Def Orbit and Isotropy Subgroup

If G is a Lie Group and G acts on a manifold M then the orbit through $x \in M$ is $G \cdot x = \{g \cdot x \mid g \in G\}$. If $x \in M$ then the set of all $g \in G$ s.t. $g \cdot x = x$ is called the isotropy subgroup of G at x. We denote the isotropy subgroup by G_x.

G_x a Lie Group?

$x \in M$ and $G_x \subseteq G$ is G_x closed?

Let $\{G_N\}$ be a sequence in G_x such that $g_n \longrightarrow g_0$ in G then we have

$$g_n \cdot x = x$$

$$\psi(g_n, x) = x \quad \forall n$$

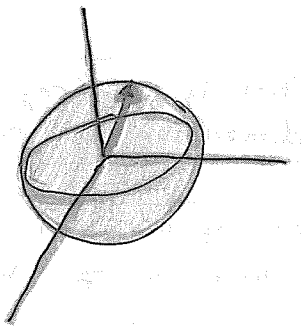
$$\Rightarrow \psi(g_0, x) = x$$

$$\Rightarrow g_0 \in G_x \quad \therefore \text{ closed subgroup } \therefore \underline{G_x \text{ a Lie subgroup.}}$$

Example

$G = SO(3)$

$M = \mathbb{R}^3$



$(A, y) \longrightarrow Ay$

If $x \in \mathbb{R}^3$, $x \neq 0$ then $G \cdot x$ is a sphere of radius $\|x\|$. We can foliate the manifold $\mathbb{R}^3$ with these submanifolds we call leaves

Example

$G = GL(n)$

$M = S(n)$ = symmetric $n \times n$

$G \times S(n) \longrightarrow S(n)$

$(g, A) = gAg^T$

$$G \cdot A = \left\{ g D_A g^t \;\middle|\; D_A \text{ is diagonal}, g \in G \right\}$$

2/14/2001

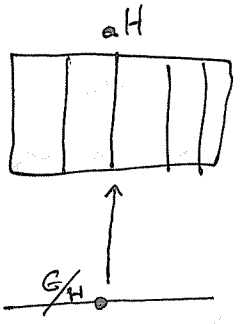
THEOREM

Fix $x \in M$. $\varphi : G \longrightarrow G \cdot x$, $\varphi(g) = g \cdot x$, $\varphi(g) = x$, $gx = x$

$\{g \mid \varphi(g) = \varphi(e)\} = G_x$ $\qquad \varphi(e) = x$

$$\frac{G}{G_x} \cong G \cdot x$$

$$\frac{SO(3)}{SO(2)} = S^2$$



~~Fiber~~
principle fibre bundle
we get local sections
not global section

$H \subset G$

$\varphi(g) = aH \in G/H$

φ projects cosets to points

φ^{-1} projects points to surfaces in G

Can get manifold from subset H even though $H \not\subset G$.

Def^n/ If G acts on M then the action is *semiregular* iff all the orbits have the same dimension as submanifolds

Def^n/ THE ACTION is regular iff it is semiregular and $\forall x \in M$ $\exists$ arbitrarily small open sets U about x $\ni$ $\forall y \in M$ either $G \cdot y \cap U = \phi$ or $G \cdot y \cap U$ is path connected

We shall return to these topics later

Def^n/ A function f is in $C^\infty_p(M)$ iff M is a manifold, $p \in M$, $p \in U$ $f: U \to \mathbb{R}$ is a smooth function for some open subset U of M $\ni p \in U$

We can define function addition and multiplication if we appropriate restrict the domain of the result

$$(f+g)(x) = f(x) + g(x)$$
$$(cf)(x) = c\,f(x)$$
$$(fg)(x) = f(x)\,g(x)$$

This would be an algebra but we don't have an additive identity

$$\hat{0}(x) = 0$$
$$f + \hat{0} = f = \hat{0} + f$$
$$f + (-1)f = \hat{0}|_U \quad \text{where } \operatorname{dom} f = U$$

Hilbert Space : identify to functions to be the same if they differ on a set of measure zero.

Def^n/ We say that $\mathbb{X}$ is a first order (linear) differential operator at p iff $\mathbb{X}: C^\infty_p M \to \mathbb{R}$ such that

(1) $\mathbb{X}(f+g) = \mathbb{X}(f) + \mathbb{X}(g)$

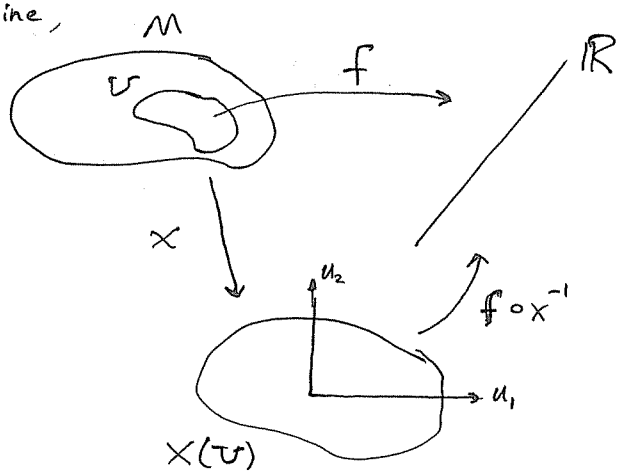
(2) $\mathbb{X}(cf) = c\,\mathbb{X}(f)$

(3) $\mathbb{X}(fg) = f(p)\,\mathbb{X}(g) + g(p)\,\mathbb{X}(f)$

Example

Let (U, x) be a chart and $p \in U$. Define,

$$\left.\frac{\partial}{\partial x^i}\right|_p (f) = \frac{\partial (f \circ x^{-1})}{\partial u^i}(x(p))$$



Now observe the following

$$[(fg)\circ x^{-1}](u) = (fg)(x^{-1}(u)) = f(x^{-1}(u))\,g(x^{-1}(u))$$
$$= [(f\circ x^{-1})(g\circ x^{-1})](u)$$

$$\therefore (fg\circ x^{-1}) = (f\circ x^{-1})(g\circ x^{-1})$$

This helps us show the 3rd property

$$\frac{\partial}{\partial u^i}((fg)\circ x^{-1}) = \frac{\partial}{\partial u^i}[(f\circ x^{-1})(g\circ x^{-1})] = (f\circ x^{-1})\frac{\partial}{\partial u^i}(g\circ x^{-1}) + (g\circ x^{-1})\frac{\partial}{\partial u^i}(f\circ x^{-1})$$
$$= (f\circ x^{-1})\frac{\partial}{\partial u^i}(g\circ x^{-1}) + (g\circ x^{-1})\frac{\partial}{\partial u^i}(f\circ x^{-1})$$

$$\therefore \left.\frac{\partial}{\partial x^i}\right|_p (fg) = f(p)\left.\frac{\partial}{\partial x^i}\right|_p (g) + g(p)\left.\frac{\partial}{\partial x^i}\right|_p (f) \quad \text{(3) holds.}$$

Def^{n} Addition and scalar multiplication of Linear Maps at P.
Let X_p and Y_p be firstorder linear operators

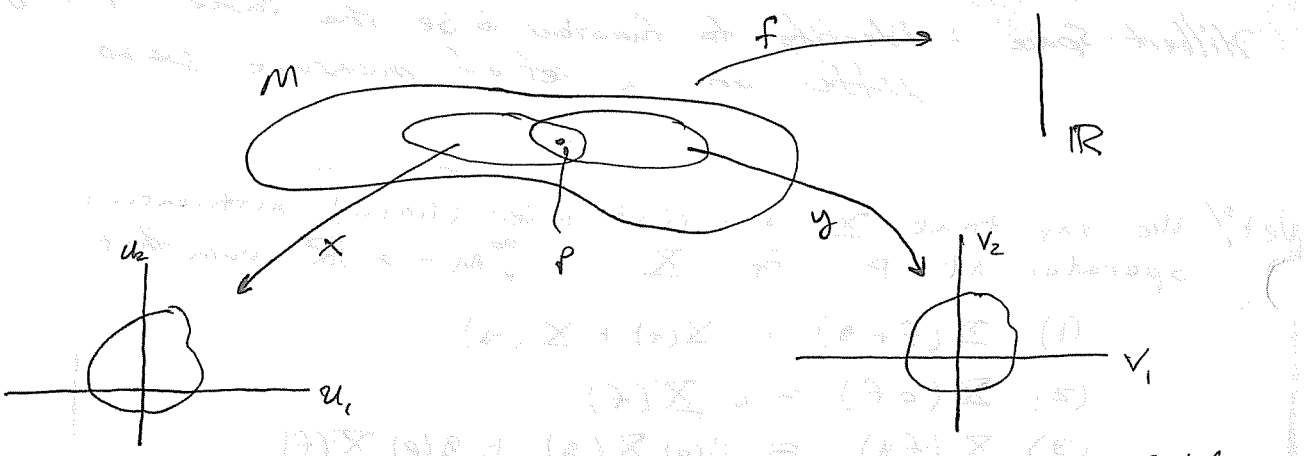
$$(i)\ (X_p + Y_p)(f) = X_p(f) + Y_p(f)$$

$$(ii)\ (c X_p)(f) = c\, X_p(f)$$

We get a linear space at p with these 1st order linear maps which we call Tangent Vectors

THEOREM

$$X_p = \sum_{i=1}^{m} a^i \left(\frac{\partial}{\partial x^i}\Big|_p \right)$$, $a^i \in \mathbb{R}$ we may expand any tangent vector by the $\frac{\partial}{\partial x^i}|_p$ vectors once we have chosen a basis at p.



Euclidean

Let (V, y) and (U, x) be charts at P. Note $(f \circ x^{-1})(u^1, u^2, \ldots) = (x \circ y^{-1})(v^1, v^2, \ldots)$

$$\frac{\partial}{\partial y^j}\Big|_p (f) = \frac{\partial}{\partial v^j}\left(f \circ y^{-1}\right)(y(p))$$

$$= \frac{\partial}{\partial v^j}\left((f \circ x^{-1})(x \circ y^{-1})(y(p))\right)$$

$$= \sum_{i=1}^{m} \frac{\partial}{\partial u^i}\left(f \circ x^{-1}\right)(x(p)) \frac{\partial (x^i \circ y^{-1})}{\partial v^j}(y(p))$$

$$= \sum_{i=1}^{m} J_{(x \circ y^{-1})}\,(y(p))^i_j \left.\frac{\partial}{\partial x^i}\right|_p (f)$$

$$\frac{\partial}{\partial y^j} = \sum J_{(x \circ y^{-1})}{}^{i}_{j} \frac{\partial}{\partial x^i}$$

$$X_p = \sum_j b^j \left.\frac{\partial}{\partial y^j}\right|_p$$

$$X_p = \sum_i \sum_j J^i_j \, b^j \frac{\partial}{\partial x^i}$$

$(b^1, b^2, \ldots, b^m)_y$

$(a^1, a^2, \ldots, a^m)_x$

$$\boxed{a^i = J^i_j \, b^j}$$

[γ]

$$V_x^i = \frac{d(x^i \circ \gamma)}{dt}$$

$$V_y^j = \frac{d(y^j \circ \gamma)}{dt}$$

$$\boxed{V_x^i = J^i_j \, V_y^j}$$

$$\frac{\hbar}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{l(l+1)}{r^2}\right)R=(E-V)R$$

$$\frac{\hbar^2}{2m}(\nabla^2+V)\psi=E\psi$$

$$R_{10}=2\left(\frac{Z}{a_0}\right)^{3/2}e^{-Zr/a_0}$$

$$\langle r^n\rangle=\left(\frac{a_0}{Z}\right)\frac{1}{2}\left(3n^2-l(l+1)\right)$$

$$\langle r^2\rangle=\left(\frac{a_0}{Z}\right)^2\frac{1}{2}\left(5n^2-3l(l+1)\right)$$

$$\left\langle\frac{1}{r}\right\rangle=\frac{Z}{a_0}\frac{1}{n^2}$$

$$\left\langle\frac{1}{r^2}\right\rangle=\left(\frac{Z}{a_0}\right)^2\frac{1}{n^3(l(l+1))}$$

$$\left\langle\frac{1}{r^3}\right\rangle=\left(\frac{Z}{a_0}\right)^3\frac{1}{n^3(l(l+1)(l+1/2))}$$

$$E_n=-\frac{Z^2e^2}{2a_0}\frac{1}{n^2}$$

$$a_0=\frac{\hbar^2}{\mu e^2}$$

$$\text{Degeneracy}=n^2$$

$$r_{max}=\frac{a_0n^2}{Z}$$

$$(l=n-1)\quad=0$$

$$E_n^{(0)}=\langle E\rangle$$

$$E_n^{(1)}=-\sum_{m\neq n}\frac{|\langle m|\hat{H}'|n\rangle|^2}{E_m^0-E_n^0}$$

$$\psi(x)=\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$

$$E_n=n^2\left(\frac{\hbar^2\pi^2}{2ma^2}\right)$$

$$\psi(x,y)=\frac{2}{a}\sin(\quad)\sin(\quad)$$

$$E_n=\frac{\hbar^2\pi^2}{2ma^2}(n_x^2+n_y^2)$$

$$\frac{-\hbar^2}{2m}\frac{d^2u}{dr^2}+\left\{V(r)+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right\}u=Eu$$

2/18/2008

Derivations are First order Linear Differential operators

$$X_p = \sum_i X_x^i \left(\frac{\partial}{\partial x^i}\Big|_p \right) \qquad X_x^i \in \mathbb{R}$$

$$X_p = \sum_j X_y^i \left(\frac{\partial}{\partial y^i}\Big|_p \right) \qquad (V, y) \text{ chart at } P.$$

$$\boxed{X_x^k = X_y^i \, J_{x \circ y^{-1}}(y(P))_j^k}$$

Now recall that before we thought of vector $v = [\gamma]$, $\gamma(0) = P$ that is a vector was an equivalence class of curves

$$V_x^i = \frac{d(x^i \circ \gamma)}{dt}(0)$$

$$\boxed{V_x^i = V_y^j \, J_{(x \circ y^{-1})}(y(P))_j^i}$$

Now X_p and $[\gamma]$ are both abstract geometric ideas but when we identify how their coordinate representatives vary under a change of coordinates we see that they behave the same.

