Extending the Poincaire Group

The Poincaire Algebra is generated by the following:

$$\left.\begin{array}{r}\text{Spatial Translations}\\ \text{Time Translations}\end{array}\right\} \longrightarrow P_m$$

$m, n = 0, 1, 2, 3$ spacetime indices.

$$\left.\begin{array}{l}\text{Rotations in Space}\\ \text{Lorentz Boosts}\end{array}\right\} \longrightarrow J_{mn}$$

Then the Poincaire Algebra is given by:

1) $[J_{mn}, J_{po}] = i(\eta_{np} J_{mo} - \eta_{mp} J_{no} + \eta_{mo} J_{np} - \eta_{no} J_{mp})$

2) $[P_m, J_{po}] = i(\eta_{mp} P_o - \eta_{mo} P_p)$

3) $[P_m, P_n] = 0$

Now we extend the Poincaire Alg. to the Superpoincaire Alg, in addition to the relations 1.) 2.) and 3.):

4) $\{Q_\alpha, Q_\beta\} = \{\bar{Q}^{\dot\alpha}, \bar{Q}^{\dot\beta}\}$

5) $\{Q_\alpha, \bar{Q}^{\dot\beta}\} = 2\sigma^{n\dot\beta}_{\alpha} P_n$

6) $\{Q^\alpha, \bar{Q}^{\dot\beta}\} = 2\bar{\sigma}^{m\alpha\dot\beta} P_m$

7) $[Q_\alpha, P_m] = 0$

8) $[J_{mn}, Q_\alpha] = -i(\sigma_{mn})_\alpha{}^\beta Q_\beta$

9) $[J_{mn}, \bar{Q}_{\dot\alpha}] = -i(\bar{\sigma}_{mn})_{\dot\alpha}{}^{\dot\beta} \bar{Q}_{\dot\beta}$

$\alpha, \beta = 0, 1$ undotted Weyl Indices.

$\dot\alpha, \dot\beta = \dot 0, \dot 1$ dotted Weyl Indices.

Altogether J, P, Q plus these 9 rules form a $\mathbb{Z}_2$ graded Lie Algebra, or a Super Lie Algebra.

UNIQUENESS OF SUSY CONSTRUCTION

So we just found the Superpoincaire Algebra (which exponentiates to the superpoincaire group) by demanding that SUSY closes onto translations P,

QUESTION: Could we construct a similar symmetry which instead closed onto rotations ($\{Q, \bar{Q}\} \propto J$) to make a physically sensible model?

ANSWER: No. For a technical exposition of this see the classic paper by Hagg, Lopuszanski and Sohnius, Nuclear Physics B 88 257 (1975)

Essentially what happens is that if you make $\{Q, \bar{Q}\} \propto J$ then higher spin states ($s = 3/2$) get coupled to lower spin states ($s=0$, $s=1/2$)

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} \propto J_{mn} (\sigma^m \sigma^n)_{\alpha\dot{\beta}}$$

And if that happens it violates the earlier paper by Coleman and Mandula Phy. Rev. 159, 1251 (1967) which was a very powerful NO-GO paper placing strict limits on further physical symmetries. SUSY as we constructed it gets around this NO-GO theorem by allowing a Super Lie Alg. structure as opposed to the implicity assumed Lie Alg. structure in the 1967 paper.

Extensions of SUSY, $N>1$

Can consider N supersymmetries Q_α^A where $A = 1, 2, \ldots, N$ ranges over the supercharges and:

$$\{Q_\alpha^A, \bar{Q}_{\dot{\beta} B}\} = 2\sigma^M_{\alpha\dot{\beta}} P_M \delta^A_B$$

Simplifications that occur in $N=1$ SUSY

(1) No central charges : $\{Q_\alpha^a, Q_\beta^b\} = \epsilon_{\alpha\beta} Z^{ab}$

Clearly the central charge $Z = 0$ if $a = b = 1$ (Z^{ab} must be antisymmetric)

(2) R-SYMMETRY is ABELIAN. We can consider a global phase rotation of the spinorial charges Q_α^A

$N=1 \Rightarrow$ R-SYMMETRY JUST $U(1)$

For $N>1$ you find a $U(N)$ symmetry between the charges. (Important element of the $N=2$ susy models)

