Conventions and Basic Relations :

$m, n, \ldots$: SPACETIME INDICES , $m = 0,1,2,3$ $\longrightarrow$ $\eta^{mn} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$\alpha, \beta, \gamma, \ldots$: Undotted Indices : $\alpha = 1,2$

$\dot\alpha, \dot\beta, \dot\gamma, \ldots$: dotted Weyl Indices : $\dot\alpha, \dot\beta = 1,2$

Grassman Coordinates θ^α and $\bar\theta_{\dot\alpha}$ are added, Anticommuting Variab

$$\{\theta^\alpha, \theta^\beta\} = \{\bar\theta_{\dot\alpha}, \bar\theta_{\dot\beta}\} = \{\theta^\alpha, \bar\theta_{\dot\beta}\} = 0$$

Up/Down Weyl indices related by,

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon^{\dot\alpha\dot\beta} \quad \text{and} \quad \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \epsilon_{\dot\alpha\dot\beta}$$

Specifically, $\epsilon^{\dot\alpha\dot\delta}\epsilon_{\dot\delta\dot\beta} = \epsilon^{\dot\alpha 1}\epsilon_{1\dot\beta} + \epsilon^{\dot\alpha 2}\epsilon_{2\dot\beta} = \begin{cases} \dot\alpha = 1, \dot\beta = 1 \Rightarrow 1 \cdot 1 \\ \dot\alpha = 1, \dot\beta = 2 \Rightarrow 0 \\ \dot\alpha = 2, \dot\beta = 2 \Rightarrow -1 \cdot -1 = 1 \end{cases}$

$$\theta^\alpha = \epsilon^{\alpha\beta}\theta_\beta \qquad \bar\theta^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\theta_{\dot\beta}$$

$$\theta_\alpha = \epsilon_{\alpha\beta}\theta^\beta \qquad \bar\theta_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\theta^{\dot\beta}$$

$$\boxed{\epsilon^{\dot\alpha\dot\delta}\epsilon_{\dot\delta\dot\beta} = \delta^{\dot\alpha}_{\dot\beta}}$$

also

$$\epsilon^{\beta\gamma}\epsilon_{\gamma\alpha} = \delta^\beta_\alpha$$

Hence, we define fermionic "scalar products"

$$\theta\theta = \theta^2 = \theta^\alpha\theta_\alpha = -2\theta_1\theta_2 = -2\theta^1\theta^2$$

$$\bar\theta\bar\theta = \bar\theta^2 = \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha} = 2\bar\theta_1\bar\theta_2 = 2\bar\theta^1\bar\theta^2$$

Derivatives are denoted as usual,

$$\partial_\alpha = \frac{\partial}{\partial\theta^\alpha} \qquad \bar\partial_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}} \qquad \xrightarrow{\text{Consistency}} \qquad \partial^\alpha = -\epsilon^{\alpha\beta}\partial_\beta$$

$$\partial^\alpha = \frac{\partial}{\partial\theta_\alpha} \qquad \bar\partial^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}} \qquad \bar\partial^{\dot\alpha} = -\epsilon^{\dot\alpha\dot\beta}\bar\partial_{\dot\beta}$$

Finally the Pauli-Matrices link Weyl and Spacetime indices,

$\sigma^m = (I, \sigma^i)$ which has undotted-dotted indices $\sigma^m_{\alpha\dot\beta}$

$\bar\sigma^m = (I, -\sigma^i)$ which instead dotted-undotted indices $\bar\sigma^{m\dot\alpha\beta}$

$$\bar\sigma^m_{\dot\alpha\beta} = \epsilon_{\dot\alpha\dot\sigma}\epsilon_{\beta\rho}\bar\sigma^{m\dot\sigma\rho}$$

$$\sigma^{m\dot\alpha\beta} = \epsilon^{\dot\alpha\dot\sigma}\epsilon^{\beta\rho}\sigma^m_{\rho\dot\sigma}$$

• Indices raise and lower like coordinates

$$\sigma^m_{\alpha\dot\beta} = \bar\sigma^m_{\dot\beta\alpha}$$

$$\mathrm{Trace}(\sigma^m\bar\sigma^n) = \sigma^m_{\alpha\dot\beta}\bar\sigma^{n\dot\beta\alpha} = \boxed{2\eta^{mn} = \sigma^m_{\alpha\dot\beta}\sigma^{n\alpha\dot\beta}}$$

$$\sigma_{m\alpha\dot\beta}\bar\sigma^{m\dot\gamma\delta} = \boxed{2\delta^\delta_\alpha\delta^{\dot\gamma}_{\dot\beta} = \sigma_{m\alpha\dot\beta}\sigma^{m\delta\dot\gamma}}$$

