Abstract

COOK, JAMES STEVEN. Foundations of Supermathematics with Applications to N=1 Supersymmetric Field Theory. (Under the direction of Ronald Owen Fulp.)

We use the term "supermathematics" to encompass all the various extensions of Alice Roger's original work on G^∞ supermanifolds. Background on how numbers, functions, linear algebra, matrix calculations, real analysis, complex analysis, manifold theory and Lie theory generalize to the context of supermathematics is provided. We use countably many Grassmann generators so this work is within the realm of infinite dimensional Banach space theory.

We find that Lie's Third Theorem holds for G^∞ super Lie groups. We also prove that the exponential mapping and other standard constructions in Lie theory apply equally well in the G^∞ setting. Portions of this work are similar to existing research, but our proofs are distinct and we have focused on the G^∞ category with infinitely many Grassmann generators. Other workers typically either use finitely many Grassmann generators or focus attention to the superanalytic category.

We provide a supersmooth principle fiber bundle framework for super gauge theory. Special sections are constructed and provide pure gauge solutions on zero curvature submanifolds. Quotient spaces and bundles are used to implement certain physical constraints. We apply these general geometric constructions to recover the superfield transformation laws of $N = 1$ super Yang-Mills theory.

We develop a gauged Wess-Zumino model in noncommutative Minkowski superspace. This is a natural extension of the work of Carlson and Nazaryan, who extended $N = 1/2$ supersymmetry over deformed Euclidean superspace to Minkowski superspace. Noncommutativity is implemented by replacing products with star products. As in the $N = 1/2$ theory, a reparameterization of the gauge parameter, vector superfield and chiral superfield are necessary to write standard C-independent gauge theory. However, our choice of parametrization differs from that used in the $N = 1/2$ supersymmetry, which leads to some unexpected new terms.

Foundations of Supermathematics with Applications to N=1 Supersymmetric Field Theory

by
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Biography

The author began his post secondary education at Mayland Community College in Spruce Pine North Carolina. He completed a A.A.S. in electronics engineering technology in May of 2000. In 1998 the author transferred to North Carolina State University in Raleigh. He completed a double major in mathematics and physics in May of 2001. In August of 2001 the author began graduate studies in physics at the State University of New York at Stony Brook. In May of 2003 he completed a M.A. in physics. During his time at Stony Brook the author was introduced to the subject of supersymmetry. Toward the end of his final semester he gave a Friday talk on the basics of N=1 supersymmetry. This talk motivated the author to study supermath in the years to follow.

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Chapter 1

Introduction

In the first section we give an account of some of the physical motivations and history of supersymmetry. We explain how supersymmetry extends current physical law and predicts the existence of particles which have yet to be observed. Then in the second section we give a survey of some of the mathematical works which share similar methods or goals with this dissertation. Finally, we conclude this chapter with a summary of the dissertation.

1.1 Physical Background

Grassmann variables have wide application throughout modern field theory. For example, they are used in path integrals involving fermionic fields and the BRST cohomology. The mathematics explored throughout this thesis is generally aimed towards gaining a concrete understanding of what precisely is a Grassmann or supervariable. Our interest in this section is quite narrow. We just want to discuss what superspace is and how it encodes N=1 supersymmetry. This is interesting because N=1 supersymmetry forms the basis of what is known as the Minimal Supersymmetric Standard Model (MSSM). This model has predictions which differ from the current Standard Model (SM) of particle physics. It is possible that the Large Hadron Collider (LHC) at CERN will detect supersymmetry as early as 2010. Of course, if it is not detected

Table 1.1: Predictions of Supersymmetry

SM Particle	Spin	SUSY	superpartner in MSSM	Spin
electron	1/2	$\leftrightarrows$	selectron	0
photon	1	$\leftrightarrows$	photino	1/2
quark	1/2	$\leftrightarrows$	squark	0
gluon	1	$\leftrightarrows$	gluino	1/2
Higgs	0	$\leftrightarrows$	Higgino	1/2

Table 1.2: Component Field Content of Superfield

scalar fields	f, m, n, d	spin 0	commuting fields
Weyl spinors	$\phi, \bar{\mathcal{X}}, \bar{\lambda}, \psi$	spin 1/2	anticommuting fields
vector field	v_n	spin 1	commuting field

the theorists can always push off its discovery a few more TeV's (or in experimental terms a few decades). The details of how supersymmetry makes contact with our everyday existence are rather involved. For example, see [84] for some of the phenomenological implications of supersymmetry.

A function U of $N = 1$ rigid superspace is called super field and it has the form,

$$\boxed{U = f + \theta\phi + \bar{\theta}\bar{\mathcal{X}} + \theta\theta m + \bar{\theta}\bar{\theta}n + \theta\sigma^n\bar{\theta}v_n + \theta\theta\bar{\theta}\bar{\Lambda} + \bar{\theta}\bar{\theta}\theta\psi + \theta\theta\bar{\theta}\bar{\theta}d.}$$

Each of the component fields $f, \phi, \bar{\mathcal{X}}, m, n, v_n, \bar{\lambda}, \psi, d$ is an ordinary relativistic quantum field and the θ's are anticommuting variables. However, there are several inequivalent representations of the Poincare group that appear here. Scalar fields f, m, n, d (spin zero), Weyl spinor fields $\phi, \psi, \bar{\lambda}, \bar{\mathcal{X}}$ (spin 1/2), and the vector field v^m (spin one). Contained in this single superfield we have all the necessary fields to construct known particle physics. Assembling them in this one superfield assumes an additional symmetry of physics which is called supersymmetry. Supersymmetry requires that there be a balance between the number of bosons and the number of fermions in a theory. A representation of supersymmetry then necessarily has that property. As we indicated above there are 8 bosonic degrees of freedom (4 scalars plus one 4-vector), and there are 8 fermionic degrees of freedom (4 Weyl spinors). Until we place further constraints on the system, these are all complex degrees of freedom.

