$\boxed{\begin{matrix}\S 8.6 \\ \#12\end{matrix}}$ a.) $f(x) = \ln(1+x)$

$$f'(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$$

$$f(x) = \int f'(x)\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^n\,dx$$

$$= \sum_{n=0}^{\infty}(-1)^n \int x^n\,dx$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} + C \quad , \quad f(0) = \ln(1) = C \Rightarrow \underline{C=0}$$

Thus $f(x) = \boxed{\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots}$

(here's a picture to explain what just happened)

$$\begin{tikzcd}
\boxed{f(x)} \arrow[d, "d/dx"] & & \boxed{\text{power series version of } f(x).} \\
\boxed{f'(x)} \arrow[r, "\text{geometric}", "\text{series}"'] & \boxed{\text{power series representation of } f'(x)} \arrow[ur, "\int"] &
\end{tikzcd}$$

---

b.) $f(x) = x\ln(1+x)$

$$= x\sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1} x^{n+1}$$

just bring the x inside the sum $x x^{n+1} = x^{n+2}$

$$= \boxed{\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{n+1}\right)x^{n+2}}$$

$$= \boxed{x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 - \cdots = x\ln(1+x)}$$

c.) see notes, we did this one in class.

$\boxed{\begin{matrix}\S 8.6 \\ \#14\end{matrix}}$ $f(x) = \frac{x^2}{(1-2x)^2}$. Let $g(x) = \frac{1}{(1-2x)^2}$. If we can find the power series for $g(x)$ then $f(x)$ is easy.

$$\int g(x)\,dx = \int \frac{dx}{(1-2x)^2} = \int \frac{-du/2}{u^2} = \frac{1}{2u} + C = \frac{1}{2(1-2x)} + C$$

Then note $\int g(x)\,dx = \frac{1/2}{1-2x} + C = \sum_{n=0}^{\infty}\frac{1}{2}(2x)^n + C = \sum_{n=0}^{\infty}2^{n-1}x^n + C$

Undo the $\int$ by $\frac{d}{dx}$, $g(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}2^{n-1}x^n + c\right) = \sum_{n=0}^{\infty} n 2^{n-1}x^{n-1}$.

Now note $f(x) = x^2 g(x)$

$$= x^2 \sum_{n=0}^{\infty} n 2^{n-1} x^{n-1}$$

$$\boxed{\frac{x^2}{(1-2x)^2} = \sum_{n=1}^{\infty} n 2^{n-1} x^{n+1} = x^2 + 4x^3 + 12x^4 + \cdots}$$

WHY USE GEOMETRIC SERIES? SEE BELOW

§8.6 #14

Discussion: our logic was if we set aside the factor of x^2 then we could apply our standard geometric series trick, then once that was done we multiplied by x^2 to obtain the desired power series of $f(x)$.

- Lets compare to the task of calculating directly by Taylor's $Th^{\underline{m}}$:

these are time consuming w/o my TI-89.

$$f(x) = \frac{x^2}{(1-2x)^2}$$

$$f'(x) = \frac{2x(1-2x)^2 - x^2(1-2x)(-2)}{(1-2x)^4} = \frac{-2x}{(2x-1)^3}$$

$$f''(x) = \frac{-2(2x-1)^3 + 2x\cdot 3(2x-1)^2\cdot 2}{(2x-1)^6} = \frac{2(4x+1)}{(2x-1)^4}$$

$$f'''(x) = \frac{-24(2x+1)}{(2x-1)^5}$$

$$f''''(x) = \frac{96(4x+3)}{(2x-1)^6}$$

Then use Taylor's $Th^{\underline{m}}$,

$$f(x) = f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2 + \tfrac{1}{3!}f'''(0)x^3 + \tfrac{1}{4!}f''''(0)x^4 + \cdots$$

$$= 0 + 0\cdot x + \tfrac{1}{2}(2)x^2 + \tfrac{1}{6}(+24)x^3 + \tfrac{1}{24}(96)(3)x^4 + \cdots$$

$$= \boxed{x^2 + 4x^3 + 12x^4 + \cdots = \frac{x^2}{(1-2x)^2}}$$

(Compare to previous page's work)

We know that the series expansion is unique (when there is one) but I hope you can see why the brute-force direct method is much more painful computationally then the geometric series trick.

- there are problems for which the direct application of Taylor's $Th^{\underline{m}}$ is appropriate, we'll see those later on. Usually if we just wanted a few terms then it's a relatively easy task. If we want the whole series then often Direct application of Taylor's $Th^{\underline{m}}$ is the wrong approach (in my course)

§8.6 #24

$\int \tan^{-1}(x^2)\,dx$ ← try to do this without series. Good luck sir.