COSET SPACE VIEW OF SPACETIME

It is usefull to let the coordinates X^m of $\mathcal{M}$ = Minkowski Space $(\mathbb{R}^4, \eta = (^{-1}{}_{1}{}_{1}{}_{1}))$ parametrize the Poincaire Group. From finite transformations on $\mathcal{M}$ we can deduce the Poincaire Algebra through the correspondance of the Lie Algebra $= \mathcal{L}$ and the tangent space at the identity of the Lie group. First consider the group of rotations on $\mathbb{R}^3$:

$$
\begin{aligned}
x' &= x\cos\theta + y\sin\theta = x + y\theta \\
y' &= -x\sin\theta + y\cos\theta = -x\theta + y \\
z' &= z
\end{aligned}
$$

Rotation of θ about z-axis

(for θ small)

Let $\theta \to 0$ to find the generator J_z's action on $f(x,y,z)$

$$
\begin{aligned}
J_z f(x,y,z) &= i \lim_{\theta\to 0}\left(\frac{f(x',y',z') - f(x,y,z)}{\theta}\right) \\
&= i \lim_{\theta\to 0}\left(\frac{f(x+y\theta,\, y-x\theta,\, z) - f(x,y,z)}{\theta}\right) \\
&= i\, y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}
\end{aligned}
\qquad \therefore \boxed{J_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)}
$$

Similarly: $J_x = -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$ AND $J_y = -i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$ generate rotations around the x-axis and y-axis. These generators form a Lie algebra;

$$
\begin{aligned}
[J_x, J_y] &= i J_z \\
[J_y, J_z] &= i J_x \\
[J_z, J_x] &= i J_y
\end{aligned}
$$

Next we will outline the same for Lorentz Boosts. First note that for parameters a^α we get generators:

$$
i\frac{\partial x'^\mu}{\partial a^\alpha}\frac{\partial}{\partial x^\mu} = X_\alpha = i\left(\frac{\partial x'}{\partial a^\alpha}\bigg|_{a=0}\frac{\partial}{\partial x} + \frac{\partial y'}{\partial a^\alpha}\bigg|_{a=0}\frac{\partial}{\partial y} + \frac{\partial z'}{\partial a^\alpha}\bigg|_{a=0}\frac{\partial}{\partial z} + \frac{\partial t'}{\partial a^\alpha}\bigg|_{a=0}\frac{\partial}{\partial t}\right)
$$

Note that $a=0$ maps to the group identity. We add $\partial/\partial t$ since our group acts on time as well.

Below on the RHS are the finite velocity transformations (pure boosts), and on the LHS the Lie Algebra generators which serve to generate boosts in the parameters (x, y, z, t),

$$x' = \gamma(x + vt) \qquad K_x = i(t\partial_x + x\partial_t) \qquad [K_x, K_y] = -iJ_z$$
$$y' = y \,,\; z' = z \qquad K_y = i(t\partial_y + y\partial_t) \qquad [K_x, J_y] = iK_z$$
$$t' = \gamma(t + vx) \qquad K_z = i(t\partial_z + z\partial_t) \qquad [K_x, J_x] = 0$$

Just as Q did form algebra on its own so to the pure boosts K_i are not themselves a Lie Algebra, however we can extend the rotation alg. with boosts to form the Lorentz Algebra:

$$J_{\mu\nu} = \left\{ \begin{array}{l} J_{ij} = -J_{ji} = \epsilon_{ijk} J_k \\ J_{i0} = -J_{0i} = -K_i \end{array} \quad i = 1,2,3 \right\} = \begin{pmatrix} 0 & \leftarrow \vec{K} \rightarrow \\ -\vec{K} & \begin{matrix} 0 & J \\ J & 0 \end{matrix} \end{pmatrix}$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\nu\rho} J_{\mu\sigma} - g_{\mu\rho} J_{\nu\sigma} + g_{\mu\sigma} J_{\nu\rho} - g_{\nu\sigma} J_{\mu\rho}) \quad (\text{oops } g = \eta)$$

Finally we add the generators of translations → Poincaire Alg.