Consider 3 classes of objects:

(1) Equivalence classes of curves through P; $\gamma_1(0) = \gamma_2(0)$
$\gamma_1 \sim \gamma_2$ means $\frac{d}{dt}(x \circ \gamma_1)(0) = \frac{d}{dt}(x \circ \gamma_2)(0)$ $\forall$ charts x.
Call the set of such equivalence classes Ξ

(2) Let $\mathcal{M}_p$ denote the class of all maps ν ∋ dom(ν) is the set of all charts of M at P, the range of ν is $\mathbb{R}^m$ and $\forall$ pair of charts x, y at P

$$\nu(x) = J_{x \circ y^{-1}}(y(P))\, \nu(y)$$

(3) Let D_p denote the class of all derivations of $C^\infty_p(M)$
$X_p \in D_p$ linear and Leibnitz

Now then define $\Phi : \Xi \longrightarrow \mathcal{M}_p$ by $\Phi([\gamma]) = \nu_\gamma$

$$\nu_\gamma(x) = \left(\frac{d(x^1 \circ \gamma)}{dt}, \frac{d(x^2 \circ \gamma)}{dt}, \frac{d(x^3 \circ \gamma)}{dt}, \ldots, \frac{d(x^m \circ \gamma)}{dt} \right)$$

$$\nu_\gamma(x) \equiv (v^1_x, v^2_x, \ldots, v^m_x)$$

$$\nu_\gamma(y) \equiv (v^1_y, v^2_y, \ldots, v^m_y)$$

$$\nu_\gamma(x) = J_{(x \circ y^{-1})}(y(P))\, \nu_\gamma(y) \quad \therefore \nu_\gamma \in \mathcal{M}_p$$

Φ is injective, $\Phi([\gamma_1]) = \Phi([\gamma_2])$

$\Rightarrow \nu_{\gamma_1} = \nu_{\gamma_2} \Rightarrow (x \circ \gamma_1)'(0) = (x \circ \gamma_2)'(0) \Rightarrow \gamma_1 \sim \gamma_2 \Rightarrow [\gamma_1] = [\gamma_2]$

Φ is also onto, Choose $\nu \in M_p$. Is there γ such that $\nu = \nu_\gamma$?
Define $\gamma(t) = x^{-1}(x(P) + t\,\nu(x))$. Then

$$(x \circ \gamma)'(0) = \nu(x) \Longrightarrow \nu_\gamma(x) = \nu(x) \quad \forall x \Rightarrow \nu_\gamma = \nu.$$

So Φ is an isomorphism. We can transport structure from one language of tangent vectors to another

Now we propose a second isomorphism $\psi : \mathcal{D}_p \to \mathcal{M}_p$ by

$$\psi(\Sigma_p) = V_{\Sigma_p}$$

Choose a chart x then we expand derivation into ∂_i's,

$$\Sigma_p = \sum \Sigma_x^i \left(\frac{\partial}{\partial x^i}\Big|_p\right)$$

$$V_{\Sigma_p}(x) = \left(\Sigma_x^1, \Sigma_x^2, \ldots, \Sigma_x^m\right)$$

$$\boxed{V_{\Sigma_p}(x) = J_{(x \circ y^{-1})}(y(p))\, V_{\Sigma_p}(y)}$$

1-1 ?] $\psi(\Sigma_p) = \psi(\vartheta_p)$ $\Sigma_p, \vartheta_p \in \mathcal{D}_p$ then

$$V_{\Sigma_p}(x) = \left(\Sigma_x^1, \Sigma_x^2, \ldots, \Sigma_x^m\right)$$

$$V_{\vartheta_p}(x) = \left(\Upsilon_x^1, \Upsilon_x^2, \ldots, \Upsilon_x^m\right)$$

$$\Sigma_x^i = \Upsilon_x^i \quad \forall \text{ charts } x \text{ and } \forall i$$

$\therefore$ ~~$V_{\Sigma_p}(x) = J_{(x \circ y^{-1})}(y(p))\, \Sigma_p = \sum \Sigma^i$~~

$$\Sigma_p = \sum_i \Sigma_x^i \frac{\partial}{\partial x^i} = \sum \Upsilon_x^i \frac{\partial}{\partial x^i} = \Upsilon_p$$

$\therefore \Sigma_p = \Upsilon_p$ so ψ is injective

ONTO] Choose $v \in \mathcal{M}_p$, choose a chart x at p define

$$\Sigma_p = \sum_i v(x)^i \frac{\partial}{\partial x^i}\Big|_p$$

$$\psi(\Sigma_p)(x) = v(x)$$

$\psi(\Sigma_p) = v$. so ψ is surjective.

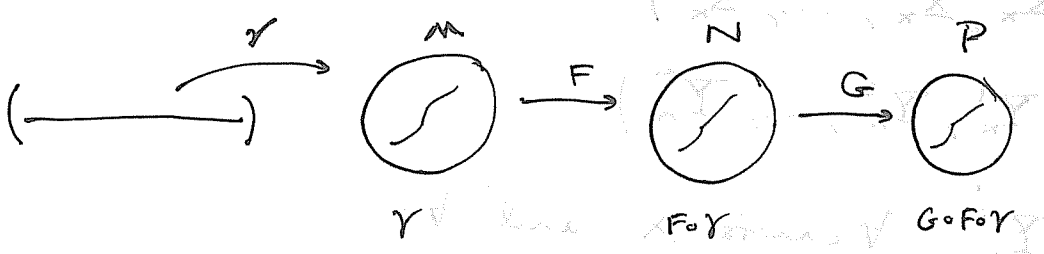
$$V_1 + V_2 = \Psi\left(\Psi^{-1}(V_1) + \Psi^{-1}(V_2)\right) \quad \text{and} \quad cV = \Psi(c\Psi^{-1}(V))$$

If we define addition and scalar multiplication in a fashion similiar to the above then we can place linear structure on all 3 ideas of the tangent. The tangent space is a vector space then.

Let M, N be manifolds and $F: M \longrightarrow N$ a smooth mapping. If $p \in M$ then d_pF is

$$d_pF : T_pM \longrightarrow T_{F(p)}N$$

$$d_pF([\gamma]_p) \equiv [F \circ \gamma]$$



$$d(G\circ F)[\gamma] = [G\circ F\circ\gamma] = d_{F(p)}G([F\circ\gamma]) = d_{F(p)}G\left(d_pF([\gamma])\right)$$

$$d(G\circ F)[\gamma] = (dG \circ dF)[\gamma]$$

Now T_pM is identified with derivations and $T_{F(p)}N$ also then how is d_pF defined? The derivation associated to $[\gamma]$, $\gamma(0)=p$ is

$$\gamma'(0) \equiv \sum_i \frac{d(x^i\circ\gamma)}{dt}(0)\left(\frac{\partial}{\partial x^i}\Big|_p\right)$$ where (U, x) a chart of M at p

So then

$$d_pF([\gamma]) = F\circ\gamma$$

The derivation associated to $[F\circ\gamma]$ is

$$(F\circ\gamma)'(0) = \sum_j \frac{d(y^j\circ F\circ\gamma)}{dt}(0)\left.\frac{\partial}{\partial y^j}\right|_{F(p)}$$

Where (V, y) is a chart of N at $F(p)$

$$d_p F\left(\sum_i \frac{d(x^i \circ \gamma)}{dt}(0)\, \frac{\partial}{\partial x^i}\Big|_p\right) = \sum_j \frac{d(y^j \circ F \circ \gamma)}{dt}(0)\, \frac{\partial}{\partial y^j}\Big|_{F(p)}$$

$$d_p F(X_p) = d_p F\left(\sum X_x^i \frac{\partial}{\partial x^i}\Big|_p\right) = \sum_{j,i} \frac{\partial(y^j \circ F \circ x^{-1})}{\partial u^i}(x(p)) \frac{d(x^i \circ \gamma)}{dt} \frac{\partial}{\partial y^j}\Big|_{F(p)}$$

$$= \sum_i \underbrace{X_x^i \; J_{y \circ F \circ x^{-1}}(x(p))_i^j}_{Y_y^j} \frac{\partial}{\partial y^j}\Big|_{F(p)}$$

The above is an example of how to Δ def$^{\underline{n}}$ from one notation to another.

2/18/2001

Vector Fields on Manifolds

A vector field X on a manifold M is a function from M into the tangent bundle of M, TM such that,

(1.) $X(p) \in T_pM \quad \forall p \in M$

(2.) for every point $p \in M$ $\exists$ a chart (U, x) at p and smooth functions $X^i : U \longrightarrow \mathbb{R}$ such that

$$X(p) = X_p = \sum_{i=1}^{m} X^i(p) \left.\frac{\partial}{\partial x^i}\right|_p$$

~~Alternate that if $x^i : U \to \mathbb{R}$, $p \to d_p x^i(X(p))$~~

$$TM = \{ (p, v) \mid p \in M, v \in T_pM \} = \bigcup_{p \in M} T_pM$$

M is the configuration space, p would be a position while v is the velocity. These are precisely generalized positions and velocity, thus we define the lagrangian here.

If X is a vector field and $\bullet$ then the flow of X is a function $\varphi : (\mathbb{R} \times M) \longrightarrow M$ such that

$$\frac{d}{dt}[\varphi(t, x)] = X(\varphi(t, x))$$

$$\varphi(0, x) = x$$

If X is a vector field on a manifold then the flow of X is a mapping from an open subset $\mathcal{D}$ of $\mathbb{R} \times M$ into M such that

$$\frac{d}{dt}\left[\varphi(t,x)\right] = X(\varphi(t,x))$$

$$\varphi(0,x) = x$$

$$\{0\} \times M \subseteq \mathcal{D}$$

Now often we just let $\mathcal{D} = I \times M$. A vector field is called complete iff $\mathcal{D} = \mathbb{R} \times M$.

An Integral Curve of X is a map $\gamma : I \longrightarrow M$ (smooth) such that forall $t \in I$,

$$\gamma'(t) = X(\gamma(t))$$

Choose a chart (U, x) at $\gamma(t_0)$ for some $t_0 \in I^\circ$

$$X(p) = \sum_{i=1}^{n} X_x^i(p) \left(\frac{\partial}{\partial x^i}\Big|_p\right) \qquad \forall\, p \in U$$

$$X_x^i : U \longrightarrow \mathbb{R} \quad \text{is smooth } \forall\, i$$

$$p \longrightarrow \left(X_x^1(p), X_x^2(p), \ldots, X_x^m(p)\right)$$

So we may write $\gamma'(t)$ another way

$$\gamma'(t) = \sum_{i=1}^{m} \frac{d(x^i \circ \gamma)}{dt}(t) \left(\frac{\partial}{\partial x^i}\Big|_p\right) = X(\gamma(t))$$

$$X(\gamma(t)) = \sum_{i=1}^{m} X_x^i(\gamma(t)) \left(\frac{\partial}{\partial x^i}\Big|_p\right)$$

$$\Rightarrow \frac{d(x^i \circ \gamma)}{dt} = X_x^i(\gamma(t)) \qquad 1 \le i \le m$$

$$\therefore \boxed{\frac{d(x^i \circ \gamma)}{dt} = \left(X_x^i \circ x^{-1}\right)\left(x(\gamma(t))\right)}$$

$$\frac{d(X^i \circ \gamma)}{dt} = \left(\Sigma^i_x \circ x^{-1}\right)\left(x(\gamma(t))\right)$$

$$X^i \circ \gamma = f^i : (\text{interval}) \longrightarrow \mathbb{R}$$
$$\Sigma^i_x \circ x^{-1} = F^i : \; x(U) \longrightarrow \mathbb{R}$$
$$\frac{df^i}{dt} = F^i\left(f^1(t), f^2(t), \ldots, f^m(t)\right)$$

} plain old ordinary DE's. We have many results proven here already.

