LECTURE 6

- In this lecture we study projectile motion in one, two and three dimensions. Projectile motion is simply constant acceleration motion due to gravity. We <u>assume</u> all problems in this lecture concern motion near the <u>surface</u> of the earth hence $\vec{a} = -g\hat{j}$ or $\vec{a} = -g\hat{k}$ (I sometimes use $\hat{k}$ for vertical direction in 3-dim'l problem).

- Suppose $\vec{a} = -g\hat{j}$ then $\vec{V}(t) = \vec{V}_o - gt\hat{j}$ and $\vec{r}(t) = \vec{r}_o + t\vec{V}_o - \frac{1}{2}gt^2\hat{j}$. It follows,

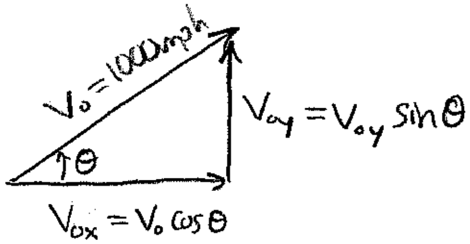
$$x(t) = x_o + V_{ox}t$$
$$y(t) = y_o + V_{oy}t - \frac{1}{2}gt^2$$

Since $a_x = 0$ and $a_y = -g$ we also have,

$$V_{fy}^2 = V_{oy}^2 - 2g(y_f - y_o) \quad \text{and} \quad \underbrace{V_{fx} = V_{ox}}_{x\text{-velocity constant.}}$$

<u>Remark</u>: it is important to notice that we <u>derive</u> these formulas via calculus from $\vec{a} = -g\hat{j}$. I expect you understand this point carefully. Of course, you can memorize ad-nauseum but, I'd rather you understood.

$\boxed{E1}$ If we fire a 1000mph bullet at θ above horizontal then what is V_x and V_y at time of firing?

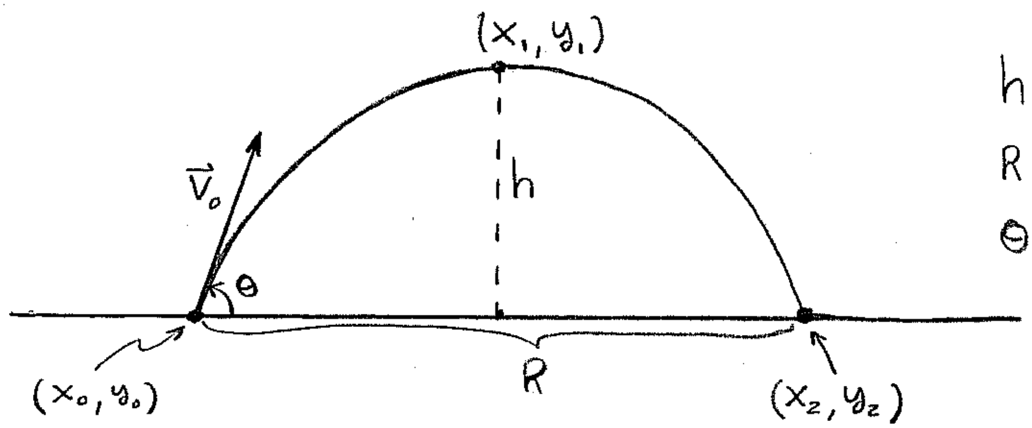


$$\vec{V}_o = V_o\cos\theta\,\hat{i} + V_o\sin\theta\,\hat{j}.$$

$$\vec{V}(t) = (V_o\cos\theta)\hat{i} + (V_o\sin\theta - gt)\hat{j}$$

while we're at it here's V at time t

E2 Find range and max height for a bullet fired at speed V_o and angle θ on a horizontal plane.



h = max. height.
R = range.
θ = angle fired above horizontal

I'll use notation $x_j(t) = x_j$ for $j = 0, 1, 2$. The flight pictured begins at $t = t_o = 0$ and ends at $t = t_2$. Since $\vec{a} = -g\hat{j}$

$x(t) = x_o + V_{ox} t$	$V_x(t) = V_{ox}$
$y(t) = y_o + V_{oy} t - \frac{1}{2} g t^2$	$V_y(t) = V_{oy} - gt$