$$x'^{\mu} = x^{\mu} + a^{\mu} \qquad P_{\mu} = i\partial_{\mu} \qquad [P_{\mu}, P_{\nu}] = 0$$
$$[P_{\mu}, J_{\rho\sigma}] = i(g_{\mu\rho} P_{\sigma} - g_{\mu\sigma} P_{\rho})$$

Now see how group multiplication generates motions on parameters. Recall that the expontial map gives group; $\exp : \mathfrak{g} \longrightarrow G$,

$$e^{iy^m P_m} x^n = (1 + iy^m P_m) x^n = x^n + y^n \Rightarrow \boxed{P_m = -i\partial_m}$$

Suppose we were given P_m and $[P_m, P_n] = 0$ we could deduce that it generates translations by the following:

$$e^{ix^m P_m} e^{iy^n P_n} = e^{i(x^m + y^m) P_m} \qquad x^m \longmapsto x^m + y^m$$

Notice $e^A e^B = e^{A + B + \frac{1}{2}[A,B] + \cdots}$ was used above. What we will do now is see how to parametrize a space so that the Q_α's we found last time generate a translation in that parameter space.

(this will be Super Space)

GRASSMAN VARIABLES

Notice the index indicates what kind of object it is

$m, n, \ldots$: SPACETIME INDICES : $m = 0,1,2,3$
$\alpha, \beta, \gamma, \ldots$: Undotted Weyl Indices : $\alpha = 1,2$
$\dot\alpha, \dot\beta, \dot\gamma, \ldots$: Dotted Weyl Indices : $\dot\alpha = 1,2$

Grassman variables are anticommuting; take θ and η to be grassman's we would define

$$\theta\eta = -\eta\theta \longleftrightarrow \{\theta, \eta\} = 0$$
$$\theta\theta = -\theta\theta \longleftrightarrow \{\theta, \theta\} = 0 \rightarrow \boxed{\theta^2 = 0}$$

Grassmans are nilpotent. Our interest lies in several grassman dimensions, SPECIFICALLY 2-undotted dimensions (θ^α) and 2-dotted dimensions ($\bar\theta_{\dot\alpha}$)

$$\theta^\alpha = (\theta^1, \theta^2)$$
$$\bar\theta_{\dot\alpha} = (\bar\theta_1, \bar\theta_2)$$

We define these to be subject to the following:

$$\{\theta^\alpha, \theta^\beta\} = \{\bar\theta_{\dot\alpha}, \bar\theta_{\dot\beta}\} = \{\theta^\alpha, \bar\theta_{\dot\beta}\} = 0$$

Next we prescribe a convention for raising and lowering indices, this is ~ like describing an innerproduct on θ's except this product is skew : $\left(\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon^{\dot\alpha\dot\beta}, \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \epsilon_{\dot\alpha\dot\beta}\right)$

$$\theta^\alpha = \epsilon^{\alpha\beta}\theta_\beta \qquad \bar\theta^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\theta_{\dot\beta}$$
$$\theta_\alpha = \epsilon_{\alpha\beta}\theta^\beta \qquad \bar\theta_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\theta^{\dot\beta}$$

Notice the suppressed index notation for $\theta\theta$ & $\bar\theta\bar\theta$,

$$\theta^2 = \theta\theta = \theta^\alpha\theta_\alpha = -2\theta_1\theta_2 = -2\theta^1\theta^2$$
$$\bar\theta^2 = \bar\theta\bar\theta = \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha} = 2\bar\theta_1\bar\theta_2 = 2\bar\theta^1\bar\theta^2$$

Conventional slant of indices

$$\theta^\alpha\theta_\alpha = -\theta_\alpha\theta^\alpha$$
$$\bar\theta_{\dot\beta}\bar\theta^{\dot\beta} = -\bar\theta^{\dot\beta}\bar\theta_{\dot\beta}$$

Unconventional !!! (LHS understood for $\theta\theta$ AND $\bar\theta\bar\theta$)

