OVERVIEW & MOTIVATIONS FOR SUSY

SUSY is explored as a possible extension of current physics. To date it lacks experimental verification but ∃ hope that LHC in CERN may detect SUSY physics around 2010. A brief summary of SUSY is not difficult:

BOSONS	$\xleftrightarrow{\text{SUSY}}$	FERMIONS	
$s = 0, 1$	⟷	$s = 1/2$	(SPIN)
photon	⟷	photino	
selectron	⟷	electron	
W	⟷	Wino	(superpartners)
B	⟷	Bino	
gluon	⟷	gluino	
squark	⟷	quark	

A theory which respects SUSY must balance the number of bosonic and the number of fermionic states. This requirement points to the existence of the so-far undiscovered particles listed above.

Who Cares? & Why

High Energy Particle Theorists & Experimentalists, String Theorists to begin. If LHC finds SUSY then many who have spent the last 20 years developing SUSY will be happy with their life. If not... (?)... It might "just be math"

1. Only known extension of Poincaire group which makes sense physically $(\{,\})$

2. Provides nice sol$^{\underline{n}}$ to the "Hiearchy Problem". That is it explains why the Higg's Boson has such a small mass. $(m_{Higgs} \overset{?}{<<} m_{plank})$

3. Arises naturally & usefully in Superstring theory.

These are the main motivations for hoping SUSY is not "just math". Generally speaking adding SUSY to a problem makes it easier to solve. For instance, $N=4$ supersymmetric Yang Mills theory allows calculations in the non-perturbative or strongly coupled sector via perturbative calculations in quantum gravity (String theoretic). Such a correlation is also thought to hold w/o SUSY but its more difficult. There are also many examples of finite QFT's thanks to SUSY.

Our goal is to understand SUSY enough to see how the MSSM is built and what new physics we should see at LHC if $\exists$ SUSY (at $E \sim$ a couple TeV). In doing this hopefully if you are interested in more sophisticated physics or math that involves SUSY you will know where to start your studies.

WESS-ZUMINO MODEL

Consider the set of fields, A, B, Ψ. These fields form the "Chiral Multiplet". Notice that:

A has spin $= 0$ (scalar field)
B has spin $= 0$ (psuedo-scalar field)
Ψ has spin $= 1/2$ (majorana fermion)

Then consider the following (very typical) action:

$$S = \int d^4x \left(\tfrac{1}{2}(\partial_\mu A)^2 + \tfrac{1}{2}(\partial_\mu B)^2 + i\bar{\Psi}\not{\partial}\Psi \right)$$

This integration is taken over Minkowski Space and the terms $(\partial_\mu A)^2 = (\partial_\mu A)(\partial^\mu A)$ with usual einstein conv. Σ. also $\not{\partial} \equiv \gamma^\mu \partial_\mu$ where γ^μ are the Dirac Matrices.

SUSY transform. δ

$$\begin{cases} \delta A = \bar{\varepsilon}\Psi \\ \delta B = i\bar{\varepsilon}\gamma^5\Psi \\ \delta \Psi = -i\gamma^\mu \varepsilon(\partial_\mu A) + i\gamma^\mu\gamma^5\varepsilon(\partial_\mu B) \\ \delta\bar{\Psi} = i\bar{\varepsilon}\gamma^\mu(\partial_\mu A) - \bar{\varepsilon}\gamma^5\gamma^\mu(\partial_\mu B) \end{cases}$$

The transformation δ given above mixes bosonic and fermionic fields, this is unique to SUSY since other symmetries in the Poincaire or internal symmetry groups preserve the Lorentz structure. However here generator of the transformation is itself a spinor $(\bar{\varepsilon})$ which allows the SUSY transformation δ to mix fields of different spin.

PROPOSITION: The action S is invariant under a SUSY transformation δ:

$$\boxed{\delta S = 0}$$

Proof:

$$S = \int d^4x\, \mathcal{L} = \int d^4x \left(\tfrac{1}{2}(\partial_\mu A)^2 + \tfrac{1}{2}(\partial_\mu B)^2 + i\bar{\Psi}\gamma^\mu \partial_\mu \Psi \right)$$

$$\bar{\epsilon}\Psi = \bar{\Psi}\epsilon \qquad \bar{\epsilon}\gamma^5\Psi = \bar{\Psi}\gamma^5\epsilon$$

Majorana Properties.