1.1.1 Poincare Algebra

The Poincare algebra is a Lie algebra that is formed by the four generators of spacetime translations (P_m) and the six generators of the Lorentz transformations ($J_{mn} = -J_{nm}$). For now we can view the Poincare algebra as an abstract Lie algebra over $\mathbb{C}$ defined by the following relations, note η_{ij} is the Minkowski metric tensor with $diag(\eta) = \{-1, 1, 1, 1\}$

$$\begin{aligned} [P_m, P_n] &= 0 \\ [P_m, J_{nk}] &= i(\eta_{mn}P_k - \eta_{mk}P_n) \\ [J_{mn}, J_{lk}] &= i(\eta_{nl}J_{mk} - \eta_{ml}J_{nk} + \eta_{mk}J_{nl} - \eta_{nk}J_{ml}). \end{aligned} \tag{1.1}$$

The indices $l, k, m, n = 0, 1, 2, 3$. Lorentz transformations include ordinary rotations in three dimensions as well as boosts. Boosts are transformations to moving frames of reference; they can be viewed as hyperbolic rotations of time and space. In particular,

$$\begin{array}{lll} J_{ij} & = \epsilon_{ijk} J_k & i, j, k = 1, 2, 3 \quad \textit{generate rotations} \\ J_{i0} & = -K_i & i = 1, 2, 3 \qquad\qquad \textit{generate boosts.} \end{array} \tag{1.2}$$

To be careful, we should emphasize that the operators above are not the transformations. Instead they are the generators of the transformations. Mathematically, they form the Lie algebra corresponding to the Lie group of transformations. Later on, we'll expand on the relation of the Lie algebra to the Lie group as it relates to the Poincare algebra and group.

For now we would like to point out that the Poincare algebra has several interesting subalgebras,

$$\begin{array}{lll} [J_i, J_j] & = \epsilon_{ijk} J_k & su(2, \mathbb{C}) \\ [P_i, P_j] & = 0 & \textit{Abelian subalgebra} \end{array} \tag{1.3}$$

The existence of the $su(2, \mathbb{C})$ subalgebra was particularly striking in the 1950's and 1960's when much of the theoretical physics communities efforts were placed in understanding the role isospin played in fundamental interactions. Since isospin also has a $su(2, \mathbb{C})$ algebra structure, it was (and is) tempting to try to identify the $su(2, \mathbb{C})$ of isospin with the $su(2, \mathbb{C})$ of the Poincare algebra. To be less naive, one might ask if there is a way to extend the Poincare algebra so that the enlarged version has subalgebras from which isospin could be derived. This would be very beautiful in the sense that it would have placed fundamental nuclear interactions on the same foundation as momentum or energy (which are associated to P_m). However, this ambitious dream to enlarge the Poincare algebra was shot down by the famous paper by Coleman and Mandula (Physical Review 159,1251 (1967)). They proved a very important no-go theorem which stated that it was not possible to enlarge the Poincare algebra without violating important symmetries of the S-matrix. The dream of understanding isospin and other "external" symmetries in a more intrinsic geometric manner lives on; this theorem merely shows that it cannot be accomplished in a strictly conventional way. The standard formalism of relativistic quantum field theory will not admit it. To give isospin a geometric (in the sense of real spatial origins) meaning will require a change in fundamental formalism like strings, twistors or perhaps noncommutative geometry.

Interestingly, the no-go theorem of Coleman and Mandula sparked a very different line of inquiry than one might have expected. Hagg, Lopuszanski and Sohnius (Nuclear Physics B 88 257 (1975)) noticed that the no-go theorem's proof assumed that the additional operators to the Poincare algebra should obey commutator brackets. Why should that be ? Why can't there be physical symmetries which are generated by anticommuting generators? Hagg, Lopuszanski and Sohnius argued that the no-go theorem was too narrow in its assumptions, that in fact it was possible to extend the Poincare algebra by adding generators which anticommute. They argued that for

physical reasons (absence of higher spin states for example) that the anticommuting generators must obey the following algebraic structure,

$$\begin{array}{ll} \{Q^A_\alpha, Q^B_\beta\} & = Z^{AB} \\ \{\bar{Q}^A_\alpha, \bar{Q}^B_\beta\} & = \bar{Z}^{AB} \\ \{Q^A_\alpha, \bar{Q}^B_\beta\} & = 2\sigma^m_{\alpha\dot{\beta}} P_m \delta^{AB}. \end{array} \tag{1.4}$$

Where the anticommutator is defined by $\{X, Y\} = XY + YX$, $\sigma^m_{\alpha\dot{\beta}}$ are the Pauli matrices for $m = 1, 2, 3$, and $A, B = 1, 2, 3, \ldots N$. Indices like α, β, γ are called "undotted indices" while indices like $\dot{\alpha}, \dot{\beta}, \dot{\gamma}$ are called "dotted indices", both types take values 1 or 2 hopefully without danger of confusion. The central charges Z^{AB} commute with everything and are antisymmetric in A and B. These relations plus the Poincare algebra form the N=1,2,3 or 4 super Poincare algebra. These are the cases of primary interest in the physical literature.

The case of interest to us is $N = 1$ for which there are no central charges and the indices A,B=1 so we omit them. We will call the generators $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ the supercharges. In total the super Poincare algebra is defined by the relations,

$$\begin{array}{l} [P_m, P_n] = 0 \\ [P_m, J_{nk}] = i(\eta_{mn} P_k - \eta_{mk} P_n) \\ [J_{mn}, J_{lk}] = i(\eta_{nl} J_{mk} - \eta_{ml} J_{nk} + \eta_{mk} J_{nl} - \eta_{nk} J_{ml}) \\ [Q_\alpha, P_m] = 0 \\ [\bar{Q}_{\dot{\alpha}}, P_m] = 0 \\ [J_{mn}, Q_\alpha] = -i(\sigma_{mn})_\alpha{}^\beta Q_\beta \\ [J_{mn}, \bar{Q}_{\dot{\alpha}}] = -i(\bar{\sigma}_{mn})_{\dot{\alpha}}{}^{\dot{\beta}} \bar{Q}_{\dot{\beta}} \\ \{Q_\alpha, Q_\beta\} = 0 \\ \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0 \\ \{Q_\alpha, \bar{Q}_\beta\} = 2\sigma^m_{\alpha\dot{\beta}} P_m \end{array} \tag{1.5}$$

The matrices σ_{mn} and $\bar{\sigma}_{mn}$ are formed from antisymmetrized products of the Pauli matrices, the details need not concern us here (see Wess and Bagger [116] for many useful formulas on such objects, generally we follow their conventions)

1.2 Survey of Supermathematics

In this section we discuss briefly a number of works on the topic of supermathematics. This survey is woefully incomplete since supermath is ubiquitous in mathematics connected to superstring theory. We focus on those works which are closer to the viewpoint and goals of this dissertation. We were not aware of some of these works until after the completion of our original work on the subject.