N = 1 Rigid Superspace

A typical point is $z = (x^m, \theta^\alpha, \bar{\theta}_{\dot\alpha})$. Consider now the product of two supertranslations:

$$\exp\left(i[a^m P_m + \xi Q + \bar{\xi}\bar{Q}]\right)\exp\left(i[x^n P_n + \theta Q + \bar{\theta}\bar{Q}]\right) = $$

$$= \exp\Big(i\{(a^m + x^m)P_m + (\xi+\theta)Q + (\bar{\xi}+\bar{\theta})\bar{Q}\} - \tfrac{1}{2}\{[\xi Q, \bar{\theta}\bar{Q}] + [\bar{\xi}\bar{Q}, \theta Q] + [\xi Q, \theta Q] + [\bar{\xi}\bar{Q}, \bar{\theta}\bar{Q}]\}\Big)$$

We used $e^A e^B = \exp\left(A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] - \frac{1}{12}[B,[B,A]] + \cdots\right)$ and the above is equality since higher order $\theta, \bar{\theta}$ terms must vanish, continuing the above;

$$= \exp\left(i\left[(a^m + x^m)P_m + (\xi^\alpha + \theta^\alpha)Q_\alpha + (\bar{\xi}+\bar{\theta})\bar{Q}\right] - \xi\sigma^m\bar{\theta}P_m + \theta\sigma^m\bar{\xi}P_m\right)$$

$$= \exp\left(i\left[(a^m + x^m + i\xi\sigma^m\bar{\theta} - i\theta\sigma^m\bar{\xi})P_m + (\xi+\theta)Q + (\bar{\xi}+\bar{\theta})\bar{Q}\right]\right)$$

Multiplication in the super Lie group generates a motion in the parameter space, a supertranslation,

$$a^m \longmapsto a^m + x^m + i(\theta\sigma^m\bar{\xi} - \xi\sigma^m\bar{\theta})$$
$$\xi^\alpha_\alpha \longmapsto \xi_\alpha + \theta_\alpha$$
$$\bar{\xi}^{\dot\alpha} \longmapsto \bar{\xi}^{\dot\alpha} + \bar{\theta}^{\dot\alpha}$$

From a calculation similar to our Poincaire P^μ discussion, we can represent P_m and $Q_\alpha, \bar{Q}^{\dot\alpha}$ as differential operators on functions of superspace $(x^m, \theta^\alpha, \bar{\theta}_{\dot\alpha})$

$$iQ_\alpha = \partial_\alpha - i\sigma^m_{\alpha\dot\alpha}\bar{\theta}^{\dot\alpha}\partial_m \qquad Q_\alpha = \partial_\alpha - i\sigma^m_{\alpha\dot\beta}\bar{\theta}^{\dot\beta}\partial_m$$
$$i\bar{Q}_{\dot\alpha} = \bar{\partial}_{\dot\alpha} - i\theta^\alpha\sigma^m_{\alpha\dot\alpha}\partial_m \qquad \bar{Q}_{\dot\alpha} = \bar{\partial}_{\dot\alpha} - i\theta^\beta\sigma^m_{\beta\dot\alpha}\partial_m$$
$$P_m = -i\partial_m \qquad P_m = i\partial_m$$

(Ryder's Conventions) $\qquad$ (Lykken's Conventions)

$$e^{i[x^\mu P_\mu + \theta Q + \bar{\theta}\bar{Q}]} \qquad e^{i[-x^m P_m + \theta Q + \bar{\theta}\bar{Q}]}$$

Covariant Derivative D_α

We studied left multiplication and found Q_α and $\bar{Q}^{\dot\alpha}$ as diff. op. on superspace. If instead you look at the right group action one finds D_α and $\bar{D}^{\dot\alpha}$

$$D_\alpha = \partial_\alpha + i\sigma^m_{\alpha\dot\rho}\bar\theta^{\dot\rho}\partial_m$$
$$\bar{D}_{\dot\alpha} = \bar\partial_{\dot\alpha} + i\theta^\rho\sigma^m_{\rho\dot\alpha}\partial_m$$