: $\theta\sigma^m\bar\theta = \theta^\alpha\sigma^m_{\alpha\dot\beta}\bar\theta^{\dot\beta}$ and $\bar\theta\bar\sigma^m\theta = \bar\theta_{\dot\beta}\bar\sigma^{m\dot\beta\alpha}\theta_\alpha$

we can assemble our list of basic identities

$$\partial_\alpha \theta^\beta = \delta_\alpha^\beta \qquad \bar\partial_{\dot\alpha}\bar\theta^{\dot\beta} = \delta_{\dot\alpha}^{\dot\beta}$$

$$\partial^\alpha \theta_\beta = \delta_\beta^\alpha \qquad \bar\partial^{\dot\alpha}\bar\theta_{\dot\beta} = \delta_{\dot\beta}^{\dot\alpha}$$

$$\partial_{\dot\alpha}\partial^{\dot\alpha}(\theta\theta)$$
$$\partial_{\dot\alpha}(2\theta^{\dot\alpha}) = 2\partial_{\dot\alpha}\theta^{\dot\alpha} = 4$$

$$\partial^\alpha \theta^\beta = -\epsilon^{\alpha\beta} \qquad \partial_\alpha\theta_\beta = -\epsilon_{\alpha\beta}$$

$$\bar\partial^{\dot\alpha}\bar\theta^{\dot\beta} = -\epsilon^{\dot\alpha\dot\beta} \qquad \bar\partial_{\dot\alpha}\bar\theta_{\dot\beta} = -\epsilon_{\dot\alpha\dot\beta}$$

$$\partial_\alpha(\theta\theta) = 2\theta_\alpha \qquad \bar\partial_{\dot\alpha}(\bar\theta\bar\theta) = -2\bar\theta_{\dot\alpha}$$

$$\partial^\alpha(\theta\theta) = -2\theta^\alpha \qquad \bar\partial^{\dot\alpha}(\bar\theta\bar\theta) = 2\bar\theta^{\dot\alpha}$$

$$\theta^\alpha\theta_\beta = \epsilon_{\beta\delta}\,\theta^\alpha\theta^\delta = -\tfrac{1}{2}\epsilon_{\beta\delta}\epsilon^{\alpha\delta}(\theta\theta) = +\tfrac{1}{2}\epsilon^{\alpha\delta}\epsilon_{\delta\beta}(\theta\theta) = \tfrac{1}{2}\delta^\alpha_\beta(\theta\theta)$$

$$\theta^\alpha\theta^\beta = -\tfrac{1}{2}\epsilon^{\alpha\beta}(\theta)^2 \qquad \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = \tfrac{1}{2}\epsilon^{\dot\alpha\dot\beta}(\bar\theta)^2$$

$$\theta^2 = -2\theta^1\theta^2 \qquad \bar\theta^2 = 2\bar\theta^1\bar\theta^2$$

$$\theta^2 = \theta^\alpha\theta_\alpha \qquad \bar\theta^2 = \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}$$

$$\partial_\alpha\partial_\beta(\theta\theta) = \partial_\alpha(2\theta_\beta) = -2\epsilon_{\alpha\beta}$$

$$(\theta A)(\theta B) = -\tfrac{1}{2}\theta^2(AB)$$

$$(\bar\theta\bar A)(\bar\theta\bar B) = -\tfrac{1}{2}\bar\theta^2(\bar A\bar B)$$

$$A\sigma^m\bar B = -\bar B\bar\sigma^m A$$

$$(\theta\sigma^m\bar\theta)(\theta\sigma^n\bar\theta) = \tfrac{1}{2}\eta^{mn}\theta^2\bar\theta^2$$

$$(\theta A)(\bar\theta\bar B) = \tfrac{1}{2}(\theta\sigma^m\bar\theta)(A\sigma_m\bar B)$$

$$\theta_\gamma\theta_\delta = \epsilon_{\gamma\alpha}\theta^\alpha\epsilon_{\delta\beta}\theta^\beta = -\tfrac{1}{2}\epsilon_{\gamma\alpha}\epsilon_{\delta\beta}\epsilon^{\alpha\beta}\theta^2$$

$$\theta_\gamma\theta_\delta = \tfrac{1}{2}\epsilon_{\gamma\alpha}\underbrace{\epsilon^{\alpha\beta}\epsilon_{\beta\delta}}_{\delta^\alpha_\delta}\theta^2 = \boxed{\tfrac{1}{2}\epsilon_{\gamma\delta}(\theta\theta) = \theta_\gamma\theta_\delta}$$