Mathematicians and physicists have been developing the theory of supermanifolds for over a quarter of a century. From almost the beginning, there have been at least two distinct approaches to the foundations of the superanalysis underlying the theory. Chronologically, the first of these is based on techniques reminiscent of ideas from algebraic geometry. We think of this approach as the sheaf theoretic development of supermathematics even when the theory of sheaves may not explicitly appear in some specific treatments of the subject. Certainly, Berezin, Leites, and Kostant [13], [76] were forerunners of this method and for that matter of the entire theory.

A second approach to the formulation of superanalysis and supermanifolds was initiated separately and differently by Rogers [98], Jadczyk and Pilch [68], and DeWitt [39]. Their work is more closely related to traditional ideas in manifold theory. Much work has been done describing both the sheaf theoretic and manifold theoretic descriptions of supermanifolds and how they are related, but we mention only a few whose work has directly impacted our work here, namely Rogers' [98], [99], [100], Batchelor's [11], and Bruzzo's [23]. The paper by Boyer and Gitler also deals with Rogers' G^∞ supermanifolds [18].

The body of a supermanifold is the part of the space which has no soul; it is an ordinary manifold. The paper [31] Catenacci, Reina and Teofilatto shows that the body of a supermanifold is well-defined only if certain topological restrictions are satisfied for a Rogers' supermanifold. We do not deal with this issue in this dissertation, but it is probably wise to keep these restrictions in mind if one was to write a more physically comprehensive mathematical model of supersymetric physics over supermanifolds. It would be interesting to try to merge the ideas in [53] with our work on super Yang-Mills theory in Chapter 8 of this dissertation.

Supernumbers are generated with sums and products of Grassmann generators. Our definitions assume an infinite number of Grassmann generators. However, much of the literature has been developed for finitely generated supernumbers. For example, Rabin and Crane worked with finitely generated supernumbers, and it is interesting to note the similarity to some of our work which was completed independently. Rabin and Crane's paper [95] on global topology of supermanifolds also suggested imposing constraints through a quotient construction. They also found interesting topologically nontrivial Rogers' manifolds in [96] where they contrasted the topology invented by DeWitt to that of Rogers'.

Also, Kostelecky, Nieto, and Truax studied the Baker-Campbell-Hausdorff relations for the supergroups in [77] and [78]. Bonora, Pasti and Tonin studied gauge theory on supermanifolds using finitely generated supernumbers [17].

The "generalized supermanifolds" of Hoyos, Quiros, Mittelbrunn, de Urries, allow

both the finite and infinitely generated supernumbers in their theory (see [56],[57],[58]). They also studied gauge theory and Fadeev-Popov fields from this viewpoint in [59].

Many supermanifolds can be viewed as a vector bundle over a ordinary manifold with odd fibers. In that view the topology of the odd fibers is fairly trivial. Rogers' definition allows for the odd directions of the supermanifold to have nontrivial topology. It is not certain that the exotic topology Rogers' allows in the fermionic directions is physically meaningful. For example, see [45]. Physical significance aside, there is a wealth of interesting mathematics to explore. For example, see the discussion of "body" "soul" and "aura" in [30].

We recommend E.A. Ivanov's overview of the work completed by Ogievetsky's students and collaborators in [65]. Also the text *Superspace, or One Thousand and one lessons in supersymmetry* by Gates, Grisaru, Rocek and Siegel has a great wealth of physics and mathematics. While the references mentioned here are certainly incomplete, we hope that one could get a fairly broad picture of geometric supermathematics if one pursued the references mentioned in this section.

1.3 Summary of Thesis

Chapters 2-5 are mostly background. Chapter 2 defines supernumbers and their properties. Chapter 3 discusses super linear algebra. Chapter 4 introduces super derivatives and supersmoothness, it is essentially the generalization of Ma 426 at NCSU for supermathematics. Chapter 5 tackles the question of conjugate and chiral variables in superspace. There are some new results mixed throughout, but we have not published those at this time. Probably, these things are known by experts but not all the details appear in the literature. In particular, Chapter 5 may form the basis for a later paper on complex chiral supermanifolds. We believe the idea to treat conjugate variables in superspace via the methods of Remmert is original.

Chapter 6 discusses supermanifolds and sub super manifolds. Then Chapter 7 discusses super Lie groups. Lie's Third Theorem is found for G^∞ super Lie groups. A number of standard theorems and constructions in Lie theory are shown to work in the G^∞ category. Chapters 6 and 7 are based largely on [37] which was a joint work of the author and R.O Fulp. Some proofs were modified slightly with the help of [68].

The author's main goal in studying supermath was to understand the geometry of Super Yang-Mills theory. Chapter 8 provides an explanation that is in fairly close analogy to the traditional principle fiber bundle formulation of Yang-Mills theory. This chapter was a joint work with R.O. Fulp, and we plan to publish once a little

more material is added.

Chapter 9 is not original work. The purpose of Chapter 9 is to show the reader how physicists describe N=1 superspace as a coset space.

Finally, Chapter 10 is closely based on [36]. This chapter is written at the level of rigor common in the physical literature. It is likely that earlier chapters together with algebraic geometry could be used to construct a more concrete description of the mathematics employed throughout Chapter 10, but we make no attempt to do that in this dissertation.

Chapter 2

Supernumbers

Supernumbers form the conceptual core of geometric supermathematics. Essentially, in supermathematics one simply replaces numbers with supernumbers. This stands in contrast to the sheaf theoretic approach where the generalization is made at the level of the function sheaf. It is likely that these approaches are categorically equivalent, but we prefer the geometric viewpoint since we would like to think more about point sets and less about mappings. Alice Rogers has a good discussion of the geometric verses the algebraic geometric approaches to supermathematics in her recent text *Supermanifolds, Theory and Applications* [102].

We first take care of some technical preliminaries. We define Grassmann generators and describe how supernumbers are constructed over an arbitrary field $\mathbb{K}$. Then we consider the general algebraic properties of Λ_∞^{alg}. In particular, commuting and anticommuting numbers are defined and the concept of parity is introduced. Typically we take either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, in either case we have that $\mathbb{K}$ is a complete, normed linear algebra; that is $\mathbb{K}$ is a Banach algebra. Following Alice Rogers, we introduce a norm on Λ_∞^{alg}, the set of all supernumbers for which the norm is finite is denoted $\Lambda(\mathbb{K})$ or simply Λ when there is no danger of ambiguity. We offer a proof that $\Lambda(\mathbb{K})$ is complete.