But, $V_{ox} = V_o \cos\theta$ and $V_{oy} = V_o \sin\theta$ therefore,

$x(t) = x_o + V_o \cos\theta\, t$	$V_x(t) = V_o \cos\theta$
$y(t) = y_o + V_o \sin\theta\, t - \frac{1}{2} g t^2$	$V_y(t) = V_o \sin\theta - gt$

The maximum height is reached as $V_y(t_1) = 0$,

$$V_o \sin\theta - g t_1 = 0 \Rightarrow t_1 = \frac{V_o \sin\theta}{g}$$

Notice that $h = y_1 - y_o$ thus we find that:

$$h = y_o + V_o \sin\theta \left(\frac{V_o \sin\theta}{g}\right) - \frac{1}{2} g \left(\frac{V_o \sin\theta}{g}\right)^2 - y_o$$

$$\therefore \boxed{h = \frac{V_o^2 \sin^2\theta}{2g}}$$

notice this only applies if we are given the specific set-up of this problem. otherwise additional thought req^{d}.

The range R can be found from solving $y(t) = y_o \Rightarrow \underbrace{0 = V_o \sin\theta\, t - \frac{1}{2} g t^2}$

$$0 = t(V_o \sin\theta - tg/2) \quad \begin{cases} t_o = 0 \text{ (start)} \\ t_2 = \frac{2V_o \sin\theta}{g} \text{ (end)} \end{cases}$$

$$R = x(t_2) - x_o = (V_o \cos\theta)\left(\frac{2V_o \sin\theta}{g}\right) \quad \therefore \boxed{R = \frac{V_o^2 \sin 2\theta}{g}}$$

$\boxed{E3}$ Suppose a cannon is fired at $0 \le \theta \le 90°$ with a speed of V_o. Suppose the gun is on a horizontal plane and it fires from the level of the plane. Find maximum height and range for the cannon.

We found $R(\theta) = \dfrac{V_o^2 \sin(2\theta)}{g}$ in $\boxed{E2}$. Use calculus to optimize R with respect to θ,

$$\frac{dR}{d\theta} = \frac{2V_o^2 \cos(2\theta)}{g} = 0 \quad \text{for } 0 \le \theta \le 90°$$

$$\Rightarrow \theta = 45° \text{ is critical angle.}$$

$$\frac{d^2R}{d\theta^2} = \frac{-4V_o^2 \sin 2\theta}{g} \Rightarrow R''(45°) = \frac{-4V_o^2}{g} < 0$$

$\therefore$ $\theta = 45°$ yields max. By 2^{nd} derivative test.

We find $\boxed{R_{max} = \dfrac{V_o^2}{2g} \text{ for } \theta = 45°}$

Maximum height found from maximizing $h = \dfrac{V_o^2 \sin^2\theta}{2g}$

$$\frac{dh}{d\theta} = \frac{2V_o^2 \sin\theta \cos\theta}{2g} = 0 \quad \text{for } 0 \le \theta \le 90°$$

Observe that

1.) $\theta = 0 \Rightarrow \sin\theta = 0$
2.) $\theta = 90° \Rightarrow \cos\theta = 0$

We have <u>two</u> critical angles for $h(\theta)$. Note that

$$\frac{dh}{d\theta} = \frac{V_o^2 \sin 2\theta}{2g} \Rightarrow \frac{d^2h}{d\theta^2} = \frac{2V_o^2 \cos(2\theta)}{2g}. \text{ Consequently}$$

1.) $h''(0) = \dfrac{2V_o^2}{2g} > 0$ $\therefore$ $\boxed{\theta = 0° \text{ yield min. } h = 0}$

2.) $h''(90°) = \dfrac{-2V_o^2}{2g} < 0$ $\therefore$ $\boxed{\theta = 90° \text{ yields max. } h = \dfrac{V_o^2}{2g}}$

(I don't reccomend either option.)

<u>Remark:</u> In $\boxed{E2}$ and $\boxed{E3}$ the flight is symmetric. It takes double the time to the middle to reach the end; $t_2 = 2t_1$. Moreover, the time of flight is <u>independent</u> of V_x. The initial y-velocity indicates the time of flight.

- The examples past this point are just twists on $\boxed{E2}$ and $\boxed{E3}$. One popular twist is to put the problem on the moon.