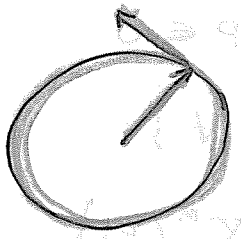
Example

Let $M = S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$

$$\Sigma = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$P \in S^2$ then let Σ act to find

$$\Sigma(P) = -y(P)\left(\frac{\partial}{\partial x}\Big|_P\right) + x(P)\left(\frac{\partial}{\partial y}\Big|_P\right)$$



$$(x,y,z)\cdot(-y,x,0) = 0$$

A chart on $S^2 = (U, \chi)$ for $U = \{P \mid z(P) > 0\}$ and

$$\chi(P) \equiv (x(P), y(P)) = (u,v)$$

$$\Sigma(\chi^{-1}(u,v)) = -u\left(\frac{\partial}{\partial x}\Big|_P\right) + v\left(\frac{\partial}{\partial y}\Big|_P\right)$$

$$\Sigma^1_\chi(u,v) = -u \quad \text{and} \quad \Sigma^2_\chi(u,v) = v$$

Another chart could be

$$\tilde{\chi}(P) = (x(P), z(P)), \quad \tilde{U} = \{P \mid y(P) > 0\}$$

$$\Sigma(\tilde{\chi}^{-1}(u,v)) = -\sqrt{1-u^2-v^2}\left(\frac{\partial}{\partial x}\Big|_P\right) + u\left(\frac{\partial}{\partial y}\Big|_P\right)$$

Exc... to show it is smooth.

$$\gamma'(t) = -y(\gamma(t))\left.\frac{\partial}{\partial x}\right|_{\gamma(t)} + x(\gamma(t))\left(\left.\frac{\partial}{\partial y}\right|_{\gamma(t)}\right)$$

$$\gamma'(t) = \frac{d(x(\gamma(t)))}{dt}\left.\frac{\partial}{\partial x}\right|_{\gamma(t)} + \frac{d(y(\gamma(t)))}{dt}\left.\frac{\partial}{\partial y}\right|_{\gamma(t)}$$

$$\frac{d(x\circ\gamma)}{dt} = -(y\circ\gamma) \qquad \frac{d(y\circ\gamma)}{dt} = (x\circ\gamma)$$

$$u(t) = x\circ\gamma$$
$$v(t) = y\circ\gamma$$

$$\frac{du}{dt} = -v$$
$$\frac{dv}{dt} = u$$

$$\longrightarrow \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\det\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} = \lambda^2 + 1 \longrightarrow \lambda = \pm i \longrightarrow \text{sol's } \cos t,\ \sin t$$

$$u = c_1\cos t + c_2\sin t$$
$$v = c_1\cos t + c_2\sin t$$

$$\longrightarrow \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

What is $\gamma(t)$ now?

$$\gamma(t) = ((x\circ\gamma)(t),\ (y\circ\gamma)(t),\ (z\circ\gamma)(t))$$
$$= (c_1\cos t + c_2\sin t,\ -c_1\sin t + c_2\cos t,\ \sqrt{1-c_1^2+c_2^2})$$

The flow

$$\varphi(t, (x_0, y_0, z_0)) = \gamma(t) \quad \text{for choice of } c_1 \text{ and } c_2$$
$$\varphi(0, (x_0, y_0, z_0)) = (x_0, y_0, z_0)$$

Want $\gamma(0) = (x_0, y_0, z_0)$

$$\varphi(t, (x_0, y_0, z_0)) = (x_0\cos t + y_0\sin t,\ -x_0\sin t + y_0\cos t,\ z_0)$$

$$\varphi(t, (x_0, y_0, z_0)) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

rotates about the z-axis

Given a vector field X we denote the flow φ as follows

$$\varphi(t,x) = \exp(tX)(x)$$

Notice that its intuitively clear that

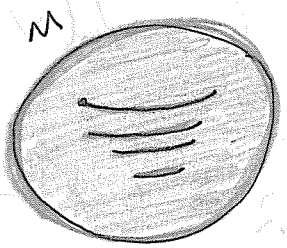
$$\varphi(t+s,x) = \varphi(t,\varphi(s,x)) \quad \leftarrow \text{reminds us of } \exp.$$

(?)

$$\frac{d}{dt}\left(\varphi(t,x)\right) = X\left(\varphi(t,x)\right)$$

Now if p that the vector field X is complete

$$\exp(tX) : M \longrightarrow M$$
$$\exp(tX)(x) = \varphi(t,x)$$



$\exp(tX)$ follows for t a sol^n to DE

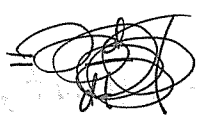
$$\exp\left((t+s)X\right) = \exp(tX) \circ \exp(sX)$$
$$\ell^{tX+sX} = \ell^{tX} \circ \ell^{sX}$$

This notation is very prevalent in Riemannian Geometry, and Lie Group theory. Its a notation that is sometimes properly suggestive.

$$\frac{d}{dt}\left(\varphi(t,x)\right) = X\left(\varphi(t,x)\right)$$

$$X\left(\varphi(t,x)\right)(f) = \frac{d}{dt}\left(\varphi(t,x)\right)(f)$$

derivations act on $f \in C^\infty_{\varphi(t,x)} M$



THEOREM $X(x) f = \frac{d}{dt} f\left(\exp(tX)(x)\right)\Big|_{t=0}$

$$\frac{d}{dt}\left(e^{tA}\right) = A e^{tA}$$

$$\frac{d}{dt}\left(e^{tA}\right)\Big|_{t=0} = A$$

$$\frac{d}{dt}\left(e^{tA}\right) = A\left(e^{tA}\right)$$

$$X\left(\varphi(t,x)\right)(f) = \frac{d}{dt}\left(\varphi(t,x)\right)(f)$$

$$= \frac{d}{dt}\left(f\left(\varphi(t,x)\right)\right)$$

$$X(x)(f) = \frac{d}{dt} f\left(\exp(tX)\right)\Big|_{t=0}$$

So this is just like matrix DE's

$$X(x) = Ax$$

$$\gamma'(t) = A\gamma(t) = X(\gamma(t))$$

The sol^{n} to this eq^{n} is e^{tA} where e^{tA} is give by series expansion. Well exp is just the abstraction of this animal.

X : vector field on $\mathbb{R}^n$

fix $A \in \mathfrak{gl}(n)$ then define the vector field X by

$$X(x) = Ax$$

Then a sol^{n} to the above DE is $\gamma: I \to \mathbb{R}^n$

$$\gamma'(t) = X(\gamma(t))$$

$$\gamma'(t) = A\gamma(t)$$ then using matrix exp. the sol^{n} is just,

Let $x_0 \in \mathbb{R}^n \Rightarrow \gamma(t) = e^{tA}x_0$, $\gamma'(t) = Ae^{tA}x_0 = A\gamma(t)$

$\gamma(t) = e^{tA}x$ is a sol^{n} $\gamma(0) = x_0 \Rightarrow \boxed{\varphi(t,x) = e^{tA}\cdot x_0}$

We had found sol^n to $Ax = X(x)$ given by ~~$\gamma(t) = Ae$~~

$\gamma(t) = e^{tA}x_0$ where the flow is $\varphi(t, x_0) = e^{tA}x_0$

$$\frac{d}{dt}\left(\varphi(t, x_0)\right) = \frac{d}{dt}\left(e^{tA} \cdot x_0\right)$$

$$= A e^{tA} x_0$$

$$= \frac{d}{dt}\left(e^{tA} x_0\right)\Big|_{t=0}$$

$$= Ax_0 = X(x)$$

Generators what are they? $\varphi(t): M \to M$
It is a vector field that yields a complete flow? If we have a vector field then we can generate a curve, flow. Conversely if we have a curve or flow we may generate a vector field.

THEOREM

$$X(x)(f) = \frac{d}{dt} f\left(\exp(tX)x\right)\Big|_{t=0}$$

$$X(\varphi(t,x))(f) = \frac{d}{dt}\left(\varphi(t,x)\right)(f)$$

$$= \frac{d}{dt}\left(f(\varphi(t,x))\right) \quad \text{since} \quad \overbrace{\gamma'(t)}^{\text{derivation}}(f) = \underbrace{\frac{d}{dt}\left(f(\gamma(t))\right)}_{\text{curve of \#'s}}$$

$f \in C_p^\infty M$, $p \in M$

$$= \frac{d}{dt} f\left(\exp(tX)(x)\right)\Big|_{t=0}$$

A vector field on M is a smooth function $X : M \to TM$ such that
$$X(p) = X_p \in T_pM.$$
where smooth means $\forall$ chart (U, x) of M $\exists$ smooth maps $X^i_x : U \to \mathbb{R}$ such that $X|_U = \sum X^i_x \frac{\partial}{\partial x^i}$

Operations on Vector field on Manifold

$$(X + Y)(p) = X_p + Y_p$$
$$(cX)(p) = c X_p$$
$$(fX)(p) = f(p) X_p$$

- Note that $f_1 f_2 = 0 \not\Rightarrow f_1 = 0$ or $f_2 = 0$

If X is a vector field on M and $f \in C^\infty(M)$ then define $X(f)$ to be the function from $M \to \mathbb{R}$ defined by $X(f)(p) = X_p(f)$

Example

$M = \mathbb{R}^3$

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}$$

$$X_p = \left(\frac{\partial}{\partial y}\Big|_p - \frac{\partial}{\partial x}\Big|_p\right) \quad \text{for} \quad p = (1,1,0)$$

$$X_p(f) = \frac{\partial f}{\partial y}\Big|_p + \frac{\partial f}{\partial x}(p)$$

$$X(f)(p) = x(p)\frac{\partial f}{\partial y}(p) - y(p)\frac{\partial f}{\partial x}(p) + z(p)\frac{\partial f}{\partial z}(p)$$

$$X(f) = x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x} + z\frac{\partial f}{\partial z} \in C^\infty(\mathbb{R}^3)$$

$$X : C^\infty(M) \longrightarrow C^\infty(M)$$

$$X(f+g) = X(f) + X(g)$$
$$X(cf) = c\,X(f)$$
$$X(fg) = f\,X(g) + g\,X(f)$$

where fg is multiplication in the ring of continous functions.

Assume that X and Y are vector fields on M. We define $[\,,\,] : C^\infty(M) \longrightarrow C^\infty(M)$ defined by

$$[X,Y](f) = X(Y(f)) - Y(X(f))$$

Let (U, x) be a chart of M at p. Then since Y, X are vector fields

$$X = \sum_{i=1}^{m} X_x^i \frac{\partial}{\partial x^i}$$
$$Y = \sum_{j=1}^{m} Y_x^j \frac{\partial}{\partial x^j}$$

$\forall f \in C^\infty(U)$, $X(f) \in C^\infty(U)$ and $X(f)(q) = \sum_i X_x^i(q) \frac{\partial f}{\partial x^i}(q)$ and also with like qualification $Y(f)(q) = \sum_j Y_x^j(q) \frac{\partial f}{\partial x^j}(q)$ $\forall q \in U$.

$$X(Y(f)) = \sum_i X_x^i \frac{\partial}{\partial x^i}\left(Y(f)\right)$$
$$= \sum_i X_x^i \frac{\partial}{\partial x^i}\left(\sum_j Y_x^j \frac{\partial f}{\partial x^j}\right) \qquad \left(Y_x^j \frac{\partial f}{\partial x^j} \in C^\infty(U)\right)$$
$$= \sum_i \sum_j X_x^i \left[\frac{\partial Y_x^j}{\partial x^i}\frac{\partial f}{\partial x^j} + Y_x^j \frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial x^j}\right)\right]$$
$$= \sum_{i,j} X_x^i \frac{\partial Y_x^j}{\partial x^i}\frac{\partial f}{\partial x^j} + \sum_{i,j} X_x^i Y_x^j \left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)$$

Likewise exchange X with Y to find $Y(X(f))$

$$Y(X(f)) = \sum_{i,j} Y_x^j \frac{\partial X_x^i}{\partial x^j} \frac{\partial f}{\partial x^i} + \sum_{i,j} X_x^j Y_x^i \frac{\partial^2 f}{\partial x^i \partial x^j}$$

Notice that the second order terms in $X(Y(f)) - Y(X(f))$ cancel and so we generate,

$$[X, Y](f) = \sum_j X(Y_x^j) \frac{\partial f}{\partial x^j} - \sum_i Y(X_x^i) \frac{\partial f}{\partial x^i}$$

$$[X, Y](f) = \sum_i \left[X(Y_x^i) - Y(X_x^i) \right] \frac{\partial}{\partial x^i}(f)$$

$$\therefore \boxed{[X, Y]_v = \sum_i \left[X(Y_x^i) - Y(X_x^i) \right] \frac{\partial}{\partial x^i}}$$

So then clearly $[X, Y]$ is also a vector field.