Strategy, change the integrand $f(x) = \tan^{-1}(x^2)$ to a power series so we can integrate!

$$f'(x) = \frac{d}{dx}\left(\tan^{-1}(x^2)\right) = \frac{2x}{1+x^4} = \sum_{n=0}^{\infty}(2x)(-x^4)^n = \sum_{n=0}^{\infty} 2(-1)^n x^{4n+1}$$

$$f(x) = \int f'(x)\,dx = \int \sum_{n=0}^{\infty} 2(-1)^n x^{4n+1}\,dx$$
$$= \sum_{n=0}^{\infty} 2(-1)^n \int x^{4n+1}\,dx$$
$$= \sum_{n=0}^{\infty} 2(-1)^n \frac{x^{4n+2}}{4n+2} + C$$

(notice that $C=0$ as $\tan^{-1}(0)=0$)

Now we can calculate the integral

$$\int \tan^{-1}(x^2)\,dx = \int \sum_{n=0}^{\infty} 2(-1)^n \frac{1}{4n+2}\, x^{4n+2}\,dx$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}\,\frac{1}{4n+3}\, x^{4n+3} + C$$
$$= \boxed{\frac{1}{3}x^3 - \frac{1}{21}x^7 + \frac{1}{55}x^{11} + \cdots + C}$$

(The radius of convergence comes from where we applied the geometric series result $|x^4|<1 \to |x|<1$ so $R=1$ integrating and/or differentiating doesn't change R.)

Remark: this is the power of power series, we replace an impossible problem with a simple almost polynomial problem. The only catch is that to be exactly correct we'd have to keep an infinite # of terms. However, in most applications we just need results that are correct to a few decimals, to achieve that we can keep enough terms to make the approximation good.

§8.7 #6

Recall $\sin(u) = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!}$ thus

$$\sin(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty} (-1)^n 2^{2n+1} \frac{1}{(2n+1)!} x^{2n+1}}$$

Explicitly the first few terms are,

$$\sin(2x) = 2x - \frac{1}{3!}(2x)^3 + \frac{1}{5!}(2x)^5 + \cdots$$

§8.7 #8

Recall $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$

$$\boxed{xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x + x^2 + \frac{1}{2}x^3 + \frac{1}{3!}x^4 + \cdots}$$

§8.7 #12

$f(x) = \ln(x)$ expand about $a = 2$.

Oh here we'll actually use Taylor's Th^{m} directly.

$n=1$	$f'(x) = \frac{1}{x}$	$f'(2) = 1/2$
$n=2$	$f''(x) = \frac{-1}{x^2}$	$f''(2) = -1/4$
$n=3$	$f'''(x) = 2/x^3$	$f'''(2) = 2/8$
$n=4$	$f''''(x) = -3\cdot 2/x^4$	$f''''(2) = -3\cdot 2/16$
$n=5$	$f^{(5)}(x) = 4\cdot 3\cdot 2/x^5$	$f^{(5)}(2) = 4!/32$
$\vdots$	$\vdots$	
n	$f^{(n)}(x) = (-1)^{n+1}(n-1)!/x^n$	$f^{(n)}(2) = (-1)^{n+1}(n-1)!/2^n$

$$\therefore f(x) = f(2) + f'(2)(x-2) + \frac{1}{2}f''(2)(x-2)^2 + \frac{1}{3!}f'''(2)(x-2)^3 + \cdots$$
$$= \ln(2) + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{6}\frac{2}{8}(x-2)^3 + \cdots$$

In better notation,

$$\ln(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$$
$$= \left(\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n-1)!}{n!\,2^n}(x-2)^n\right) + \ln(2)$$
$$= \boxed{\ln(2) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n2^n}(x-2)^n = \ln(x)}$$

$\boxed{\begin{matrix}\S 8.7 \\ \#20\end{matrix}}$

$$e^{-x/2} = e^{u} \qquad : u = -x/2$$

$$= \sum_{n=0}^{\infty} \frac{u^n}{n!} \qquad : \text{recall}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-x}{2}\right)^n = \boxed{\sum_{n=0}^{\infty}(-1)^n \frac{1}{n!\,2^n} x^n}$$

Explicitly $e^{-x/2} = 1 - x/2 + x^2/8 - x^3/48 + \cdots$

$\boxed{\begin{matrix}\S 8.7 \\ \#22\end{matrix}}$

$$\sin(x^4) = \sin(u)$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{u^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{(x^4)^{2n+1}}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!} x^{8n+4}}$$

Explicitly, $\sin(x^4) = x^4 - \frac{1}{3!}x^{12} + \frac{1}{5!}x^{20} - \cdots$

$\boxed{\begin{matrix}\S 8.7 \\ \#36\end{matrix}}$

$$\int \frac{e^x - 1}{x}\,dx = \int \frac{1}{x}\left(\cancel{1} + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots - \cancel{1}\right)dx$$

$$= \int\left(1 + \frac{1}{2}x + \frac{1}{3!}x^2 + \cdots\right)dx$$

$$= \boxed{x + \frac{1}{4}x^2 + \frac{1}{3\cdot 3!}x^3 + \cdots + C}$$

same.

$$\int \frac{1}{x}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!} - 1\right)dx = \sum_{n=1}^{\infty}\frac{1}{n!}\int x^{n-1}\,dx = \boxed{\left(\sum_{n=1}^{\infty}\frac{1}{n\,n!}x^n\right) + C}$$

(sigma notation)

$\boxed{\begin{matrix}\S 8.7 \\ \#50\end{matrix}}$

$$\sum_{n=0}^{\infty}\frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty}(-1)^n \frac{(\pi/6)^{2n}}{(2n)!}$$

AH HA! $\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ So the above is just $x = \pi/6$ substituted into cosine. Hence,

$$\sum_{n=0}^{\infty}(-1)^n \frac{(\pi/6)^{2n}}{(2n)!} = \cos(\pi/6) = \boxed{\sqrt{3}/2}$$

<u>Remark:</u> #49 → #54 are all the same sillyness