$$\bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = \epsilon_{\dot\alpha\dot\delta}\epsilon_{\dot\beta\dot\gamma}\bar\theta^{\dot\delta}\bar\theta^{\dot\gamma} = \tfrac{1}{2}\epsilon_{\dot\alpha\dot\delta}\epsilon_{\dot\beta\dot\gamma}\epsilon^{\dot\delta\dot\gamma}(\bar\theta\bar\theta) = -\tfrac{1}{2}\epsilon_{\dot\beta\dot\gamma}\epsilon^{\dot\gamma\dot\delta}\epsilon_{\dot\delta\dot\alpha}\bar\theta\bar\theta = \tfrac{1}{2}\epsilon_{\dot\beta\dot\gamma}\delta^{\dot\gamma}_{\dot\alpha}\bar\theta\bar\theta = \tfrac{1}{2}\epsilon_{\dot\beta\dot\alpha}\bar\theta\bar\theta = \boxed{-\tfrac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta = \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta}}$$

$$\partial_\alpha(\theta^\beta\lambda_\beta) = \lambda_\alpha$$

$$\bar\theta^{\dot\alpha}\theta^\beta\sigma^m_{\beta\dot\beta}\bar\theta^{\dot\beta} = \theta^\beta\sigma_{\beta\dot\beta}\bar\theta^{\dot\beta}\bar\theta^{\dot\alpha} = \tfrac{1}{2}\theta^\beta\sigma_{\beta\dot\beta}\epsilon^{\dot\beta\dot\alpha}(\bar\theta\bar\theta)$$

$$\partial_\alpha(\theta^\beta\theta_\beta) =$$

$$\theta^\alpha\partial_\alpha\theta\theta = \theta^\alpha 2\theta_\alpha = 2(\theta\theta)$$

$$\left(\sigma_\alpha^{mn\beta} = \frac{i}{4}\left[\sigma^m_{\alpha\dot\gamma}\bar\sigma^{n\dot\gamma\beta} - \sigma^n_{\alpha\dot\gamma}\bar\sigma^{m\dot\gamma\beta}\right]\right)$$

$$\partial_\alpha(\theta^\gamma\theta^\beta) = -\tfrac{1}{2}\epsilon^{\gamma\beta}2\theta_\alpha$$

$$\partial_{\dot\alpha}(\bar\theta\bar\psi) = \partial_{\dot\alpha}(\bar\theta_{\dot\beta}\bar\psi^{\dot\beta}) = -\epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta} = \boxed{-\bar\psi_{\dot\alpha} = \partial_{\dot\alpha}(\bar\theta\bar\psi)}$$

$$\bar\theta^{\dot\beta}(\bar\lambda_{\dot\alpha}\bar\theta^{\dot\alpha})$$
$$-\bar\lambda_{\dot\alpha}\bar\theta^{\dot\beta}\bar\theta^{\dot\alpha}$$
$$-\bar\lambda_{\dot\alpha}\tfrac{1}{2}\epsilon^{\dot\beta\dot\alpha}\bar\theta\bar\theta$$
$$-\tfrac{1}{2}\bar\lambda^{\dot\beta}(\bar\theta\bar\theta)$$

$$\theta^\alpha(\theta\lambda) = \theta^\alpha(\theta^\gamma\lambda_\gamma) = -\tfrac{1}{2}\epsilon^{\alpha\gamma}\theta\theta\,\lambda_\gamma = -\tfrac{1}{2}\theta\theta\,\epsilon^{\alpha\gamma}\lambda_\gamma = \boxed{-\tfrac{1}{2}(\theta\theta)\lambda^\alpha = \theta^\alpha(\theta\lambda)}$$

$$\partial_{\dot\alpha}(\bar\psi_{\dot\beta}\bar\theta^{\dot\beta}) = -\bar\psi_{\dot\beta}\partial_{\dot\alpha}\bar\theta^{\dot\beta} = -\bar\psi_{\dot\beta}\delta^{\dot\beta}_{\dot\alpha} = -\bar\psi_{\dot\alpha}$$

Supercharges : Q and the Supercovariant Derivatives : D

Q we derived last time, D is chosen such that,

$$\{D_\alpha, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot\beta}\} = 0$$
$$\{\bar{D}_{\dot\alpha}, Q_\beta\} = \{\bar{D}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$$

This makes $D_\alpha \Phi$ a superfield. Also we can realize both Q and D as actions on a coset-space, Q is the left-group action while D is the right group action.