Jacobi Identity: If $[A, B] = AB - BA$

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

Then if $X : C^\infty(M) \longrightarrow C^\infty(M)$ then we have $[X, Y] = X \circ Y - Y \circ X$
We can check the Jacobi identity for our $[,]$ which we refer to as the Lie Derivative

Alternating: $[X, Y] = -[Y, X]$ clear from def^n.

Define the adjoint: $L_X(Y) = [X, Y]$ then from the Jacobi Identity

$$L_X[Y, Z] = [Y, L_X Z] + [L_X Y, Z]$$

$$L_x(ab) = a(L_x b) + (L_x a) b$$

Example x, y, z on all of $\mathbb{R}^3$, these are independent.

$$Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

$$Y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}$$

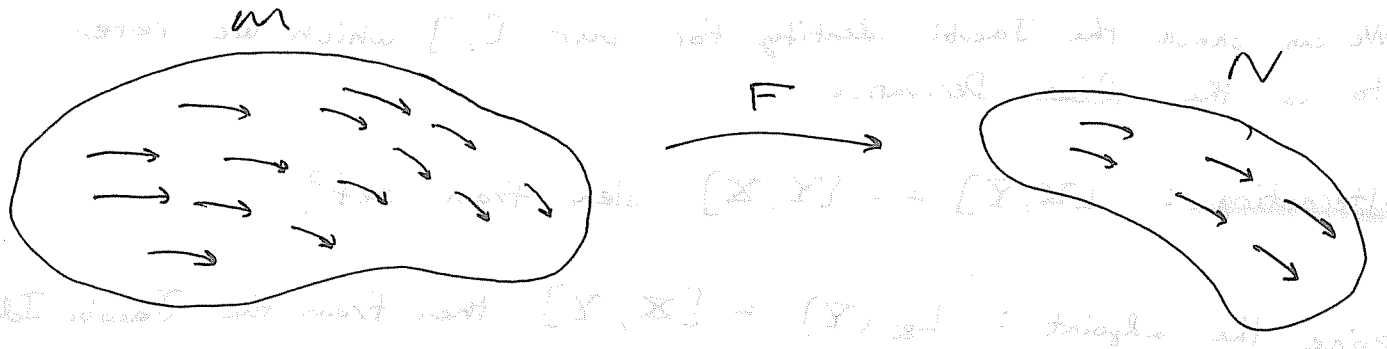
$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$$

$$\begin{aligned}[X, Y] &= \left(X(x)\frac{\partial}{\partial z} - X(z)\frac{\partial}{\partial x}\right) - \left(Y(y)\frac{\partial}{\partial z} - Y(z)\frac{\partial}{\partial y}\right) \\ &= \left(0 - y\frac{\partial}{\partial x}\right) - \left(0 - x\frac{\partial}{\partial y}\right) \\ &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \\ &= Z\end{aligned}$$

This is the infinitesimal algebra of the group of rotations.

Def^n/ Assume that M and N are manifolds and that $F: M \to N$ is smooth. Two vector fields X on M and Y on N are F related iff $\forall p \in M$,

$$d_p F(X(p)) = Y(F(p))$$



THEOREM

If $X \overset{F}{\sim} Z$ and $Y \overset{F}{\sim} W$ then we have

$$[X, Y] \overset{F}{\sim} [Z, W]$$

$L_a : G \to G$

$L_a(x) = ax$

Left invariant if $d_x L_a(\mathbb{X}(x)) = \mathbb{X}(ax) = \mathbb{X}(L_a(x))$

So if we identify L_a with F and $\mathbb{X}$ left invariant vector field on G so we get $\mathbb{X} \overset{L_a}{\sim} \mathbb{X}$ and $\mathbb{Y} \overset{L_a}{\sim} \mathbb{Y}$ $\forall a$

So then $[\mathbb{X}, \mathbb{Y}]$ is left invariant $\forall a$. ....

$X_x^i \in C^\infty(U)$

$$X(q) = \sum_{i=1}^{n} X_x^i(q) \left.\frac{\partial}{\partial x^i}\right|_q$$

X and Y vector fields, $f \in C^\infty U$, $X(f) \in C^\infty U$

$$[X, Y] = \left[\sum_{i=1}^{n} X_x^i \left(\frac{\partial}{\partial x^i}\right), \sum_{j=1}^{n} Y_x^j \left(\frac{\partial}{\partial x^j}\right)\right]$$

$$= \sum_{i=1}^{n} \left[X(Y_x^i) - Y(X_x^i)\right] \frac{\partial}{\partial x^i}$$

$[X(q), Y(q)]$ = ~~doesn't~~ make sense but $[X, Y]$ - vector field

$$[X, Y](q) = \sum_{i=1}^{n} \left[X(q)\left(Y_x^i\right) - Y(q)\left(X_x^i\right)\right] \left.\frac{\partial}{\partial x^i}\right|_q$$

$\varphi : V \longrightarrow X^V$

$V, W \in T_e G$ then $[V, W] = \varphi^{-1}\left([X^V, X^W]\right)$

$[V, W] = \varphi^{-1}\left([\varphi(V), \varphi(W)]\right)$

$$\begin{array}{ccc} X & \overset{LA}{\sim} & X \\ Y & \overset{LA}{\sim} & Y \\ [X,Y] & \overset{LA}{\sim} & [X,Y] \end{array}$$

$$(dt)f = tf$$

THEOREM

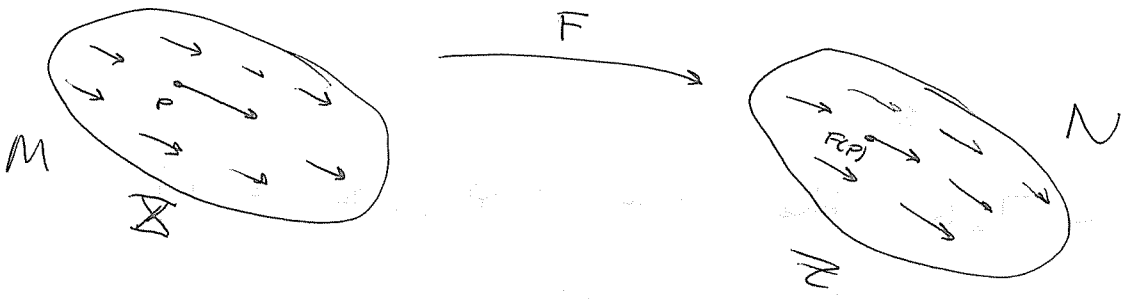
Let M and N be manifolds and $F : M \to N$ a smooth map. If X and Y are vector fields on M and Z and W are vector fields on N such that

$$\begin{array}{ccc} X & \overset{F}{\sim} & Z \\ Y & \overset{F}{\sim} & W \\ [X,Y] & \overset{F}{\sim} & [Z,W] \end{array}$$

$\overset{F}{\sim}$ not an equivalence relation.

Proof

$$X \overset{F}{\sim} Z \iff d_p F(X(p)) = Z(F(p)) \quad \forall p \in M$$
$$Y \overset{F}{\sim} W \iff d_p F(Y(p)) = W(F(p)) \quad \forall p \in M$$



Lemma: $dF(X_p)(f) = X_p(f \circ F) \quad \forall f \in C^\infty_{F(p)}(N)$

$f \circ F \in C^\infty_p(M)$

Proof: Let (U,x) be chart at p and (V,y) a chart at $F(p)$,

$$d_p F(X_p) = \sum_i X^i_x(p) \, J^{(x(p))\,i}_{y \circ F \circ x^{-1}} \left(\frac{\partial}{\partial y^i}\Big|_{F(p)} \right)$$

$$d_p F(X_p) f = \sum_i X^i_x(p) \, J^{(x(p))\,i}_{y \circ F \circ x^{-1}} \, \frac{\partial f}{\partial y^i}(F(p))$$

$$= \sum_i \sum_j X^i_x(p) \, J^{(x(p))\,i}_{y \circ F \circ x^{-1}} \, \frac{\partial}{\partial u^j}\left(y^j \circ f\right)(p)$$

$$X_p(f\circ F) = \sum_i X_x^i(p)\frac{\partial}{\partial x^i}(f\circ F)(p)$$

$$= \sum_i X_x^i(p)\,\frac{\partial(f\circ y^{-1}\circ y\circ F\circ x^{-1})}{\partial u^i}(x(p))$$

$$= \sum_{i,j} X_x^i(p)\,\frac{\partial(f\circ y^{-1})}{\partial v^j}\,\frac{\partial(y^j\circ F\circ x^{-1})}{\partial u^i}(x(p))$$

$$= \sum_{i,j} X_x^i(p)\left(J_{y\circ F\circ x^{-1}}\right)_i^j\frac{\partial f}{\partial y^j}$$ lemma done.

Proof of Theorem

Let $p\in M$ and $f\in C^\infty_{F(p)}(N)$. Then

$$W(f)(F(q)) = W_{F(q)}(f)$$

$$= d_qF(Y(q))f$$

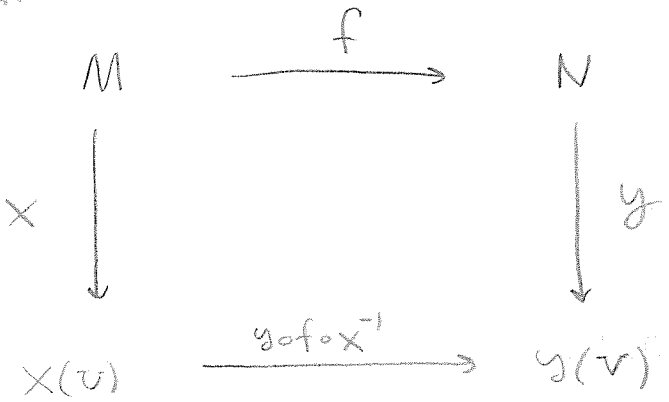
$$= Y_q(f\circ F)$$

$$= Y(f\circ F)(q)$$ $\forall q$ in nbrd of p true anytime

$$\therefore \boxed{W(f)\circ F = Y(f\circ F)}$$

Missing somehere

$T_e G \to$ (Left invariant vector field)



$$d_p f\left(\sum v^i \frac{\partial}{\partial x^i}\Big|_p\right) = \sum w^j \frac{\partial}{\partial y^j}\Big|_{f(p)}$$

$$D_{x(p)}(y \circ f \circ x^{-1})(v^1, v^2, \ldots, v^m) = (w^1, w^2, \ldots, w^m)$$

$$w^j = \sum_i \left(J_{y \circ f \circ x^{-1}}\right)^j_i v^i$$

$$gl(n) \cong \mathbb{R}^{n^2}$$

$$T_A\, gl(n) = gl(n), \quad x^i_j(A) = A^i_j, \quad x : Gl(n) \to Gl(n).$$

$$\sum_{j,i} B^i_j \left(\frac{\partial}{\partial x^i_j}\Big|_A\right) \longrightarrow \sum_{i,j} B^i_j E^j_i$$

$$\frac{\partial}{\partial x^i_j}\Big|_A \in T_A\, Gl(n)$$

$$T_a \mathbb{R}^m = \left\{ \sum_i a^i \frac{\partial}{\partial u^i}\Big|_u \;\middle|\; a^i \in \mathbb{R} \right\} \cong \mathbb{R}^m$$

$$\sum_i a^i \frac{\partial}{\partial u^i}\Big|_u \longrightarrow \sum a^i e_i$$

$L_A : \mathfrak{gl}(n) \longrightarrow \mathfrak{gl}(n)$

$L_A(B) = AB$

$L_A : \mathfrak{Gl}(n) \longrightarrow \mathfrak{Gl}(n)$

$X^V(a) = d_I L_a(V)$

$d_I L_A(B) = X^B(A)$ $\quad B \in T_I \mathfrak{Gl}(n)$

X^B left invariant vector field

$$X^B = \sum_{k,l,i} B_i^k \, x_l^i \, \frac{\partial}{\partial x_l^k}$$

$$= \sum_{k,l} \left(\sum_i B_i^k \, x_l^i \right) \frac{\partial}{\partial x_l^k}$$

