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ is $\underline{\text{constant}}$ along the sol$^{\underline{n}}$s to Newton's Eq$^{\underline{n}}$ $m\ddot{x} + kx = 0$.



$$\underbrace{\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = 0}_{\text{ellipse in } x\dot{x}\text{-plane.}}$$

(circle too big will break spring but math does not show this here)