Next, the issue of superconjugation is addressed. We briefly compare the conjugation found in Bryce DeWitt's *Supermanifolds* [39] to that of Alice Rogers' text *Supermanifolds, Theory and Applications* [102]. We choose Dewitt's convention in order to make closer contact to the physics literature. Real, imaginary and complex supernumbers are defined and related. We should mention that we are also indebted to Buchbinder and Kuzenko's *Ideas and Methods of Supersymmetry and Supergravity* [29] which we found to be an invaluable resource in our exploration of this topic.

2.1 Multi-index Notation

Define the set of all increasing strings of N-indices of length k to be $\mathcal{I}_k(N)$. Introduce the multi-index I where $I \in \mathcal{I}_k(N)$ by $I = (i_1, i_2, \ldots i_k)$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq N$. For example if $N = 4$ then $I_2(4) = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$.

Next, define the set of all increasing strings of arbitrarily many ($N = \infty$) indices of length k to be $\mathcal{I}_k$. Introduce the multi-index I where $I \in \mathcal{I}_k \implies I = (i_1, i_2, \ldots i_k)$ with $1 \leq i_1 < i_2 < \cdots < i_k < \infty$. For convenience define $I \in \mathcal{I}_0$ to be the null-index which means we simply put the label "0" on that element. For example,

$$\begin{array}{l} z_I = z_{i_1,i_2,\ldots i_k} \quad I \in \mathcal{I}_k \quad k \geq 1 \\ z_I = z_0 \quad I \in \mathcal{I}_0 \end{array} \tag{2.1}$$

Clearly $\mathcal{I}_k$ is an infinite set for $k \geq 1$. In contrast to the finite example, notice that $\mathcal{I}_2 = \{(1,2),(1,3),(1,4),\ldots(2,3),(2,4),\ldots(3,4),(3,5),\ldots\}$. Finally, let the union of all $\mathcal{I}_k$ for $k = 0, 1, 2, \ldots$ be denoted by $\mathcal{I}(\infty)$.

2.2 Grassmann Generators

We define Grassmann generators ζ^i to be anticommuting indeterminates;

$$\zeta^i \zeta^j = -\zeta^j \zeta^i. \tag{2.2}$$

Take $i = j$ to see that the square of any Grassmann generator is zero. We also define the generators to be linearly independent over $\mathbb{K}$;

$$c_i \zeta^i = 0 \implies c_i = 0. \tag{2.3}$$

where the repeated index i is to be summed over. More often we are interested in the linear independence of products of the Grassmann generators, we also assume these to be linearly independent,

$$\sum_{k=0}^{\infty} c_{i_1 i_2 \ldots i_k} \zeta^{i_1} \zeta^{i_2} \ldots \zeta^{i_k} = 0 \implies c_{i_1 i_2 \ldots i_k} = 0. \tag{2.4}$$

Repeated indices are summed over all values of i_k. The coefficients $c_{i_1 i_2 \ldots i_k}$ are in $\mathbb{K}$ and are assumed to be completely antisymmetric. If we had a sum over a symmetric coefficient tensor then all data about that tensor would be lost in that such a tensor vanishes upon contraction with the completely antisymmetric product of the Grassmann generators. Thus it is reasonable to assume that the coefficients are antisymmetric from the beginning. We leave the proof of the existence of such an algebra

and generators as an exercise for the reader. For $I \in \mathcal{I}_k$, define

$$\zeta^I = \zeta^{(i_1,i_2,\ldots,i_k)} = \zeta^{i_1}\zeta^{i_2}\ldots\zeta^{i_k}. \tag{2.5}$$

In the case $I \in \mathcal{I}_0$ define $\zeta^I = 1$. We now compactly express the linear independence of the Grassmann generators,

$$\sum_{k=0}^{\infty}\sum_{I\in\mathcal{I}_k} c_I\zeta^I = 0 \implies c_I = 0 \quad \forall I \in \mathcal{I}(\infty). \tag{2.6}$$

2.3 Λ_N, the Grassmann Polynomials

First we consider the supernumbers that can be built using just the first N Grassmann generators, $\zeta^1, \zeta^2, \ldots, \zeta^N$. A supernumber is formed by taking a $\mathbb{K}$-linear combination of these generators and their products. The set of all supernumbers built from just N Grassmann generators is denoted Λ_N. If $z \in \Lambda_N$ then

$$z = \sum_{k=0}^{N} \frac{1}{k!} z_{i_1 i_2 \ldots i_k}\zeta^{i_1}\zeta^{i_2}\ldots\zeta^{i_k}. \tag{2.7}$$

For example,

$$\begin{aligned}
&\Lambda_0 = \mathbb{K}\\
&\Lambda_1 = \{z_0 + z_1\zeta^1\}\\
&\Lambda_2 = \{z_0 + z_1\zeta^1 + z_2\zeta^2 + z_{12}\zeta^1\zeta^2\}\\
&\Lambda_3 = \{z_0 + z_1\zeta^1 + z_2\zeta^2 + z_3\zeta^3 + z_{12}\zeta^1\zeta^2 + z_{13}\zeta^1\zeta^3 + z_{23}\zeta^2\zeta^3 + z_{123}\zeta^1\zeta^2\zeta^3\}.
\end{aligned} \tag{2.8}$$

The sums in the above were taken over increasing indices so no $\frac{1}{k!}$ factors appeared. Notice that the dimension of each of the above (as a vector space over $\mathbb{K}$) is simply 2^N, and the natural basis for Λ_N is simply monomials of the first N-Grassmann generators. Rogers denotes supernumbers generated by L Grassmann generators by B_L, also in her notation β plays the role of our ζ.

2.4 Λ_∞^{alg}, Formal Algebraic Supernumbers

Algebraic supernumbers are formal power series of arbitrarily many Grassmann generators. They are formal in the sense that we do not suppose any notion of convergence in the infinite sums below.

Definition 2.4.1. *Let z be a supernumber then*

$$z = \sum_{k=0}^{\infty} \frac{1}{k!} z_{i_1 i_2 \ldots i_k} \zeta^{i_1} \zeta^{i_2} \ldots \zeta^{i_k} \tag{2.9}$$

where $z_0 \in \mathbb{K}$ and $z_{i_1 i_2 \ldots i_k} \in \mathbb{K}$ are called the Grassmann coeffients of z. Repeated indices are summed over all values and these are infinite sums because we are allowing arbitrarily many Grassmann generators, $\zeta^1, \zeta^2, \ldots$. The set of all formal algebraic supernumbers is denoted Λ_∞^{alg}.