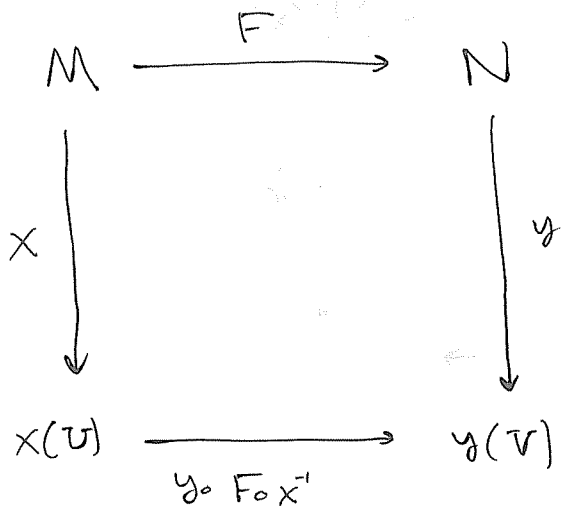
linear thus are indeed smooth.

$$X = \sum \left(X_x^i \right) \frac{\partial}{\partial x^i}$$

supposed to be smooth

$\vdots$

$$[X^B, X^C] = X^{[B,C]}$$



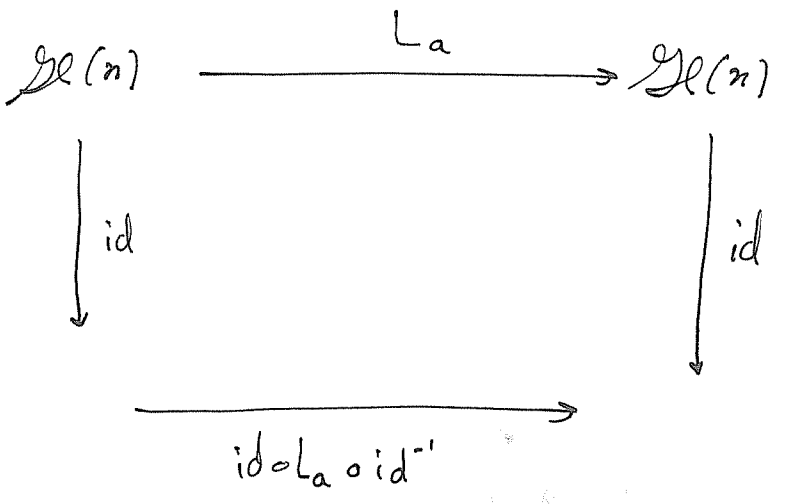
$$d_p F\left(\sum v^i \left.\frac{\partial}{\partial x^i}\right|_p\right) = \sum w^i \left.\frac{\partial}{\partial y^i}\right|_{F(p)}$$

$$w^i = \sum_j J_{y\circ F\circ x^{-1}}(x(p))^i_j \, v^j$$

$$D_{x(p)}(y\circ F\circ x^{-1})(\underbrace{v^1, v^2, \ldots, v^m}_{\sum v^i e_i}) = (\underbrace{w^1, w^2, \ldots, w^n}_{\sum w^i e_j})$$

$$d_p F(v) = w$$

$$D(y\circ F\circ x^{-1})(v_x) = w_y$$



$$dL_a\left(\sum_{i,j} v_j^i \frac{\partial}{\partial x_j^i}\right) = \sum_{k,l} w_l^k \frac{\partial}{\partial x_l^k}$$

$$D\left[\mathrm{id} \circ L_a \circ (\mathrm{id})^{-1}\right]\left(\sum_{i,j} v_j^i E_i^j\right) = \sum_{k,l} w_l^k E_n^l$$

$L_a : \mathfrak{gl}(n) \longrightarrow \mathfrak{gl}(n)$ is linear

$D_p L_a = L_a$

$(D_p L_a)(h) = L_a(h)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} b & 0 & 0 \\ e & 0 & 0 \\ h & 0 & 0 \end{pmatrix}$$

$\therefore$ $D\left(\sum_j^i E_i^j\right) = L_a\left(\sum v_j^i E_i^j\right)$

$= \sum v_j^i L_a(E_i^j)$

$$D L_a(v) = L_a(v) = av$$

$$dL_a\left(\sum v_j^i \left(\frac{\partial}{\partial x_j^i}\right)\right) = \sum_{i,j} (av)_j^i \frac{\partial}{\partial x_j^i}$$

$$dL_a\left(\sum v^i_j \frac{\partial}{\partial x^i_j}\right) = \sum_{i,j} (av)^i_j \frac{\partial}{\partial x^i_j}$$

$$= \sum_{i,j,k} a^i_k v^k_j \frac{\partial}{\partial x^i_j}$$

since $x^i_j(A) = A^i_j$, $a^i_k = x^i_k(a)$

$$= \sum_{i,j,k} x^i_k(a)\, v^k_j \frac{\partial}{\partial x^i_j}$$

Proven in Problem #1. $\longrightarrow$ $X^v(a) = d_I L_a(v)$

$v \in T_I G \Rightarrow v = \sum_{i,j} v^i_j \frac{\partial}{\partial x^i_j}\Big|_I$

$$= \sum_{i,j,k} x^i_k(a)\, v^k_j \frac{\partial}{\partial x^i_j}\Big|_a$$ this holds for all

$$\boxed{X^v = \sum_{i,j,k} v^k_j\, x^i_k \frac{\partial}{\partial x^i_j}}$$

This an explicit formula for the left invariant vector field determined by a matrix

This formula allows us to calculate $[X^v, X^w] = X^{[v,w]}$

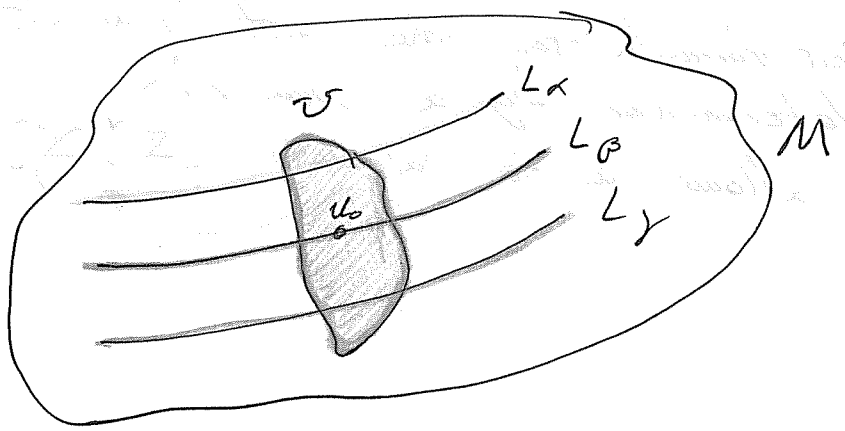
Foliations : Frobenius Theorem Exc...

Def 2/ A smooth foliation of a manifold M of dimension $0 < p \leq m$ is decomposition of M into disjoint connected subsets, $M = \bigcup_{\alpha \in A} L_\alpha$ of $\{L_\alpha\}_{\alpha \in A}$ called the leaves of the foliation such that for $u_0 \in M$ $\exists$ an open set U of M and a chart $(x, y) : U \longrightarrow \mathbb{R}^p \times \mathbb{R}^q$ such that $\forall \alpha \in A$ the components of $U \cap L_\alpha$ are given by equations, ($m = q + p$ and $q = m - p$)

$$y^i = c^i = \text{constants}$$

Thus there exist constants $c^i \in \mathbb{R}$ such that $\forall \alpha$ and $\forall$ component C of $U \cap L_\alpha$

$$u \in C \iff y^i(u) = c^i \quad \forall\ 1 \leq i \leq q$$



Example

$$M = T^2 = \{(e^{i\varphi}, e^{i\theta}) \mid \varphi, \theta \in \mathbb{R}\}$$

For $t_0 \in \mathbb{R}$

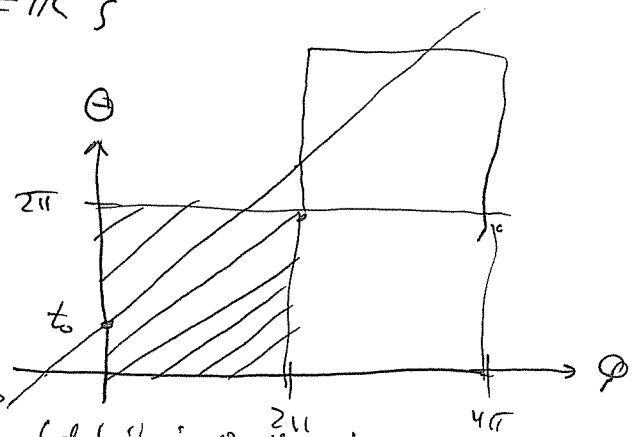
$$L_{t_0} = \{(e^{it}, e^{i(t_0 + \omega t)}) \mid t \in \mathbb{R}\}$$

ω is irrational number fixed

$$\frac{\mathbb{R}}{2\pi\mathbb{Z}} \times \frac{\mathbb{R}}{2\pi\mathbb{Z}} = \frac{\mathbb{R}^2}{2\pi(\mathbb{Z}\times\mathbb{Z})}$$

$\theta = t$

$\theta = t_0 + \omega t$



then we take some $t_0 \neq t$ and do it again, then continue till the tauras is completely covered.

$$L_{t_0} \cap L_{s_0} \neq \phi$$

$\Rightarrow \exists\, s, t$ such that $e^{is} = e^{it}$ and $e^{i(s_0+ws)} = e^{i(t_0+wt_0)}$

So then $s - t = n(2\pi)$ and $(s_0 + ws) - (t_0 + wt) = m(2\pi)$

ahen $s_0 - t_0 = m(2\pi) - w(n(2\pi)) = 2\pi(m - nw)$. For this foliation we have $p=1$, $q=1$ and $m=2$. Now on an appropriate $U \subseteq T^2$

$$(e^{i\theta}, e^{i\phi}) \longrightarrow (\theta, \phi)$$

is a chart. Let θ and ϕ vary on interval less than 2π and take inverse image and call it U. Define then

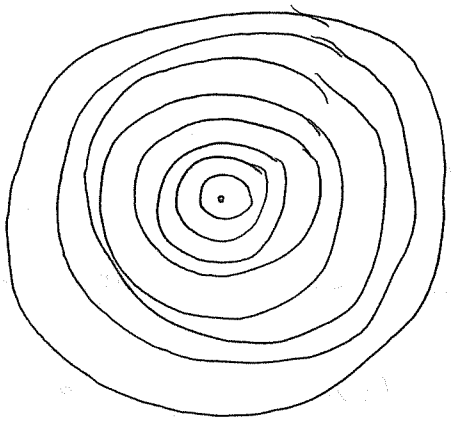
$$f(\theta, \varphi) = (\theta, \varphi - w\theta)$$

Define $(x, y) : U \longrightarrow \mathbb{R} \times \mathbb{R}$ by $(x, y)(e^{i\theta}, e^{i\phi}) = f(\theta, \phi) = \varphi$,

$f(\theta, \phi) = (\theta, \phi - w\theta)$ then Look at L_{t_0}

$$(x, y)(e^{it}, e^{i(t_0+wt)}) = (t, (t_0 + wt) - wt) = (t, t_0)$$

$\mathbb{R}^2 - 0$ — foliated punctured plane by circles.