$\boxed{E4}$ What is the height and range of flight for projectile with speed initially V_o fired at Θ above the horizon on the MOON? (assume $g_{moon} = g/6$)

Just like $\boxed{E2}$ except $\vec{a} = -g_{moon}\,\hat{j} \Rightarrow$ replace g with g_{moon},

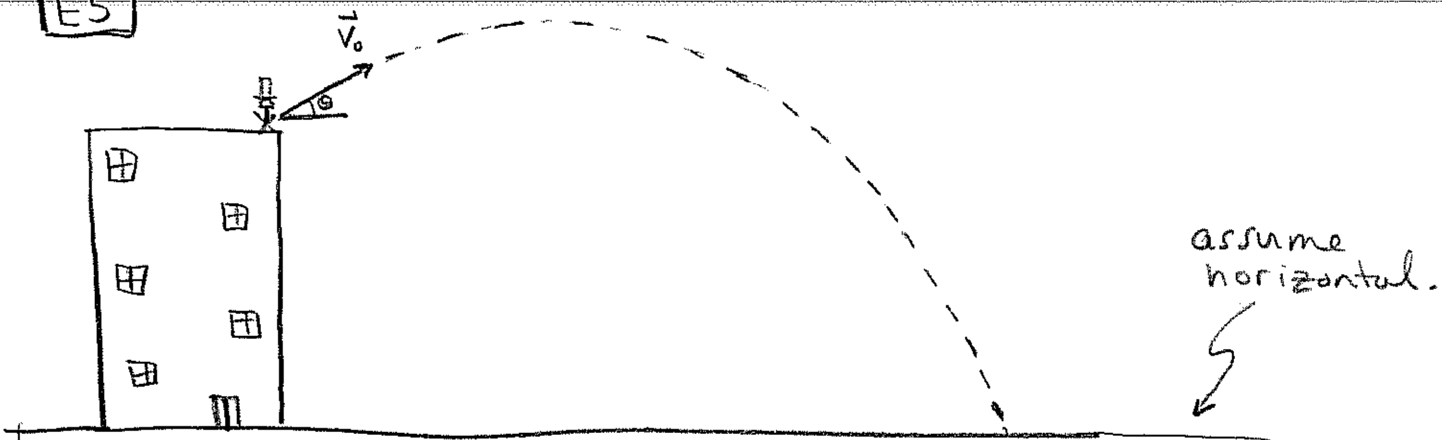
$$h_{moon} = \frac{V_o^2 \sin^2\Theta}{2 g_{moon}} = \frac{6 V_o^2 \sin^2\Theta}{2g} = 6\, h_{earth}.$$

$$R_{moon} = \frac{V_o^2 \sin(2\Theta)}{g_{moon}} = \frac{6 V_o^2 \sin 2\Theta}{g} = 6\, R_{earth}$$

We find that if the acceleration due to gravity is reduced by a factor of 6 then the height and range are increased by a factor of 6.

<u>Remark:</u> Since $V_x(t) = V_o \cos\Theta$ we'd find the same velocity on earth or moon in the horizontal direction. The reason we go 6 times as far is that it takes 6 times as much time for gravity to reverse the initial y-velocity.

E5



Show that the speed at which the ball hits the ground is independent of θ for a fixed V_o.
(you can throw the same speed for any reasonable θ)

Let y_1 = height of building + height to realease point for ball.
Let $y = 0$ at the ground where the ball lands
Since $\vec{a} = -g\hat{j}$ the usual formulas follow,

$$y(t) = y_1 + V_o \sin\theta\, t - \tfrac{1}{2} g t^2$$

We are interested in t such that $y(t) = 0$ (ball hits ground)

$$0 = y_1 + V_o \sin\theta\, t - \tfrac{1}{2} g t^2$$

$$\Rightarrow \quad 0 = t^2 - \frac{2V_o \sin\theta\, t}{g} - \frac{2y_1}{g}$$

$$\Rightarrow \quad 0 = \left(t - \frac{V_o \sin\theta}{g}\right)^2 - \frac{V_o^2 \sin^2\theta}{g^2} - \frac{2y_1}{g}$$

$$\Rightarrow \quad t_\pm = \frac{V_o \sin\theta \pm \sqrt{V_o^2 \sin^2\theta + 2y_1 g}}{g}$$

physically obvious we want (+) sol$^{\underline{n}}$ here.