We assume that $z_{i_1 i_2 \ldots i_k} \in \mathbb{K}$ are completely antisymmetric in $i_1 i_2 \ldots i_k$. We will drop the "alg" in Λ_∞^{alg} a little later when we introduce the norm for supernumbers.

Definition 2.4.2. *We define the body z_B of the supernumber z by $z_B = z_0$, and the soul z_S by*

$$z_S = \sum_{k=1}^{\infty} z_{i_1 i_2 \ldots i_k} \zeta^{i_1} \zeta^{i_2} \ldots \zeta^{i_k}. \tag{2.10}$$

Clearly, $z = z_B + z_S$.

There are other useful notations for exposing the Grassmann content of a supernumber. Let $z \in \Lambda_\infty^{alg}$ as before, then using the multi-index notation we write,

$$z = \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} z_I \zeta^I. \tag{2.11}$$

This notation has the advantage of reminding us of the doubly infinite nature of the summation. Also, it emphasizes the decomposition of the supernumber into terms with p-Grassmann generators. Such terms are said to be homogeneous of degree p. In the above sum the homogeneous term of degree p is defined by

$$z_p = \sum_{I \in \mathcal{I}_p} z_I \zeta^I \tag{2.12}$$

Hence, a supernumber can be written as a sum of homogeneous pieces,

$$z = \sum_{p=0}^{\infty} z_p. \tag{2.13}$$

The concept of degree gives Λ_∞^{alg} a natural $\mathbb{Z}$ grading. Still another method of expressing the summation is possible.

$$z = z_0 + z_1 \zeta^1 + z_2 \zeta^2 + z_{12} \zeta^{(1,2)} + z_3 \zeta^3 + z_{23} \zeta^{(2,3)} + z_{123} \zeta^{(1,2,3)} + \ldots \tag{2.14}$$

We have written the terms up to Λ_3. The ordering of the terms are $\mathbb{K}$, Λ_1, Λ_2, Λ_3, and so on. We can compactly write the sum above by,

$$z = \sum_{k=0}^{\infty} z_{I_k} \zeta^{I_k}. \tag{2.15}$$

In this summation the manifest indication of degree is lost. However, this notation could be very useful in dealing with certain analytical questions. Define the m^{th} partial sum to be $z(m)$,

$$z(m) = \sum_{k=0}^{m} z_{I_k} \zeta^{I_k}. \tag{2.16}$$

For each multi-index $I = (i_1, \ldots, i_k)$ let $top(I) = i_k$ and let

$$N_m = max\{top(I_1), \ldots, top(I_m)\}.$$

Then $z(m) \in \Lambda_{N_n}$.

2.5 Multiplicative Structure of Λ_∞^{alg}

We pause to note some important properties of Λ_∞^{alg} and to introduce some useful notations for explicit calculations. In some sense the Grassmann generators are just place holders that help encode a rather intricate multiplication of the Grassman coefficients. After all a supernumber is completely equivalent to its Grassmann coefficients. This follows directly from our assumption of linear independence of the Grassmann generators,

$$\begin{array}{rll} z = w & \iff & \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} z_I \zeta^I = \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} w_I \zeta^I \\ & \iff & \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} (z_I - w_I) \zeta^I = 0 \\ & \iff & z_I - w_I = 0 \ \ \forall I \in \mathcal{I}(\infty). \end{array} \tag{2.17}$$

Next we note that formally $z, w \in \Lambda_\infty^{alg} \implies zw \in \Lambda_\infty^{alg}$. In particular,

$$\begin{array}{rl} zw & = (\sum_{p=0}^{\infty} z_p)(\sum_{r=0}^{\infty} w_r) \\ & = \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} z_p w_r \\ & = \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} \sum_{I \in \mathcal{I}_p} \sum_{J \in \mathcal{I}_r} z_I w_J \zeta^I \zeta^J \\ & = \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} \sum_{I \in \mathcal{I}_p} \sum_{J \in \mathcal{I}_r} z_I w_J \epsilon(I|J) \zeta^{(I|J)} \\ & = \sum_{q=0}^{\infty} \sum_{K \in \mathcal{I}_q} (zw)_K \zeta^K \end{array} \tag{2.18}$$

In the last step we reordered the Grassmann generators so that they are in the canonical order. We define $\epsilon(I|J)$ to be zero if $I \cap J \neq \emptyset$ and, otherwise it is the sign of the permutation that reshuffles (I, J) to be the increasing index $(I|J)$. In

terms of homogeneous elements we note

$$\begin{aligned}(zw)_q &= \sum_{p+r=q} z_p w_r \\ &= \sum_{p+r=q} \sum_{I\in\mathcal{I}_p} \sum_{J\in\mathcal{I}_r} z_I w_J \zeta^I \zeta^J \\ &= \sum_{p+r=q} \sum_{(I|J)\in\mathcal{I}_q} \epsilon(I|J) z_I w_J \zeta^{(I|J)}\end{aligned} \tag{2.19}$$

Finally, at the level of Grassmann coefficients, for $K \in \mathcal{I}(\infty)$,

$$(zw)_K = \sum_{I\sqcup J=K} \epsilon(I|J) z_I w_J. \tag{2.20}$$

Where $\sqcup$ denotes the disjoint union of I and J. In total we can summarize our findings; Λ_∞^{alg} is an associative algebra with a unit 1 over $\mathbb{K}$. The operations of addition and multiplication are closed in the set of algebraic supernumbers. We could say more about supernumbers without a norm, but we are primarily interested in normed supernumbers.

2.6 The Norm of a Supernumber

In much of the literature one finds that supernumbers are taken to be either Λ_N or Λ_∞^{alg}. In the case of Λ_N, the issue of convergence becomes trivial as there are only finitely many Grassmann generators in the theory. However, this approach has the disadvantage that there is a maximum possible polynomial degree and some care must be taken to avoid the ambiguities that arise in this case (see Rogers' notion of the "z-mapping"). On the other hand, some individuals prefer to work with Λ_∞^{alg} but make claims that cannot be verified in this case. In particular, it is often claimed that $(z_S)^N = 0$ for some N sufficiently large. This is not generally true in Λ_∞^{alg}. However, we can show that for each $z \in \Lambda$ we find $\lim_{N\to\infty}(z_S)^N = 0$, once an appropriate notion of convergence is introduced. We first define a norm, then we prove Λ is complete with respect to the norm, and finally we conclude this section by collecting a few technical results about limits of supernumbers.