If $(x,y): U \to \mathbb{R}^p \times \mathbb{R}^q$ is a chart for a foliation of M, then $X_{u_0} \in T_{u_0}M$ is tangent to the leaf L_α through u_0 iff $\exists$ a curve $\gamma: (-a,a) \to L_\alpha \cap U$ such that $\gamma(0) = u_0$ and $\gamma'(u_0) = X_{u_0}$. This curve is in a single component of $U \cap L_\alpha$, so $y^i(\gamma(t)) = c^i$

$$dy^i(\gamma'(t)) = 0$$

$$t=0 \Rightarrow \quad dy^i(X_{u_0}) = 0$$

$$T_{u_0}L_\alpha = \left\{ X_{u_0} \in T_{u_0}M \,\middle|\, dy^i(X_{u_0}) = 0 \right\}$$

$$X_{u_0} = \sum X^i_{(x,y)} \left.\frac{\partial}{\partial x^i}\right|_{\gamma(p)} + \sum Y^j_{(x,y)} \left.\frac{\partial}{\partial y^j}\right|_{u_0}$$

$$x^1, x^2, \ldots, x^p, y^1, \ldots, y^q$$

$$0 = dy^i(X_{u_0}) = Y^i_{u_0}$$

$$T_{u_0}L_\alpha = \left\{ \sum X^i \left.\frac{\partial}{\partial x^i}\right|_{u_0} \,\middle|\, X^i \in \mathbb{R} \right\}$$

$$V \in \mathfrak{gl}(n)$$

$$V_j^i \left(\frac{\partial}{\partial x_j^i}\Big|_I \right)$$

$$\mathfrak{gl}(n) \xrightarrow{\psi} T_I \mathfrak{gl}(n)$$

$$V = \sum_{i,j} V_j^i E_i^j \longleftrightarrow \sum_{i,j} V_j^i \left(\frac{\partial}{\partial x_j^i}\Big|_I \right) \xrightarrow{\Phi} X^{\sum_{i,j} V_j^i \left(\frac{\partial}{\partial x_j^i}\Big|_I \right)}$$

$$T_e G \xrightarrow{\Phi} \text{LIVF}$$

$$V \xrightarrow{\Phi} X^V$$

$$\mathfrak{gl}(n) \xrightarrow{\Phi \circ \psi} \text{LIVF}$$

$$V \longrightarrow X^{\sum V_j^i \left(\frac{\partial}{\partial x_j^i}\Big|_I \right)}$$

$$(\Phi \circ \psi)[V, W] = \left[(\Phi \circ \psi)(V), (\Phi \circ \psi)(W) \right]$$

$$(\Phi \circ \psi)(A) = V_A$$

$$V_{[A,B]} = [V_A, V_B]$$

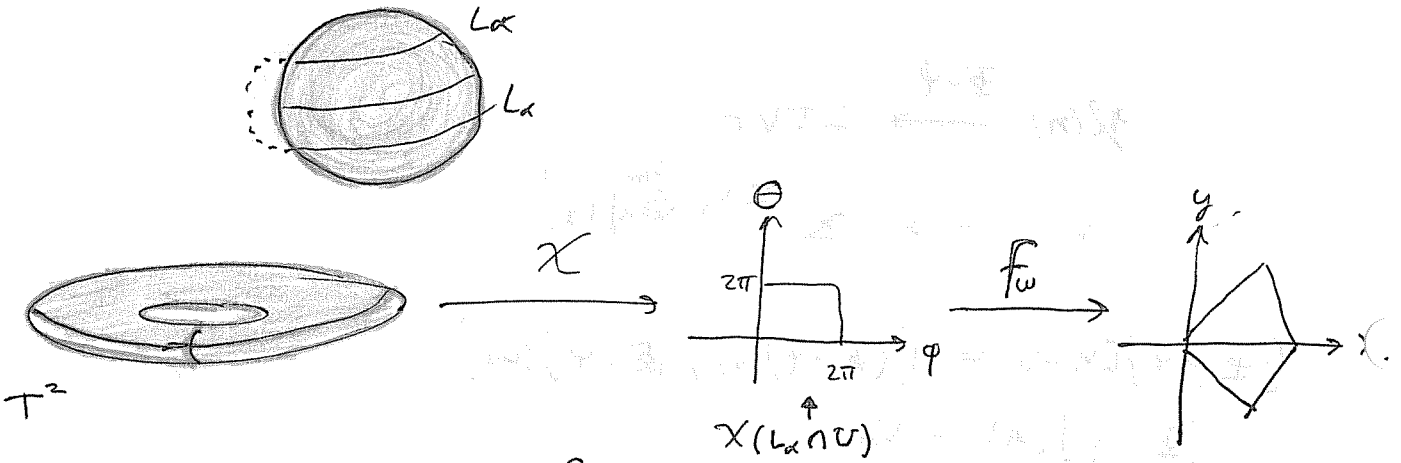
Foliations Continued

M – manifold

$\mathcal{L} = \{L_\alpha\}_{\alpha \in A}$ foliation of M

$M \in \bigcup_{\alpha \in A} L_\alpha$ with $L_\alpha \cap L_\beta = \phi$ if $\alpha \neq \beta$

$\forall u_0 \in M$ $\exists$ an open set U and a chart $(x, y) : U \longrightarrow \mathbb{R}^p \times \mathbb{R}^q$ such that the components of $U \cap L_\alpha$ are given by $y^i = c^i$.



$$(e^{i\varphi}, e^{i\theta}) \xrightarrow{\chi} (\varphi, \theta) \xrightarrow{f_\omega} (\varphi, \theta - \omega\varphi)$$

$$(\varphi, \theta) \in (0, 2\pi) \times (0, 2\pi)$$

$$U = \{ (e^{i\varphi}, e^{i\theta}) \mid (\varphi, \theta) \in (0, 2\pi) \times (0, 2\pi) \}$$

$$f_\omega(\chi(P)) = (\chi(P), y(P))$$

Module is a vector space structure except that we replace the Field with a ring

We proved that a vector field X on M is tangent to the leaves of $\mathcal{F}$ iff $\forall u$ and $\forall$ adapted chart on (x, y)

$$X_{u_0} = \sum a^i \left(\frac{\partial}{\partial x^i}\Big|_{u_0}\right)$$

Where $X_{u_0} \in T_{u_0}M$. X_{u_0} is tangent to L_α iff

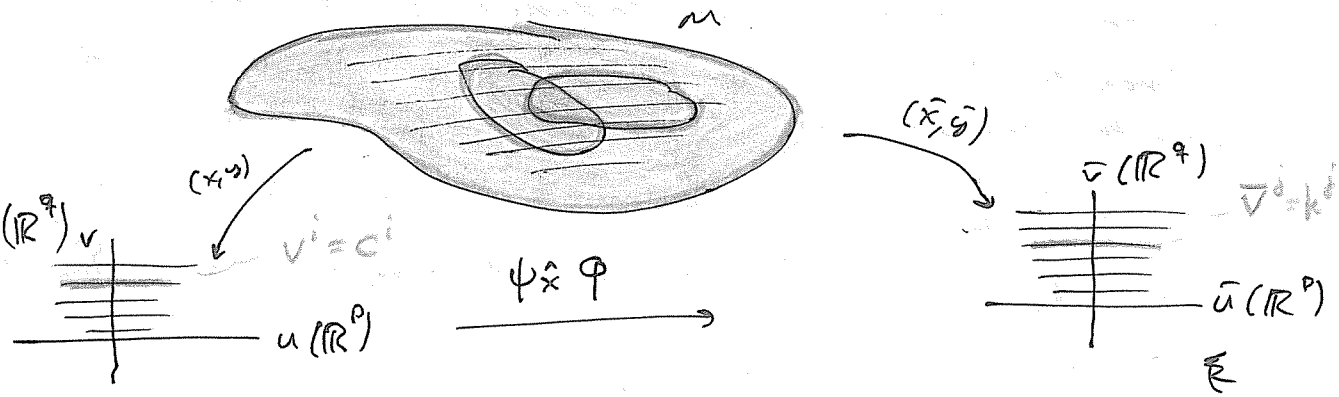
$$X_{u_0} = \sum a^i \frac{\partial}{\partial x^i}\Big|_{u_0}.$$

If (x, y) and $(\bar{x}, \bar{y})$ are both charts on M adapted to $\mathcal{F}$ and if $U \cap V \neq \phi$,

$$(x, y) : U \longrightarrow \mathbb{R}^p \times \mathbb{R}^q$$
$$(\bar{x}, \bar{y}) : V \longrightarrow \mathbb{R}^p \times \mathbb{R}^q$$

Then the coordinate change map $\psi \hat{x} \varphi$ has the property

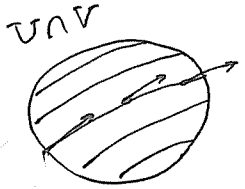
$$(\bar{x}(P), \bar{y}(P)) = \left(\psi\left(x(P), y(P)\right), \varphi(y(P))\right) \leftarrow \text{will show}$$



$$(\psi \hat{x} \varphi)(u, v) = (\bar{u}, \bar{v})$$

$$(\psi(u,v), \varphi(u,v)) = (\bar{u}, \bar{v}) \qquad \text{will show } \frac{\partial \varphi^i}{\partial u^i} = 0$$

Now then $\left(\frac{\partial}{\partial x^i}\right)$ and $\left(\frac{\partial}{\partial \bar{x}^i}\right)$ are both tangents to the leaves. Since



$\frac{\partial}{\partial x^i}\Big|_{\bar{p}}$ is tangent to leaf through $\bar{p}$ and since $\frac{\partial}{\partial \bar{x}^i}\Big|_{\bar{p}}$ are a basis of vectors (LI and spanning) tangent to the leaf through $\bar{p}$, we have that we can expand

$$\frac{\partial}{\partial x^i}\Big|_{\bar{p}} = \sum_{j=1}^{p} c_{\bar{p}}^{j} \frac{\partial}{\partial \bar{x}^j}\Big|_{\bar{p}} \quad \text{thus} \quad \bar{y}^r(q) = \varphi^r(x(q), y(q)) \quad \text{then}$$

($\bar{p}$ any point.)

$$\frac{\partial \bar{y}^r}{\partial x^i}(q) = \sum_{j=1} c_i^j(q) \frac{\partial \bar{y}^r}{\partial \bar{x}^j}(q) = \sum_{j=1} c_i^j(q) \underbrace{d_q \bar{y}^r\left(\frac{\partial}{\partial x^i}(q)\right)}_{\text{zero for numerous regions}} = 0$$

$$\frac{\partial(\bar{y}^r \circ (x, y)^{-1})}{\partial u^i} = 0$$

$$\frac{\partial(\bar{y}^{\wedge} \circ (x,y)^{-1})}{\partial u^i} = 0$$

$$(\bar{x}(q), \bar{y}(q)) = (\psi \hat{\times} \varphi)(x(q), y(q))$$

$$(\bar{x}, \bar{y}) = (\psi \hat{\times} \varphi) \circ (x, y)$$

$$(\bar{x}, \bar{y}) \circ (x, y)^{-1} = \psi \hat{\times} \hat{\varphi}$$

$$\bar{y}^{\wedge} \circ (x,y)^{-1} = \varphi^{\wedge}$$

$$\frac{\partial \varphi^{\wedge}}{\partial u^i} = 0 \quad \therefore \quad \bar{y} = \varphi \circ y \quad \therefore \quad \varphi = \bar{y} \circ y^{-1}$$

It says then the coordinate Δ map $(\bar{x}(p), \bar{y}(p)) = (\psi(x(p), y(p)), \varphi(y(p)))$.

Second Definition for Foliation

$y : U \longrightarrow \mathbb{R}^q$ & where $dy = dy^1, \ldots, dy^q$

A smooth foliation of codimension q (the leaves have dimension $p = m - q$) is a maximal family of smooth submersions $f_\alpha : U_\alpha \longrightarrow \mathbb{R}^q$ where

$\{U_\alpha\}_{\alpha \in A}$ is an open cover of M and for $u_0 \in U_\alpha \cap U_\beta$ $\exists$ a local diffeomorphism of $\varphi_{\beta\alpha}^{u_0}$ of $\mathbb{R}^q$ such that

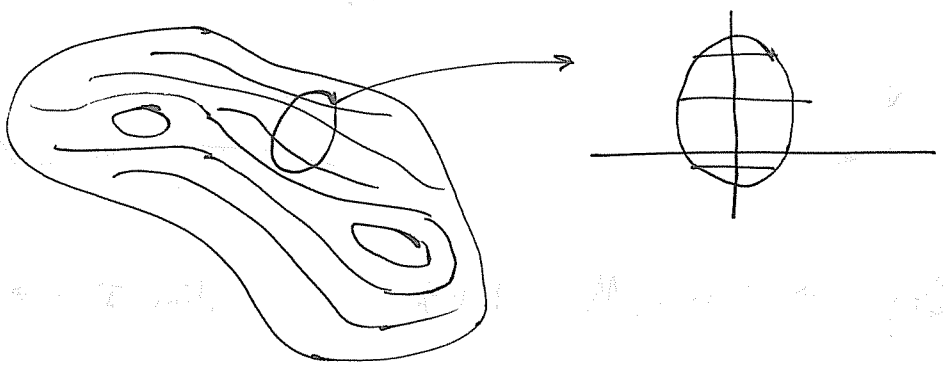
$$f_\beta = \varphi_{\beta\alpha}^{u_0} \circ f_\alpha$$

$\bar{y} = \varphi \circ y$ shows the previous def^n $\Rightarrow$ this def^n

Let $\mathcal{F}$ be a foliation of a manifold M. For $x \in M$ let

$$D(x) \equiv \{ X_x \in T_xM \mid X_x \text{ is tangent to the leaf of } \mathcal{F} \text{ trough } x \}$$