$$\delta A = \bar{\epsilon}\Psi \qquad \delta\Psi = -i\gamma^\mu\epsilon\partial_\mu A + \gamma^\mu\gamma^5\epsilon\partial_\mu B$$

$$\delta B = i\bar{\epsilon}\gamma^5\Psi \qquad \delta\bar{\Psi} = i\bar{\epsilon}\gamma^\mu\partial_\mu A - \bar{\epsilon}\gamma^5\gamma^\mu\partial_\mu B$$

$$\begin{aligned}
\delta\left(\tfrac{1}{2}(\partial_\mu A)^2 + \tfrac{1}{2}(\partial_\mu B)^2\right) &= (\partial_\mu A)\delta(\partial^\mu A) + (\partial_\mu B)\delta(\partial^\mu B) \\
&= (\partial_\mu A)\partial^\mu(\delta A) + (\partial_\mu B)\partial^\mu(\delta B) \qquad \delta \neq \delta(x^\mu) \\
&= (\partial_\mu A)\partial^\mu(\bar{\epsilon}\Psi) + (\partial_\mu B)\partial^\mu(i\bar{\epsilon}\gamma^5\Psi) \\
&= \bar{\epsilon}\left[\partial_\mu A + i(\partial_\mu B)\gamma^5\right]\partial^\mu\Psi
\end{aligned}$$

$$\begin{aligned}
\delta(i\bar{\Psi}\gamma^\mu\partial_\mu\Psi) &= i\delta(\bar{\Psi})\not{\partial}\Psi + i\bar{\Psi}\not{\partial}(\delta\Psi) \\
&= i\left[i\bar{\epsilon}\gamma^\mu\partial_\mu A - \bar{\epsilon}\gamma^5\gamma^\mu\partial_\mu B\right]\gamma^\nu\partial_\nu\Psi + i\bar{\Psi}\gamma^\mu\partial_\mu\left[-i\gamma^\nu\epsilon\partial_\nu A + \gamma^\nu\gamma^5\epsilon\partial_\nu B\right] \\
&= -\bar{\epsilon}\gamma^\mu\gamma^\nu\partial_\mu A\partial_\nu\Psi + \bar{\Psi}\gamma^\mu\gamma^\nu\epsilon\partial_\mu\partial_\nu A \\
&\qquad - i\bar{\epsilon}\gamma^5\gamma^\mu\gamma^\nu\partial_\mu B\partial_\nu\Psi + i\bar{\Psi}\gamma^\mu\gamma^\nu\gamma^5\epsilon\partial_\mu\partial_\nu B \\
\text{I.B.P in } \delta S = 0 \quad &= \bar{\epsilon}\gamma^\mu\gamma^\nu(\partial_\mu\partial_\nu A)\Psi + \bar{\Psi}\gamma^\mu\gamma^\nu\epsilon(\partial_\mu\partial_\nu A) \\
&\qquad + i\bar{\epsilon}\gamma^5\gamma^\mu\gamma^\nu(\partial_\mu\partial_\nu B)\Psi + i\bar{\Psi}\gamma^\mu\gamma^\nu\gamma^5\epsilon\partial_\mu\partial_\nu B \\
\text{symmetrize in } \mu \leftrightarrow \nu \quad &= \bar{\epsilon}\,\tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\}(\partial_\mu\partial_\nu A)\Psi + \cdots + i\bar{\Psi}\,\tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\}\gamma^5\epsilon\,\partial_\mu\partial_\nu B \\
\text{def}^{\text{n}} \text{ of Dirac } \gamma\text{'s} \quad &= \tfrac{1}{2}\bar{\epsilon}\,\eta^{\mu\nu}(\partial_\mu\partial_\nu A)\Psi + \cdots + \tfrac{i}{2}\bar{\Psi}\gamma^5\eta^{\mu\nu}\epsilon\,\partial_\mu\partial_\nu B \\
\Box = \partial^\mu\partial_\mu \quad &= \tfrac{1}{2}\bar{\epsilon}\Psi\,\Box A + \tfrac{1}{2}\bar{\Psi}\epsilon\,\Box A + \tfrac{i}{2}\bar{\epsilon}\gamma^5\Psi\,\Box B + \tfrac{i}{2}\bar{\Psi}\gamma^5\epsilon\,\Box B \\
\text{Majorana properties} \quad &= \bar{\epsilon}\Psi\,\Box A + i\bar{\epsilon}\gamma^5\Psi\,\Box B \\
\text{I.B.P AGAIN.} \quad &= -\bar{\epsilon}\left[\partial_\mu A + i(\partial_\mu B)\gamma^5\right]\partial^\mu\Psi
\end{aligned}$$

Thus $\delta S = 0$ since the $\delta\left(\tfrac{1}{2}(\partial_\mu A)^2 + \tfrac{1}{2}(\partial_\mu B)^2\right)$ will cancel the $\delta(i\bar{\Psi}\not{\partial}\Psi)$ modulo a total derivative term which by physical assumption vanishes on the boundary of integration.