Lets calculate the anticommutator of D_α and Q_α

$$\begin{aligned}\{D_\alpha, Q_\gamma\}f &= \{\partial_\alpha + i\sigma^m_{\alpha\dot\rho}\bar\theta^{\dot\rho}\partial_m, \partial_\gamma - i\sigma^m_{\gamma\dot\rho}\bar\theta^{\dot\rho}\partial_m\}f \\ &= \underbrace{\{\partial_\alpha, \partial_\gamma\}f}_{0} + \{\partial_\alpha, -i\sigma^m_{\gamma\dot\rho}\bar\theta^{\dot\rho}\partial_m\}f \\ &\quad + \sigma^m_{\alpha\dot\rho}\sigma^n_{\gamma\dot\delta}\underbrace{\{\bar\theta^{\dot\rho}, \bar\theta^{\dot\delta}\}}_{0}\partial_m\partial_n f \quad \text{, since } \bar\theta^{\dot\rho}\bar\theta^{\dot\delta} = -\bar\theta^{\dot\delta}\bar\theta^{\dot\rho} \\ &\quad + i\sigma^m_{\alpha\dot\rho}\partial_m\{\bar\theta^{\dot\rho}, \partial_\gamma\}f \\ &= i\sigma^m_{\alpha\dot\rho}\partial_m\underbrace{\{\bar\theta^{\dot\rho}, \partial_\gamma\}}_{0}f - i\underbrace{\{\partial_\alpha, \bar\theta^{\dot\rho}\}}_{0}\sigma^m_{\gamma\dot\rho}\partial_m f \\ &= i\sigma^m_{\alpha\dot\rho}\partial_m f(0) = 0 \qquad \therefore \boxed{\{D_\alpha, Q_\gamma\} = 0}\end{aligned}$$

Similar slightly less trivial calculations will show

$$\boxed{\begin{array}{ll}\{D_\alpha, Q_\rho\} = 0 & \{D_\alpha, \bar{Q}_{\dot\rho}\} = 0 \\ \{\bar{D}_{\dot\alpha}, Q_\rho\} = 0 & \{\bar{D}_{\dot\alpha}, \bar{Q}_{\dot\rho}\} = 0\end{array}}$$

Finally similar to $\{Q_\alpha, \bar{Q}_{\dot\rho}\} = -2i\sigma^m_{\alpha\dot\rho}\partial_m$ we find,

$$\begin{aligned}\{D_\alpha, \bar{D}_{\dot\alpha}\} &= \{\partial_\alpha + i\sigma^m_{\alpha\dot\rho}\bar\theta^{\dot\rho}\partial_m, \bar\partial_{\dot\alpha} + i\theta^\rho\sigma^m_{\rho\dot\alpha}\partial_m\} \\ &= \{\partial_\alpha, \bar\partial_{\dot\alpha}\} + i\sigma^m_{\alpha\dot\rho}\partial_m\{\bar\theta^{\dot\rho}, \bar\partial_{\dot\alpha}\} + i\sigma^m_{\rho\dot\alpha}\partial_m\{\partial_\alpha, \theta^\rho\} + \sigma\sigma\{\bar\theta, \theta\}\partial_m\partial_n \\ &= 0 + i\sigma^m_{\alpha\dot\rho}\partial_m\delta^{\dot\rho}_{\dot\alpha} + i\sigma^m_{\rho\dot\alpha}\partial_m\delta^\rho_\alpha + 0 \\ &= 2i\sigma^m_{\alpha\dot\alpha}\partial_m \qquad \therefore \boxed{\{D_\alpha, D_{\dot\alpha}\} = 2i\sigma^m_{\alpha\dot\alpha}\partial_m}\end{aligned}$$

Note the somewhat strange $\{\bar\theta^{\dot\rho}, \bar\partial_{\dot\alpha}\} = \delta^{\dot\rho}_{\dot\alpha}$ follows from

$$\{\bar\theta^{\dot\rho}, \bar\partial_{\dot\alpha}\}f = (\bar\theta^{\dot\rho}\bar\partial_{\dot\alpha} + \bar\partial_{\dot\alpha}\bar\theta^{\dot\rho})f = \bar\theta^{\dot\rho}\bar\partial_{\dot\alpha}f - \bar\theta^{\dot\rho}\bar\partial_{\dot\alpha}f + \bar\partial_{\dot\alpha}\bar\theta^{\dot\rho}f = \delta^{\dot\rho}_{\dot\alpha}f$$

Usually we have $[x^\mu, \partial_\nu] = \delta^\mu{}_\nu$ (for bosonic coordinates) while here $\{\theta^\alpha, \partial_\rho\} = \delta^\alpha{}_\rho$ anticommutator for ferm. coord.