If $(x,y) \longrightarrow \mathbb{R}^p \times \mathbb{R}^q$ is a chart of M compatible with $\mathcal{F}$ then $X_x \in D(x) \iff X_x = \sum_i a^i \left(\frac{\partial}{\partial x^i}\Big|_x \right)$



$$y^i = c^i$$

$$dy^i = 0$$

$$v = \sum_{i=1}^{p} a^i \frac{\partial}{\partial x^i} + \sum_{j=1}^{q} b^j \frac{\partial}{\partial y^j}$$

$$0 = dy^k \left(\frac{\partial}{\partial x^i} \right) = 0$$ x^i and y^k component functs of the same chart

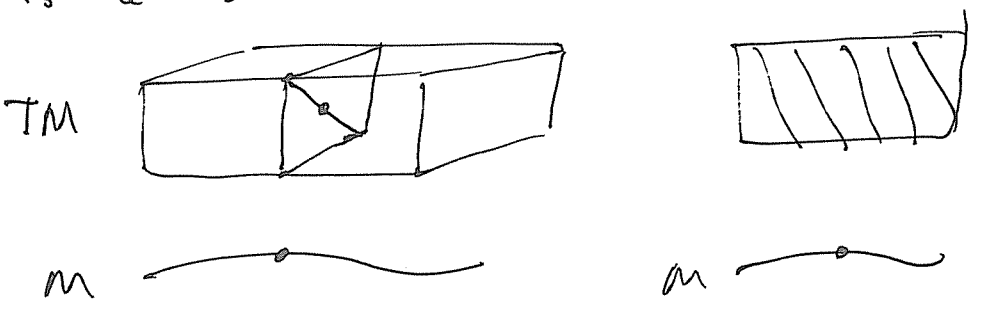
$$0 = dy^k(v) = b^k$$

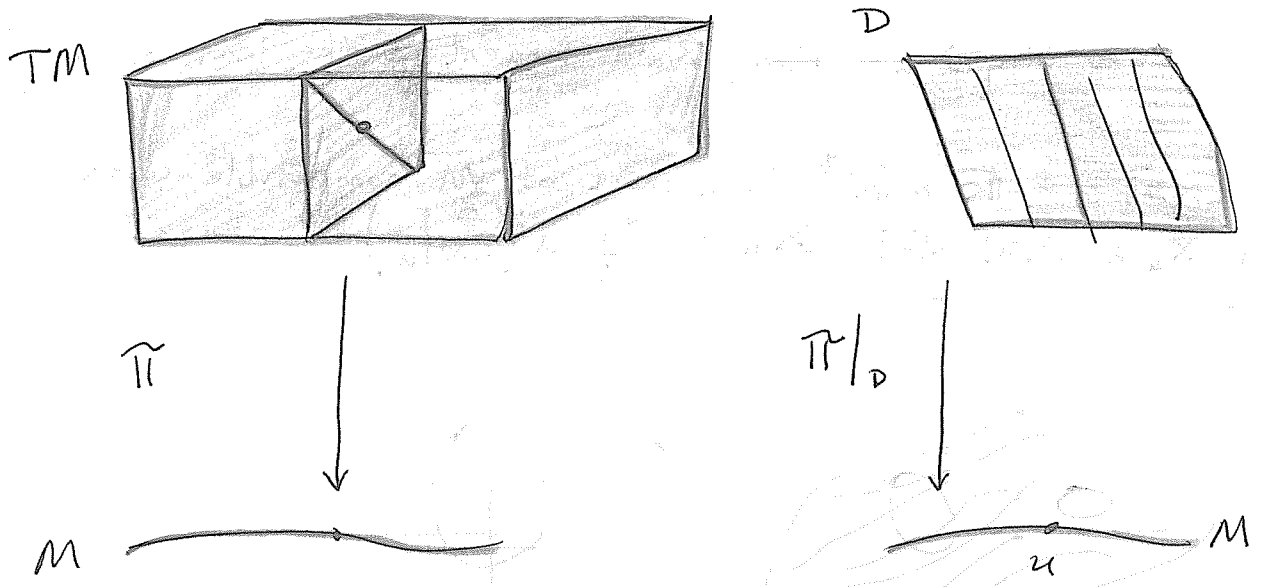
Therefore: $\left\{ \frac{\partial}{\partial x^i}\Big|_x \right\}$ is a basis of $D(x)$ $\forall x \in M$.

Can we address this geometrically?

$$D \equiv \bigcup_{x \in M} D(x) \subseteq \bigcup_{x \in M} T_xM = TM$$

D is a sub bundle





Notice $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right\} \longrightarrow \dim T_u M = p+q$ $\qquad \dim D \longrightarrow p$

Def^n/ A p-dimensional <u>distribution</u> D on M is a function on M such that $D(u)$ is a p-dimesional subspace of $T_u M$ $\forall u \in M$

Def^n/ A <u>p-dimensional distribution</u> D of M is called a subundle iff it is smooth in the sense that $\forall u_0 \in M$ $\exists$ an open set U about u_0 and smooth vector fields $X_1, X_2, \ldots, X_p$ on U such that

$$\{X_i(u) \mid u \in U\}$$

is a basis of $D(u)$ $\forall u \in U$, $X_i(u) \in D(u)$

Example

Let $M \equiv \mathbb{R}^3 - \{(0,0,0)\}$. Let $\mathcal{F}$ be the foliation of $\mathbb{R}^3$ whose leaves are spheres about the origin centered at $(0,0,0)$. Let D be the distribution tangent to the leaves of $\mathcal{F}$. Define vector fields

$$X \equiv x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} = y\partial_z - z\partial_y$$

$$Y \equiv x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} = x\partial_z - z\partial_x$$

$$Z \equiv x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = x\partial_y - y\partial_x$$

at each point $u \in M$, $X(u)$, $Y(u)$, $Z(u)$ belong to $D(u)$ Note

$$X(u) = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} = \left(-\frac{z}{x}\right)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + \left(\frac{y}{x}\right)\left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right)$$

Remember x, y, z all smooth functions of $C^\infty M$

$$X = \left(-\frac{z}{x}\right)Z + \left(\frac{y}{x}\right)Y \qquad \left(\text{Form a basis where } x \neq 0\right)$$

We have shown that the tangent distribution D is a sub bundle, at every point 2 of the three will generate a subspace, D is a sub bundle. Additionally we can calculate that

$$[X, Y] = Z \;,\; [Y, Z] = X \;,\; [Z, X] = Y$$

Def^{n}/ A subbundle E of the tangent b. TM is <u>integrable</u> iff for every pair of vector fields X, Y locally defined on TM such that $X(u), Y(u) \in E(u)$ $\forall u$ it follows that $[X, Y](u) \in E(u)$ $\forall u$

Thinking in Lie derivative notation

$$\begin{aligned}[fX, gY] &= L_{fX}(gY) \\ &= f\,X(g)\,Y + g\,L_{fX}Y \\ &= f\,X(g)\,Y - g\,L_Y(fX) \\ &= f\,X(g)\,Y - g\,Y(f)\,X + gf\,[X, Y]\end{aligned}$$

Returning to our construction with $D(u)$

$$[X_i, X_j](u) = \sum_k f_{ij}^k(u)\, X_k(u)$$

f_{ij}^k ↑ generalization of structure constants in Lie Algebra

Frobenius Theorem

If $\mathcal{F}$ is a foliation then its tangent field D is an integrable subbundle of TM. Conversely if D is an integrable subbundle of TM then $\exists$ a foliation $\mathcal{F}$ whose tangent subumdle is D

Def^{n}/ tangent field to $\mathcal{F}$ $\equiv$ tangent subbundle of $\mathcal{F}$ $\equiv$
$\equiv \{X_u \mid X_u$ is tangent to the leaf of $\mathcal{F}$ through $u\}$

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FACT EASILY PROVEN FROM FROBENIUS

If X is a vector field on a manifold M and $X(u) \neq 0$ $\forall u \in M$ then X defines a 1-dimensional foliation on M

$$D(u) = \{ \lambda X(u) \mid \lambda \in \mathbb{R} \} \subseteq T_u M$$

$$[X, X] = 0$$

THEOREM

If X is a vector field on a Lie Group G and X is either left or right invariant then it is complete.

Def^n/ X is complete means $\forall p \in M$ the sol^n of the eq^n $\gamma(0) = p$ exists for all time.

$$\frac{d}{dt}(x^i \circ \gamma)(t) = \frac{d(x^i \circ \gamma)}{dt}(t) = X^i_x \circ x^{-1}\left(x^1(\gamma(t)), x^2(\gamma(t)), \ldots, x^m(\gamma(t)) \right)$$

$$x^i(\gamma(0)) = x^i(p)$$

$$(-a, a)$$

Proof in Warner's Foundations of integrable manifolds

Corollary: If X is left or right invariant vector field then each integral curve γ of X is a Lie Group homomorphism from $(\mathbb{R}, +)$ to G.

$$\mu_s(t) = \gamma(t+s)$$

$$\lambda_s(t) = \gamma(t)\gamma(s) = R_{\gamma(s)}(\gamma(t))$$

$$(\mu_s)'(t) = \gamma'(t+s) = X(\gamma(t+s)) = X(\mu_s(t))$$

$$\lambda_s'(t) = d R_{\gamma(s)}(\gamma'(t))$$

$$= d_{\gamma(t)} R_{\gamma(s)}(X(\gamma(t)))$$

$$= X(R_{\gamma(s)}(\gamma(t)))$$

$$= X(\lambda_s(t))$$

MARSDEN'S FOUNDATIONS OF MECHANICS, ABRAHAM AND MARSDEN

$\therefore$ $\boxed{\gamma(t+s) = \gamma(t)\gamma(s)}$

Def^n/ Define $\exp : T_e G \longrightarrow G$ as follows : for $V \in T_e G$ let

$\exp(V) = \gamma_V(1)$ where γ_V is the 1-parameter group of G such that $\gamma_V'(t) = \overline{X}^V(\gamma_V(t))$, $\gamma_V(0) = e$

It can be proven that at the identity of G the differential of exp is the identity map then the implicit fnct. theorem

$\Rightarrow \exists$ U open in $T_e G$ about 0 open in G about e such that $\exp / U : U \longrightarrow V$ is a diffeomorphism.

$$[\exp | U]^{-1} : V \to U \subseteq T_e G \cong \mathbb{R}^{\dim(G)}$$

Def^n/ $\forall$ right invariant vector field η define $\exp(\eta) = \exp(\eta(e))$

$$\exp(t\eta) = \exp(t\eta(e))$$

$$\gamma_V(t) = \exp(t\eta)$$

THEOREM

If G is a Lie group acting on a manifold M and $x \in M$ then the orbit $G \cdot x$ of G through x is an immersed submanifold of M.

Proof

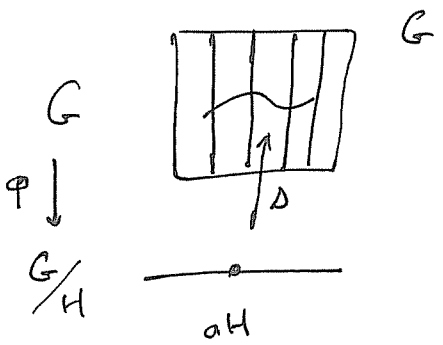
Define a map $G_x : G \longrightarrow M$ by $\sigma_x(g) \equiv g \cdot x \equiv \sigma(g, x)$

$\text{Image}(\sigma_x) = G \cdot x$ the orbit through x.

$\sigma_x = \sigma|_{G \times \{x\}}$ is smooth

It is known $\exists$ a unique manifold structure on G/G_x such that mapping $G \longrightarrow G/G_x$ has smooth local ~~actions~~ sections.