Consider then that $V_y(t) = V_o \sin\theta - gt$ thus,

$$V_y(t_+) = V_o \sin\theta - g\left[\frac{V_o \sin\theta}{g} + \frac{\sqrt{V_o^2 \sin^2\theta + 2y_1 g}}{g}\right]$$

$$V_y(t_+) = -\sqrt{V_o^2 \sin^2\theta + 2y_1 g}$$

We know $V_x(t_+) = V_o \cos\theta$ since $a_x = 0$. Thus,

$$V(t_+) = \sqrt{V_x(t_+)^2 + V_y(t_+)^2} = \sqrt{V_o^2 \cos^2\theta + V_o^2 \sin^2\theta + 2y_1 g}$$

$$\boxed{V_f = \sqrt{V_o^2 + 2y_1 g}}$$ ← θ independent!

E6) Try E5) again, but this time w/o time.

$$V_{fy}^2 = V_{oy}^2 - 2g(y_f - y_o) \quad \text{since } a_y = 0$$

$$V_{fx}^2 = V_{ox}^2 \quad \text{since } a_x = 0$$

Thus,

$$V_f^2 = V_{fx}^2 + V_{fy}^2 = V_{ox}^2 + V_{oy}^2 - 2g(y_f - y_o)$$

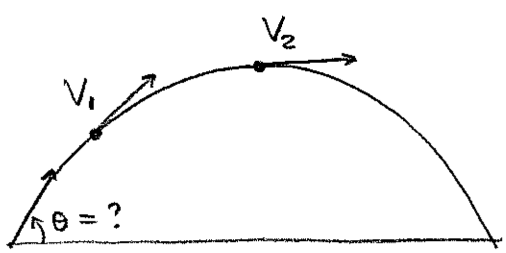
$$= V_o^2 - 2g(y_f - y_o)$$

$$= \underline{V_o^2 + 2g y_o} \quad \text{since } y_f = 0.$$

$$\Rightarrow \boxed{V_f = \sqrt{V_o^2 + 2g y_o}}$$

(obviously E6) is easier sol^{n}, but, if you understand the eq^{n}'s you could argue either way!)

E7) The speed of a projectile when it reaches max. height is 0.47 times its speed at half the max height. At what angle was the projectile fired?



$$\underbrace{V_2 = V_{2x}}_{} = V_o \cos\theta$$

at top of flight $V_{2y} = 0$.

(Note $h_1 = h_2/2$ or $h_2 = 2h_1$)

$$\underbrace{V_{2y}^2}_{0} = V_{1y}^2 - 2g(h_2 - h_1) \Rightarrow V_{1y}^2 = 2g\left(h_2 - \frac{h_2}{2}\right) = g h_2$$

We also derived $h_2 = \dfrac{V_o^2 \sin^2\theta}{2g}$ in E2) thus, $V_{1y}^2 = \dfrac{V_o^2 \sin^2\theta}{2}$.

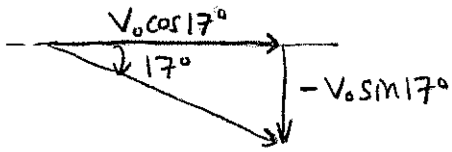
$$V_2 = 0.47 V_1 \Rightarrow V_2^2 = (0.2209) V_1^2 = (0.2209)\left[V_{1x}^2 + V_{1y}^2\right]$$

$$\Rightarrow (0.7791) V_o^2 \cos^2\theta = (0.2209)\left[\frac{V_o^2 \sin^2\theta}{2}\right] \qquad \underline{V_o^2 \text{ cancels!}}$$

Thus $\tan^2\theta = 7.0539 \Rightarrow \theta = \tan^{-1}(2.6559) = \boxed{69.37°}$

E8 A ball is thrown at 7.80 m/s at $\theta = 17°$ below the horizontal. After 4.00 s it hits the ground.
(i.) how far from building does it travel horizontally?
(ii.) what is the height from which the ball was thrown.
(iii.) how long does it take for the ball to fall the first 10 meters of its flight?