2.6.1 Definition of the Norm

Definition 2.6.1. *The norm of a supernumber z is denoted by $||z||$; it is induced from the norm of the Grassmann components z_I which we will denote $|z_I|$ (if $\mathbb{K} = \mathbb{C}$ the norm is the modulus, if $\mathbb{K} = \mathbb{R}$ the norm is absolute value). Define then,*

$$||z|| = \sum_{p=0}^{\infty} \sum_{I\in\mathcal{I}_p} |z_I| = \sum_{k=0}^{\infty} |z_{I_k}|. \tag{2.21}$$

The set of all $z \in \Lambda_\infty^{alg}$ such that $||z|| < \infty$ is defined to be Λ.

It is straightforward to prove that $||\cdot||$ is a norm.

Proposition 2.6.2. *Let $z, w \in \Lambda$ and $\alpha \in \mathbb{K}$, then*

1. $||z|| \geq 0$
2. $||z|| = 0 \iff z = 0$
3. $||z \pm w|| \leq ||z|| + ||w||$
4. $||\alpha z|| = |\alpha| \; ||z||$.

Clearly Λ is a normed linear space. Additionally, Λ is complete relative to the norm $||.||$ just defined. Hence, Λ is a Banach space. We note that Rogers' notation for $\Lambda(\mathbb{R})$ is B_∞, and in Jadczyk and Pilch [68] an abstract general Banach-Grassmann algebra Q plays the same role. The main distinction between our concept of supernumber and B_∞ is with superconjugation.

Example 2.6.3. *We allow the possibility that a supernumber has infinitely many nonzero Grassmann coefficients. Notice*

$$b = \zeta^1 + \frac{1}{2}\zeta^2 + \frac{1}{4}\zeta^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{2^n}\zeta^{n+1}$$

is in Λ since it has finite norm

$$||b|| = ||\sum_{n=0}^{\infty} \frac{1}{2^n}\zeta^{n+1}|| = \sum_{n=0}^{\infty} ||\frac{1}{2^n}\zeta^{n+1}|| = \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2.$$

The following proposition is used widely throughout our work.

Proposition 2.6.4. *Let $z, w \in \Lambda$ then $||zw|| \leq ||z||||w||$. $\Lambda(\mathbb{K})$ is a Banach algebra over $\mathbb{K}$.*

The proof can be found in [98].

2.6.2 Proof that Λ is Complete

We now supply a proof that Λ is complete. Let $\{z(n)\}$ be a Cauchy sequence in Λ then we seek to show that $z(n) \to z \in \Lambda$ as $n \to \infty$. We write,

$$z(n) = \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} z_I(n)\zeta^I. \tag{2.22}$$

Let $J \in \mathcal{I}(\infty)$ and consider,

$$\begin{aligned} |z_J(m) - z_J(n)| &\leq \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} |z_I(m) - z_I(n)| \\ &= || \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} (z_I(m) - z_I(n)) \zeta^I || \\ &= || \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} z_I(m) \zeta^I - \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} z_I(n) \zeta^I || \\ &= ||z(m) - z(n)|| \end{aligned} \tag{2.23}$$

Hence, for any $\epsilon > 0$, we can choose a positive number M such that if $m, n > M$,

$$|z_J(m) - z_J(n)| \ = ||z(m) - z(n)|| < \epsilon/2. \tag{2.24}$$

Thus, for each multi-index J we find that $\{z_J(n)\}$ is a Cauchy sequence in $\mathbb{K}$. Recall that $\mathbb{K}$ is complete so Cauchy sequences converge. That is, there exists $z_J \in \mathbb{K}$ such that,

$$\lim_{n \to \infty} z_J(n) = z_J. \tag{2.25}$$

Let us define z

$$z = \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} z_I \zeta^I. \tag{2.26}$$

We propose to show that 1.) $z \in \Lambda$ and 2.) $z(n) \to z$ as $n \to \infty$.

1.) First, we show that $z \in \Lambda$. Given $\epsilon > 0$ there exists a positive integer M such that $n, m \geq M$ we have $||z(m) - z(n)|| < \epsilon/2$ and

$$\begin{aligned} ||z(m)|| &= ||z(m) - z(M) + z(M)|| \\ &\leq ||z(m) - z(M)|| + ||z(M)|| \\ &< \epsilon/2 + ||z(M)|| \\ &< \epsilon/2 + ||z(M)|| + ||z(1)|| + \cdots + ||z(M-1)|| \end{aligned} \tag{2.27}$$

Next define K to be the finite sum below,

$$K = \epsilon/2 + ||z(M)|| + ||z(1)|| + \ldots ||z(M-1)||. \tag{2.28}$$

thus $||z(m)|| \leq K \ \ \forall n \in \mathbb{N}$. Then $\forall N \in \mathbb{N}$,

$$\sum_{k=0}^{N} |z_{I_k}(n)| \leq \sum_{k=0}^{\infty} |z_{I_k}(n)| = ||z(n)|| \leq K. \tag{2.29}$$

Also note that,

$$\lim_{n \to \infty} \sum_{k=0}^{N} |z_{I_k}(n)| = \sum_{k=0}^{N} \lim_{n \to \infty} |z_{I_k}(n)| = \sum_{k=0}^{N} |z_{I_k}| \leq K. \tag{2.30}$$

We find that $\{\sum_{k=0}^{N}|z_{I_k}|\}$ is an increasing sequence of non-negative real number terms with upper bound K. Therefore, this sequence converges and we deduce,

$$\lim_{N\to\infty}\sum_{k=0}^{N}|z_{I_k}| \leq K. \tag{2.31}$$

In other words, $||z|| \leq K < \infty$. Hence $z \in \Lambda$, the proof of 1.) is finished.

2.) We now show that $z(n) \to z$ as $n \to \infty$. Let $\epsilon > 0$ and recall we can choose M such that $m, n > M$ implies $||z(m) - z(n)|| < \epsilon/2$. Thus, assuming $m, n > M$,

$$\sum_{k=0}^{N}|z_{I_k}(n) - z_{I_k}(m)| \leq \sum_{k=0}^{\infty}|z_{I_k}(n) - z_{I_k}(m)| = ||z(m) - z(n)|| < \epsilon/2. \tag{2.32}$$

Let $m \to \infty$, then $z_{I_k}(m) \to z_{I_k}$ thus,

$$\lim_{m\to\infty}\sum_{k=0}^{N}|z_{I_k}(n) - z_{I_k}(m)| = \sum_{k=0}^{N}|z_{I_k}(n) - z_{I_k}| \leq \epsilon/2. \tag{2.33}$$

Apparently, $\{\sum_{k=0}^{N}|z_{I_k}(n) - z_{I_k}|\}$ is a positive increasing bounded sequence of real numbers, thus the limit as $N \to \infty$ exists.

$$\lim_{N\to\infty}\sum_{k=0}^{N}|z_{I_k}(n) - z_{I_k}| = ||z(n) - z|| \leq \epsilon/2 < \epsilon. \tag{2.34}$$

Summarizing, $n > M$ implies $||z(n) - z|| < \epsilon$. That is $z(n) \to z$ as $n \to \infty$. The proof of 2.) is complete.