$H \subseteq G$



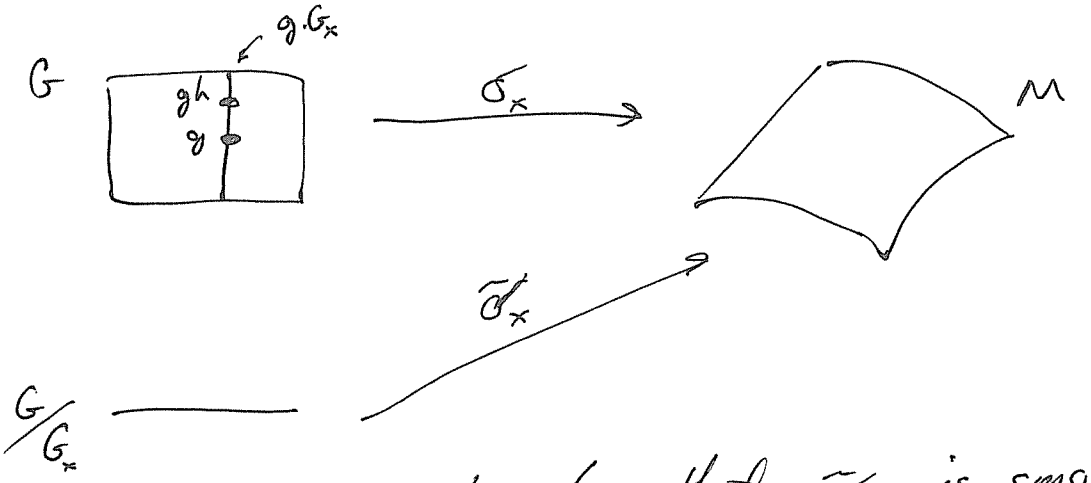
$$\varphi \circ D = id_U$$

It can be shown that the induced mapping

$$\tilde{\sigma}_x : G/G_x \longrightarrow M \quad \text{def}^{\underline{n}} \text{ by } \tilde{\sigma}_x(g G_x) = g \cdot x$$

$$gh \cdot x = g(h, x) = g \cdot x$$

$$\sigma_x(gh) = \sigma_x(g) \quad \forall h \in G_x$$



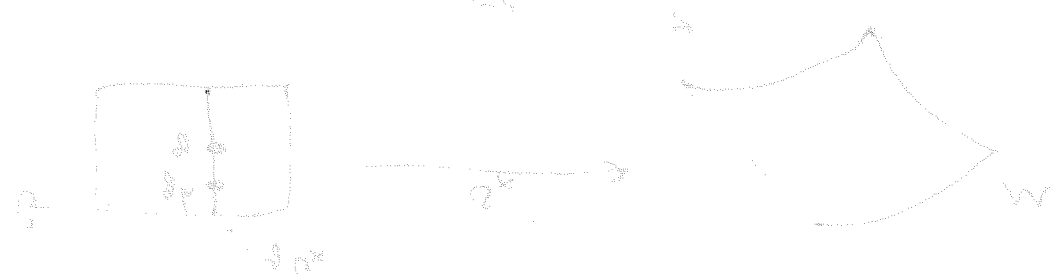
~~is smooth~~ can be shown that $\tilde{\sigma}_x$ is smooth.

Notice $g_1 \cdot x = g_2 \cdot x \Rightarrow g_2^{-1} g_1 x = x \Rightarrow g_2^{-1} g \in G_x$

$\Rightarrow g_1 \in g_2 G_x$

$\Rightarrow g_1 G_x = g_2 G_x$

thus $\tilde{\sigma}_x$ is one to one. For smoothness of $\tilde{\sigma}_x$ see warner proof of Theorem 3.62 page 123. He actually shows that if you define $\tilde{\sigma}_x : G/G_x \longrightarrow M$ . .



$$\psi(g, x) = g \cdot x$$

$$(g_1, g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$$

$$e \cdot x = x$$

"Action" additionally ψ is smooth.

Examples of Actions

$$\mathfrak{gl}(n) \times (\mathbb{R}^n)^* \longrightarrow (\mathbb{R}^n)^*$$

$$(g \cdot \alpha)(x) = \alpha(g^{-1} \cdot x)$$

or more generally,

$$\mathfrak{gl}(n) \times T^0_2 \mathbb{R}^n \longrightarrow T^0_2 \mathbb{R}^n \quad , \quad \beta \in T^0_2 \mathbb{R}^n, \ \beta : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$(g \cdot \beta)(x, y) = \beta(g^{-1}x, g^{-1}y)$$

Assume G acts on a manifold M, $\psi : G \times M \longrightarrow M$ an action and that ξ is a right-invariant vector field on G. Define a vector field ξ_M on M by:

$$\xi_M(x) = \frac{d}{dt}\left[\exp(t\xi) \cdot x\right]\Big|_{t=0}$$

In general exp is a particular sol^n to a DE and possible just the plain old matrix exp in one case. This ξ_M is the <u>linearization</u> of the action ψ which in general may be non-linear. Now is this ξ_M truly linear? Notice that if

$$\sigma_g(x) = \psi(g, x) \qquad \forall g \quad \forall x$$

$$\sigma_g : M \longrightarrow M$$

$$g \longrightarrow \sigma_g$$

$$G \longrightarrow \mathrm{Diff}\, M$$

Now ψ is smooth and σ_g is a restriction of ψ and thus smooth

Additionally, $\sigma_{g^{-1}} \circ \sigma_g = \sigma_{g^{-1}g} = \sigma_e = id_M$ thus σ_g a diffeomorphism.

Define linearization of $\psi : G \times M \to M$. Let ξ be a right-invariant vector field on G. Define vector field ξ_m on M by:

$$\xi_m(x) \equiv \frac{d}{dt}\left[\exp(t\xi)\cdot x\right]\Big|_{t=0}$$

Notice that if $\sigma_x(g) = \psi(g,x) \quad \forall g, \forall x$ then $\sigma_x : G \to M$ is a diffeomorphism.

$$\boxed{\xi_m(x) = \frac{d}{dt}\left[\sigma_x(\exp(t\xi))\right]\Big|_{t=0}}$$ geometrically "clear"

$$= d_e\sigma_x\left(\frac{d}{dt}\left(\exp(t\xi)\Big|_{t=0}\right)\right)$$

$$\boxed{\xi_m(x) = d_e\sigma_x(\xi)}$$ algebraically "clear"

$\xi, \eta \in RIVF(G)$ then we get ξ_m and η_m by construction above so then notice

(1) $$(\xi+\eta)_m(x) = d_e\sigma_x(\xi+\eta) =$$
$$= d_e\sigma_x(\xi) + d_e\sigma_x(\eta)$$
$$= \xi_m(x) + \eta_m(x)$$
$$= (\xi_m + \eta_m)(x)$$

Kobiatu and Mibestan → ξ^* fundamental vector field : the vector field induced by the Lie Algebra of the group

(2) Let $c \in \mathbb{R} \longrightarrow (c\xi)_m(x) = d_e\sigma_x(c\xi) = c\,d_e\sigma_x(\xi) = c\,\xi_m(x)$

Chain Rule revisited

$$\frac{d}{dt} f(\gamma(t)) = df(\gamma'(t)) \quad \leftarrow \text{step between boxes on last page.}$$

$$d(f \circ g) = df \circ dg$$

Or in terms of a chart ϕ $\quad f : M \longrightarrow N$

$$\frac{d}{dt}(f \circ \gamma) = \frac{d}{dt}(f \circ x^{-1} \circ x \circ \gamma)$$

$$= \sum \frac{\partial (f \circ x^{-1})}{\partial u^i} \frac{d(x^i \circ \gamma)}{dt}$$

$A^T A = I$

$df(\gamma'(t)) =$ ↗

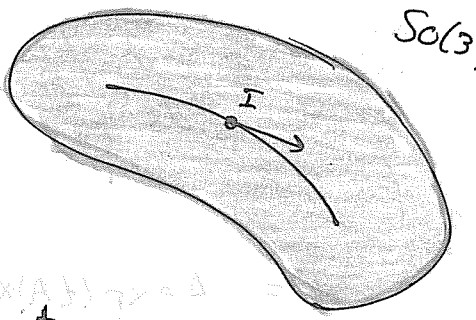
Example : $\quad SO(3) \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$\mathfrak{so}(3) = \{A \in \mathfrak{gl}(3) \mid A^t = -A\}$$

$$\Sigma^A(B) = d_I R_B(A)$$

$\xi \in \mathfrak{so}(3)$

$C^t C = I$ ☆



$$C^t(\lambda) C(\lambda) = I \quad , \lambda \in (-a, a)$$

$$C(0) = I$$

$$C(\lambda)^T \frac{dC}{d\lambda}(\lambda) + \left[\frac{d}{d\lambda} C(\lambda)^T\right] C(\lambda) = 0$$

$\lambda = 0$ $\quad \left(\frac{dC}{d\lambda}(0)\right) + \frac{dC}{d\lambda}(0)^T = 0$

$\uparrow$ A $\quad \therefore A + A^T = 0$

should run in reverse start with A

define $C(\lambda) = e^{\lambda A}$

rotate about 3 axis

$$C(\lambda) = \begin{pmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\frac{dC}{d\lambda} = \begin{pmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\cos\lambda & -\sin\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$T_I SO(3) = \mathfrak{so}(3) \ni \xi = A = \frac{dC}{d\lambda}(0) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$M = \mathbb{R}^3$

$$\xi_m(x) = \frac{d}{dt}\left[\exp(t\xi_e)\cdot x\right]\Big|_{t=0}$$

$$= \frac{d}{dt}\exp\begin{pmatrix} 0 & t & 0 \\ -t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}\Big|_{t=0} \qquad \text{now } \frac{d}{dt}\left(e^{tA}\right) = e^{tA}A$$

$$= A\exp(tA)x\Big|_{t=0}$$

$$= A \cdot x$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$= \begin{pmatrix} x^2 \\ -x^1 \\ 0 \end{pmatrix} \leftarrow \text{components of vector field}$$

remember $e_1 \leftrightarrow \frac{\partial}{\partial x^1}$, $e^2 \leftrightarrow \frac{\partial}{\partial x^2}$ therefore

$$\xi_m(x) = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2} + 0\frac{\partial}{\partial x^3} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

A is an infinitesimal rotation around some vector here the z-axis.

3/26/2001

1. Action on Manifold, $\psi : G \times M \to M$
2. $\xi \in RIVF(G)$
3. $\xi_m(x) = \frac{d}{dt}\left[\exp(t\xi)\cdot x\right]\Big|_{t=0}$ where $\exp(t\xi) \equiv \exp(t\xi_e)$
4. Algebra of ξ_m's easy in notation $\xi_m(x) = d_e\sigma_x(\xi)$
5. $\xi \longrightarrow \xi_m$ is linear operation
6. Now show $[\xi, \gamma] \longrightarrow [\xi_m, \gamma_m]$ bracket holds

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We show that $[\xi_m, \eta_m] = [\xi, \eta]_m$. To prove this I show that ξ is σ_x-related to ξ_m $\forall x$, $\forall \xi \in RIVF(G)$

$$d_g\sigma_x(\xi(g)) = d_g\sigma_x(d_eR_g(\xi_e))$$
$$= d_e(\sigma_x \circ R_g)(\xi_e)$$
$$= d_e\sigma_{gx}(\xi_e)$$
$$= \xi_m(g\cdot x)$$
$$= \xi_m(\sigma_x(g))$$

$\therefore \xi \overset{\sigma_x}{\sim} \xi_m$

also $\eta \overset{\sigma_x}{\sim} \eta_m$

$\therefore [\xi, \eta] \overset{\sigma_x}{\sim} [\xi_m, \eta_m]$

$\therefore d_e\sigma_x([\xi, \eta](e)) = [\xi_m, \eta_m]$

$[\xi, \eta]_m = [\xi_m, \eta_m]$

σ_x

$G \longrightarrow M$

$\xi \in RIVF(G) \longrightarrow \xi_M$

$\xi(g) \xrightarrow{d\sigma_x} \xi_m(\sigma_x(g))$?

$(\sigma_x \circ R_g)(a) = \sigma_x(ag)$
$= (ag)\cdot x$
$= a\cdot(g\cdot x)$
$= \sigma_{g\cdot x}(a)$

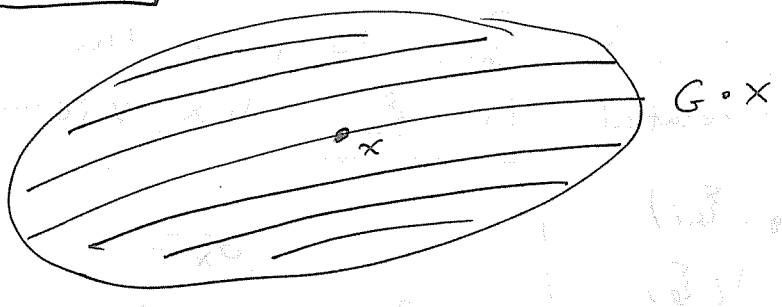
Corollary

Assume G acts *regularly* on a manifold M and that

$$D(x) = T_x(G \cdot x) \quad \forall x \in M$$

Then D is an integrable subbundle of M and $\therefore$ $\exists$ a foliation $\tilde{\mathcal{F}}$ of M whose leaves are orbits of G

Sketch of Pf



$$\xi_m(x) \in T_x(G \cdot x) \quad \forall x$$

$$\xi_m(x) \in D(x) \quad \forall x$$

$$\Rightarrow D(x) = \{ \xi_m(x) \mid \xi \in RIVF(G) \}$$

and $[\xi_m, \eta_m] = [\xi, \eta]_m$ $\therefore$ 'integrable subbundle

$\therefore$ by Frobenius $\exists$ a foliation of M whose leaves are orbits of G

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