Since $V_o = 7.80\ m/s$ and



$$V_{ox} = 7.4592\ m/s$$
$$V_{oy} = -2.2805\ m/s.$$

Now answering (i.) is trivial. We have a constant x-velocity and the time of 4.00s hence

$$\Delta x = \left(7.4592\ \tfrac{m}{s}\right)(4.00\,s) = \boxed{29.84\ m}$$ ← horizontal displacement

Notice we can calculate V_{fy} since

$$V_{fy} = V_{oy} - gt = -2.2805\ \tfrac{m}{s} - \left(9.81\ \tfrac{m}{s^2}\right)(4.00\,s) = \underline{-41.521\ \tfrac{m}{s}}$$

And, then recall $V_{fy}^2 = V_{oy}^2 - 2g\,(\underbrace{y_f - y_o}_{-\text{height of launch point.}})$

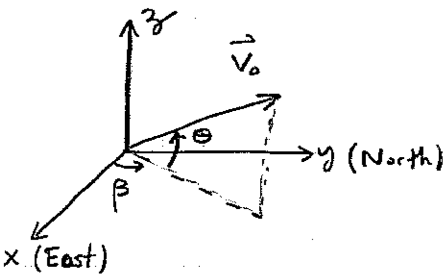
$$y_f - y_o = \frac{V_{oy}^2 - V_{fy}^2}{2g}$$
$$= \frac{\left(-2.2805\ \tfrac{m}{s}\right)^2 - \left(-41.521\ \tfrac{m}{s}\right)^2}{2\left(9.81\ m/s^2\right)}$$
$$= -87.6\,m \quad \Rightarrow \quad \boxed{y_o - y_f = 87.6\,m}$$ ← height of launch.

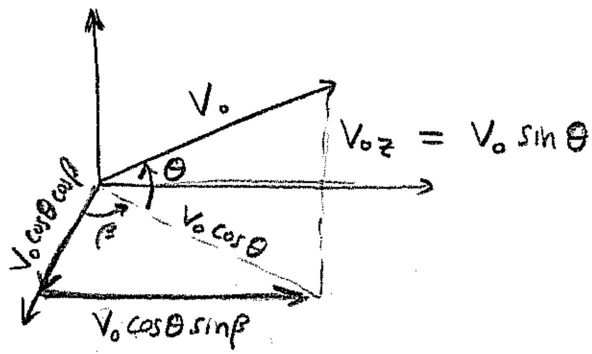
To find time for first 10 m drop can solve $77.6\,m = y(t)$ where $y(t) = 87.6\,m - \left(2.2805\ \tfrac{m}{s}\right)t - \tfrac{1}{2}gt^2$. This gives

$$-\tfrac{1}{2}gt^2 - \left(2.2805\ \tfrac{m}{s}\right)t + 10\,m = 0$$

$$\Rightarrow \boxed{t \cong 1.21\,s}$$

E9 Suppose a gun is fired at angle θ above horizontal and angle β CCW from EAST. (see picture)
Where does the bullet land?
(assume horizontal landscape... very boring)
$\vec{a} = -g\hat{k}$ is given.





$$\vec{V}_0 = (V_0\cos\theta\cos\beta)\,\hat{i} + (V_0\cos\theta\sin\beta)\,\hat{j} + (V_0\sin\theta)\,\hat{k}$$

Note, $V_{0z}(t) = V_0\sin\theta - gt \Rightarrow t = \frac{V_0\sin\theta}{g}$ at zenith hence by symmetry, $t_{flight} = \frac{2V_0\sin\theta}{g}$ and plugging into $\vec{r}(t) = \vec{r}_0 + t\vec{V}_0 - \frac{1}{2}gt^2\,\hat{k}$ we find

$$\vec{r}(t_{flight}) = \vec{r}_0 + \frac{2V_0\sin\theta}{g}\left(V_0\cos\theta\cos\beta\,\hat{i} + V_0\cos\theta\sin\beta\,\hat{j} + V_0\sin\theta\,\hat{k}\right) - \frac{1}{2}g\left(\frac{2V_0\sin\theta}{g}\right)^2\hat{k}$$

$$= \boxed{\vec{r}_0 + \frac{V_0^2\sin(2\theta)}{g}\left[\cos\beta\,\hat{i} + \sin\beta\,\hat{j}\right]}$$