$\boxed{\text{Algebra of SUSY transformations } \delta}$

$$
\begin{aligned}
\delta_1 \delta_2 A &= \delta_1(\bar{\epsilon}_2 \psi) \\
&= \bar{\epsilon}_2 \delta_1 \psi \\
&= \bar{\epsilon}_2(-i\gamma^\mu \epsilon_1 \partial_\mu A + \gamma^\mu \gamma^5 \epsilon_1 \partial_\mu B) \\
&= -i\bar{\epsilon}_2 \gamma^\mu \epsilon_1 \partial_\mu A + \bar{\epsilon}_2 \gamma^\mu \gamma^5 \epsilon_1 \partial_\mu B
\end{aligned}
$$

$$
\begin{aligned}
\delta_2 \delta_1 A &= -i(\bar{\epsilon}_1 \gamma^\mu \epsilon_2)\partial_\mu A + (\bar{\epsilon}_1 \gamma^\mu \gamma^5 \epsilon_2)\partial_\mu B \\
&= i(\bar{\epsilon}_2 \gamma^\mu \epsilon_1)\partial_\mu A + (\bar{\epsilon}_2 \gamma^\mu \gamma^5 \epsilon_1)\partial_\mu B
\end{aligned}
$$

$$\therefore [\delta_1, \delta_2]A = (\delta_1\delta_2 - \delta_2\delta_1)A = -2i\bar{\epsilon}_2 \gamma^\mu \epsilon_1 \partial_\mu A$$

A similar calculation follows for B, however for the field ψ we obtain an extra piece; compare:

$$[\delta_1, \delta_2]A = -2\bar{\epsilon}_2 \gamma^\mu \epsilon_1 P_\mu \cdot A$$

$$[\delta_1, \delta_2]B = -2\bar{\epsilon}_2 \gamma^\mu \epsilon_1 P_\mu \cdot B$$

$$[\delta_1, \delta_2]\psi = -2\bar{\epsilon}_2 \gamma^\mu \epsilon_1 P_\mu \cdot \psi + i\bar{\epsilon}_2 \gamma^\nu \epsilon_1 \gamma_\nu \gamma^\mu \partial_\mu \psi$$

We define $P_\mu = i\partial_\mu$. Notice that if $\gamma^\mu \partial_\mu \psi = 0$ then all three commutators behave the same. We will not impose the eq$^{\underline{n}}$ of motion of ψ, we would like to use a path-integral formalism so we must keep the fields "Off-Shell", that just means to leave our fields unconstrained.

$\boxed{\text{QUESTION: How can we make } [\delta_1, \delta_2] = -2\bar{\epsilon}_2 \gamma^\mu \epsilon_1 P_\mu \text{ for } A, B \text{ AND } \psi \text{ ?}}$

Auxillary Fields F & G Closes Algebra

We add the scalar fields F and G to our model. The action is modified slightly: (modifications)

$$S = \int d^4x \left(\tfrac{1}{2}(\partial_\mu A)^2 + \tfrac{1}{2}(\partial_\mu B)^2 + i\bar{\Psi}\not{\partial}\Psi + \tfrac{1}{2}F^2 + \tfrac{1}{2}G^2 \right)$$

Likewise the SUSY transformation δ for Ψ must be modified in order to maintain $\delta S = 0$.

$$\begin{aligned}
\delta A &= \bar{\epsilon}\Psi \\
\delta B &= i\bar{\epsilon}\gamma^5\Psi \\
\delta F &= i\bar{\epsilon}\gamma^\mu\partial_\mu\Psi \\
\delta G &= -i\bar{\epsilon}\gamma^5\gamma^\mu\partial_\mu\Psi \\
\delta\Psi &= -i\gamma^\mu\epsilon\,\partial_\mu A + \gamma^\mu\gamma^5\epsilon\,\partial_\mu B - \epsilon F - i\gamma^5\epsilon G \\
\delta\bar{\Psi} &= i\bar{\epsilon}\gamma^\mu\partial_\mu A - \bar{\epsilon}\gamma^5\gamma^\mu\partial_\mu B - \bar{\epsilon}F - i\bar{\epsilon}\gamma^5 G
\end{aligned}$$

The fields F and G are called auxillary because they are just put in to close the algebra. They have no dynamics, their eq^{n}'s of motion are algebraic (with S as above their eq^{n}'s of mot. are $F=0$ and $G=0$)

$$[\delta_1, \delta_2] = -2i\bar{\epsilon}_2\gamma^\mu\epsilon_1\partial_\mu$$

- This is an operator eq^{n}, it holds for the entire Chiral Multiplet $A, B, \Psi, \bar{\Psi}, F, G$.