2.6.3 Technical Properties of Supernumbers

We find the following properties to be useful for certain delicate questions.

Proposition 2.6.5. *Cancellation property: Let $z, w \in \Lambda$ if $az = aw$ for all $a \in {}^1\Lambda$ then $z = w$.*

Proof. Let $u, w, z \in \Lambda$. Notice that the following are equivalent,

1. $\{\ az = aw\ \forall\ a \in {}^1\Lambda \implies z = w\ \}$
2. $\{\ av = 0\ \forall\ a \in {}^1\Lambda \implies v = 0\ \}$

We will prove (2.). Suppose $av = 0$ for all $a \in {}^1\Lambda$. Observe that for any $k = 1, 2, 3, \ldots$

the Grassmann generator $\zeta^k \in {}^1\Lambda$. Furthermore consider,

$$\zeta^k v = \zeta^k \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} v_I \zeta^I = \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} v_I \zeta^k \zeta^I$$

Thus $\zeta^k v = 0$ implies $v_I = 0$ for each $I \in \mathcal{I}(\infty) - \{k\}$ (meaning all multi-indices without the index k). Since this holds for arbitrary k we find $v_I = 0$ for any multi-index $I \in \mathcal{I}(\infty)$. Therefore, using Equation 2.17, $v = 0$. □

It is worthwhile to pause and see why this proof fails in the finite case. For example, in Λ_2 we have that $a(3\zeta^1\zeta^2) = a(4\zeta^1\zeta^2)$ for all $a \in {}^1\Lambda_2$. Indeed, since $a \in {}^1\Lambda_2$ implies that $a = b\zeta^1 + c\zeta^2$, we can observe that $a(3\zeta^1\zeta^2) = 0 = a(4\zeta^1\zeta^2)$ since either ζ^1 or ζ^2 will be repeated. So the cancellation property fails in the finite case. The trouble stems from the fact that $\zeta^1\zeta^2$ is the "top-form" in Λ_2. In contrast, Λ has no "top-form". So given a particular product of Grassmann generators we can always find an additional generator which is distinct from the product.

Proposition 2.6.6. *Let $z \in \Lambda$ such that $z_B = 0$, then for each $0 < \eta < 1$, there exists $\alpha \geq 0$ such that $||a^n|| \leq \alpha\eta^n$ for $n \in \mathbb{N}$.*

This is also stated in Proposition 3.1 of [68], and the proof is given by Alice Rogers in Lemma 2.7b of [98].

Remark 2.6.7. *The notation employed within this remark will not cause confusion since it is used only in this remark and throughout Chapter 8. Consider the set $\tilde{\Lambda}$ of all $z \in \Lambda_\infty^{alg}$ such that the 2-norm defined by $||z|| = \sqrt{\sum_I |z_I|^2}$ is finite. We previously believed there was a counter example to the Banach algebra inequality for the 2-norm on $\tilde{\Lambda}$. However, the following calculation shows that $||zw|| \leq ||z||\,||w||$ for all $z, w \in \tilde{\Lambda}$ given the 2-norm: $||z|| = \sqrt{\sum_I |z_I|^2}$ and $||w|| = \sqrt{\sum_J |w_J|^2}$ are finite and $|\cdot|$ denotes absolute value on $\mathbb{R}$ or modulus on $\mathbb{C}$. Recall,*

$$zw = \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} \sum_{I \in \mathcal{I}_p} \sum_{J \in \mathcal{I}_r} z_I w_J \epsilon(I|J) \zeta^{(I|J)} \tag{2.35}$$

where $\epsilon(I|J) = \pm 1$ and there are many terms which are zero whenever I and J share a common index (a Grassmann generator is repeated for that component and as such it is zero). The notation $\zeta^{(I|J)}$ indicates a product of Grassmann generators with strictly increasing indices if possible and simply zero if there is an index repeated in the multi-index $(I|J)$. Thus,

$$||zw|| \leq \sqrt{\sum_{p=0}^{\infty} \sum_{r=0}^{\infty} \sum_{I \in \mathcal{I}_p} \sum_{J \in \mathcal{I}_r} |z_I w_J|^2} \tag{2.36}$$

We know that $|z_I w_J| \leq |z_I||w_J|$ for each I, J thus,

$$||zw|| \leq \sqrt{\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}\sum_{I\in\mathcal{I}_p}\sum_{J\in\mathcal{I}_r}|z_I|^2|w_J|^2}. \tag{2.37}$$

Finally notice that (suppressing the degree notation)

$$\begin{aligned} ||z||\ ||w|| &= \sqrt{\textstyle\sum_I |z_I|^2}\sqrt{\textstyle\sum_J |w_J|^2} \\ &= \sqrt{\textstyle\sum_I |z_I|^2 \sum_J |w_J|^2} \\ &= \sqrt{\textstyle\sum_I \sum_J |z_I|^2|w_J|^2} \end{aligned} \tag{2.38}$$

Therefore, $||zw|| \leq ||z||||w||$. If the Grassmann components are nonoverlapping, then we will find equality (just as we would for the one-norm) because the inequality in Equation 2.36 becomes an equality in the case that I and J do not share a common index.