$D_\alpha = \partial_\alpha + i\sigma^m_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_m$	$Q_\alpha = \partial_\alpha - i\sigma^m_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_m$
$\bar{D}_{\dot\alpha} = \bar\partial_{\dot\alpha} + \theta^\beta\sigma^m_{\beta\dot\alpha}\partial_m$	$\bar{Q}_{\dot\alpha} = \bar\partial_{\dot\alpha} - i\theta^\beta\sigma^m_{\beta\dot\alpha}\partial_m$

Raising the indices on these <u>derivatives</u> goes as $D^\alpha = -\epsilon^{\alpha\beta}D_\beta$,

$D^\alpha = \partial^\alpha - i\sigma^{m\alpha}{}_{\dot\beta}\bar\theta^{\dot\beta}\partial_m$	$Q^\alpha = \partial^\alpha + i\sigma^{m\alpha}{}_{\dot\beta}\bar\theta^{\dot\beta}\partial_m$
$\bar{D}^{\dot\alpha} = \bar\partial^{\dot\alpha} - i\theta^\beta\sigma_\beta{}^{m\dot\alpha}\partial_m$	$\bar{Q}^{\dot\alpha} = \bar\partial^{\dot\alpha} + i\theta^\beta\sigma_\beta{}^{m\dot\alpha}\partial_m$

★ also $D_m = \partial_m$, covariant derivative on the bosonic portion of superspace is just the usual derivative.

$\sigma^m\bar\sigma^n\theta_r$

Component field expansion: $\mathcal{U}$ is unconstrained Scalar Superfield

$$\mathcal{U} = f(x) + \theta\phi(x) + \bar\theta\bar\chi(x) + \theta\theta m(x) + \bar\theta\bar\theta n(x) + \theta\sigma^m\bar\theta V_m(x) + (\theta\theta)\bar\theta\bar\lambda(x) + (\bar\theta\bar\theta)\theta\psi(x) + (\bar\theta\bar\theta)(\theta\theta)d(x)$$

Chiral Superfield Φ is scalar supefield constrained by $\bar D_{\dot\alpha}\Phi = 0$

$$\Phi(y,\theta) = A(y) + \sqrt{2}\,\theta\psi(y) + \theta\theta F(y) \qquad \text{for } y^m = x^m + i\theta\sigma^m\bar\theta$$

$$\Phi(x,\theta,\bar\theta) = A(x) + \sqrt{2}\,\theta\psi(x) + \theta\theta F(x) + i\theta\sigma^m\bar\theta\partial_m A(x) + \frac{i}{\sqrt{2}}(\theta\theta)\partial_m\psi(x)\sigma^m\bar\theta - \frac{1}{4}(\theta\theta)(\bar\theta\bar\theta)\Box A(x)$$

Where by comparison these Chiral components are related to the $\mathcal{U}$ expansion above,

$$\phi \rightarrow \sqrt{2}\,\psi$$
$$m \rightarrow F$$
$$f \rightarrow A$$
$$V_m \rightarrow i\partial_m A = i\partial_m f$$

Vector Superfield V is a scalar superfield with the constraint $V = V^\dagger$

$$V_{WZ} = -\theta\sigma^m\bar\theta V_m + i(\theta\theta)\bar\theta\bar\lambda - i(\bar\theta\bar\theta)\theta\lambda + \tfrac{1}{2}(\theta\theta)(\bar\theta\bar\theta)D$$

Components of V in the Wess-Zumino gauge.

Again by comparison these components are related to $\mathcal{U}$'s components,

$$V_m \rightarrow -V_m$$
$$\bar\lambda \rightarrow i\bar\lambda$$
$$d \rightarrow \tfrac{1}{2}D$$

Spinor Superfield W_α contains several Chiral Superfields: $\bar D_{\dot\alpha} W_\beta = 0$ (Abelian Case)

$$W_\alpha = -\frac{1}{4}(\bar D\bar D)D_\alpha V(x,\theta,\bar\theta) \qquad \text{where } V \text{ is vector superfield}$$

Likewise we define $\bar W_{\dot\alpha}$ by

$$\bar W_{\dot\alpha} = -\frac{1}{4}(DD)\bar D_{\dot\alpha}V(x,\theta,\bar\theta) \qquad \text{and} \qquad D_\beta\bar W_{\dot\alpha} = 0$$

The components of W_α are

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) - \frac{i}{2}(\sigma^m\bar\sigma^n\theta)_\alpha[\partial_m V_n - \partial_n V_m](y) + \theta\theta\sigma^m_{\alpha\dot\beta}\partial_m\bar\lambda^{\dot\beta}(y)$$

These components are the "curl multiplet" or "field strength" multiplet