> Now the algebra of the SUSY generators δ is seen to close on translations. Hence SUSY is sometimes called the "square root" of a translation

Changing to Weyl Spinors

Upto now we have written everything in terms of Majorana Spinors, we can use instead 2 Weyl spinors:

$$\bar{\epsilon}^A = \begin{pmatrix} \eta^\alpha \\ \bar{\eta}_{\dot{\beta}} \end{pmatrix}^T \quad \text{AND} \quad Q_A = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot{\beta}} \end{pmatrix}$$

Majorana spinors have 4 components while Weyl have 2.

$$\delta = i\bar{\epsilon}^A Q_A = i(\eta^\alpha \ \bar{\eta}_{\dot{\beta}})\begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot{\beta}} \end{pmatrix} = (\eta^\alpha Q_\alpha + \bar{\eta}_{\dot{\beta}} \bar{Q}^{\dot{\beta}})i$$

$$\underset{\substack{\uparrow \\ \text{Majorana}}}{\bar{\epsilon} Q} = (\eta Q \underset{\substack{\uparrow \\ \text{Weyl Spinors}}}{+} \bar{\eta}\bar{Q})i \quad \text{(indices often suppressed)}$$

While δ followed $[\delta, \delta] \sim P$ we will find that Q_α and $\bar{Q}^{\dot{\beta}}$ are subject to anticommutation relations,

$$[\delta_1, \delta_2] = -[\eta_1 Q + \bar{\eta}_1 \bar{Q}, \ \eta_2 Q + \bar{\eta}_2 \bar{Q}]$$

$$= -[\eta_1 Q, \eta_2 Q] - [\eta_1 Q, \bar{\eta}_2 \bar{Q}] - [\bar{\eta}_1 \bar{Q}, \eta_2 Q] - [\bar{\eta}_1 \bar{Q}, \bar{\eta}_2 \bar{Q}]$$

$$= -(\eta_1 Q \eta_2 Q - \eta_2 Q \eta_1 Q) - \cdots - (\bar{\eta}_1 \bar{Q} \bar{\eta}_2 \bar{Q} - \bar{\eta}_2 \bar{Q} \bar{\eta}_1 \bar{Q})$$

$$= -[\eta_2^\beta (Q_\alpha Q_\beta + Q_\beta Q_\alpha)\eta_1^\alpha + \cdots + \bar{\eta}_{2\dot{\beta}}(\bar{Q}^{\dot{\gamma}}\bar{Q}^{\dot{\beta}} + \bar{Q}^{\dot{\beta}}\bar{Q}^{\dot{\gamma}})\bar{\eta}_{1\dot{\gamma}}]$$

$$= -\eta_2^\beta \{Q_\alpha, Q_\beta\}\eta_1^\alpha - \eta_2^\alpha \{\bar{Q}^{\dot{\beta}}, Q_\alpha\}\bar{\eta}_{1\dot{\beta}} - \bar{\eta}_2^{\dot{\beta}}\{Q^\alpha, \bar{Q}_{\dot{\beta}}\}\eta_{1\alpha} - \bar{\eta}_{2\dot{\beta}}\{\bar{Q}^{\dot{\beta}}, \bar{Q}^{\dot{\alpha}}\}\bar{\eta}_{1\dot{\alpha}}$$

Next we expand $-2i\,\bar{\epsilon}_2 \gamma^\mu \epsilon_1 \partial_\mu$ into the weyl notation:

$$\bar{\epsilon}_2 \gamma^\mu \epsilon_1 = (\eta_2^\alpha \ \bar{\eta}_{\dot{\beta}})\begin{pmatrix} 0 & \bar{\sigma}^{\mu \alpha \dot{\beta}} \\ \sigma^\mu_{\alpha\dot{\beta}} & 0 \end{pmatrix}\begin{pmatrix} \eta_1^\beta \\ \bar{\eta}_1^{\dot{\beta}} \end{pmatrix} = -\eta_2^\alpha \sigma^\mu{}_\alpha{}^{\dot{\beta}} \bar{\eta}_{1\dot{\beta}} + \eta_2^\beta \sigma^{\mu\dot{\beta}}{}_\beta \bar{\eta}_{1\dot{\beta}}$$

$$\boxed{\{Q_\alpha, Q_\beta\} = \{\bar{Q}^{\dot{\alpha}}, \bar{Q}^{\dot{\beta}}\} = 0 \quad \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = -2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu}$$