2.7 Commuting and Anticommuting Supernumbers

Supernumbers have all the usual properties of a number system modulo the more complicated commutation properties that go with the Grassmann generators. We define the parity $\epsilon(z)$ of $z \in \Lambda$ as follows:

$$\begin{aligned} \epsilon(z) = 0 &\ \Leftrightarrow\ \zeta^i z = z\zeta^i \qquad \forall\ \zeta^i \\ \epsilon(z) = 1 &\ \Leftrightarrow\ \zeta^i z = -z\zeta^i \qquad \forall\ \zeta^i \end{aligned} \tag{2.39}$$

Of course some supernumbers do not have a definite parity, but we can always decompose any supernumber into a commuting part z_c, with $\epsilon(z_c) = 0$, and an anticommuting part z_a, with $\epsilon(z_a) = 1$,

$$z = z_c + z_a \tag{2.40}$$

Commuting (also called even or bosonic) supernumbers are generated by even number of Grassmann generators,

$$z_c = z_B + \sum_{k\ even}^{\infty} \frac{1}{k!} z_{i_1 i_2 \ldots i_k} \zeta^{i_1}\zeta^{i_2}\ldots\zeta^{i_k} \tag{2.41}$$

Anticommuting (also called odd or fermionic) supernumbers are generated by an odd number of Grassmann generators,

$$z_a = \sum_{k\ odd}^{\infty} \frac{1}{k!} z_{i_1 i_2 \ldots i_k} \zeta^{i_1}\zeta^{i_2}\ldots\zeta^{i_k} \tag{2.42}$$

In view of this decomposition we define,

Definition 2.7.1. *Define the set of all commuting or anticommuting complex supernumbers by,*

$$\begin{aligned}\mathbb{C}_c &= \{z \in \Lambda(\mathbb{C}) \mid \epsilon(z) = 0 \} = {}^0\Lambda(\mathbb{C}) \\ \mathbb{C}_a &= \{z \in \Lambda(\mathbb{C}) \mid \epsilon(z) = 1 \} = {}^1\Lambda(\mathbb{C}).\end{aligned} \tag{2.43}$$

Notice zero is both even and odd thus $\Lambda(\mathbb{C}) = \mathbb{C}_c \oplus \mathbb{C}_a$. Similar definitions apply for other types of supernumbers (we withold details since we have yet to discuss conjugation).

2.8 The Inverse of a Supernumber

The multiplicative inverse of a supernumber z is denoted z^{-1}. When it exists it satisfies the equations,

$$zz^{-1} = 1 \qquad\qquad z^{-1}z = 1. \tag{2.44}$$

Recall the body of a supernumber $z = z_o + z_i\zeta^i + \frac{1}{2}z_{ij}\zeta^i\zeta^j + \cdots$ is the part without a Grassmann generator; that is, we define the body of z to be $z_B = z_o$. Define the mapping $b : \Lambda \to \mathbb{K}$ defined by $b(z) = z_B$. The following proposition follows easily from our previous discussion about the multiplication of supernumbers.

Proposition 2.8.1. *b preserves addition and multiplication. That is for $z, w \in \Lambda$,*

$$b(zw) = b(z)b(w) \qquad\qquad b(z+w) = b(z) + b(w). \tag{2.45}$$

Consider then what this tells us about the inverse of z,

$$1 = b(1) = b(zz^{-1}) = b(z)b(z^{-1}) \tag{2.46}$$

This shows that if z has an inverse then $b(z) \neq 0$. One can prove the following proposition.

Proposition 2.8.2. *Let $z \in \Lambda$, then z is invertible if and only if $b(z) \neq 0$.*

We do not prove this result here but we derive heuristically a formula for the inverse. (see [98]). Let $z = z_B + z_S \in \Lambda$ and suppose that $z_B \neq 0$,

$$z = z_B + z_S \implies z_B^{-1}z = 1 + z_B^{-1}z_S. \tag{2.47}$$

Now z has an inverse if and only if $z_B^{-1}z$ has an inverse. Let us denote $x = z_B^{-1}z_S$ and recall the geometric series result from calculus,

$$(1+x)^{-1} = \frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k. \tag{2.48}$$

This step is rather suspicious in our case, but let us go on and see where it leads us. Notice that $(1+x)^{-1} = (1+z_B^{-1}z_S)^{-1} = (z_B^{-1}z)^{-1} = z^{-1}z_B$ hence,

$$z^{-1}z_B = \sum_{k=0}^{\infty}(-1)^k(z_B^{-1}z)^k. \tag{2.49}$$

Thus we find,

$$z^{-1} = \sum_{k=0}^{\infty}(-1)^k(z_B^{-1}z)^k z_B^{-1}. \tag{2.50}$$

We can verify that this formula is reasonable (assuming the series converges)

$$\begin{aligned}
zz^{-1} &= z_B(1+z_B^{-1}z_S)\textstyle\sum_{k=0}^{\infty}(-1)^k(z_B^{-1}z_S)^k z_B^{-1} \\
&= z_B(1+z_B^{-1}z_S)(z_B^{-1}+\textstyle\sum_{k=1}^{\infty}(-1)^k(z_B^{-1}z_S)^k z_B^{-1}) \\
&= z_B z_B^{-1} + z_B\textstyle\sum_{k=1}^{\infty}(-1)^k(z_B^{-1}z_S)^k z_B^{-1} \\
&\quad + z_B z_B^{-1}z_S z_B^{-1} + z_B z_B^{-1} z_S \textstyle\sum_{k=1}^{\infty}(-1)^k(z_B^{-1}z_S)^k z_B^{-1} \\
&= 1 + \textstyle\sum_{k=1}^{\infty}(-1)^k z_B(z_B^{-1}z_S)^k z_B^{-1} + z_B z_B^{-1}z_S z_B^{-1} \\
&\quad - \textstyle\sum_{k=1}^{\infty}(-1)^{k+1} z_B(z_B^{-1}z_S)^{k+1} z_B^{-1} \\
&= 1 + \textstyle\sum_{k=1}^{\infty}(-1)^k z_B(z_B^{-1}z_S)^k z_B^{-1} + z_B z_B^{-1}z_S z_B^{-1} \\
&\quad - \textstyle\sum_{k=2}^{\infty}(-1)^k z_B(z_B^{-1}z_S)^k z_B^{-1} \\
&= 1 + (-1)^{-1} z_B(z_B^{-1}z_S)^1 z_B^{-1} + z_B z_B^{-1}z_S z_B^{-1} \\
&= 1
\end{aligned} \tag{2.51}$$

2.9 Exponential Function on Λ

We can define the exponential function on $\Lambda(\mathbb{K})$ via the usual power series expansion,

Definition 2.9.1. *Let $x \in \Lambda(\mathbb{K})$, then we define*

$$e^x = \sum_{k=0}^{\infty}\frac{1}{k!}x^k. \tag{2.52}$$

It can be shown that $e^x \in \Lambda(\mathbb{K})$ by showing that $\{\sum_{k=0}^{n}\frac{1}{k!}x^k\}_{n=1}^{\infty}$ is a Cauchy sequence in $\Lambda(\mathbb{K})$, hence it converges inside $\Lambda(\mathbb{K})$ which is complete.

Also, for each $x \in \Lambda(\mathbb{K})$ we can prove e^x is an invertible supernumber. This follows from the observation that the $b(e^x) = e^{x_B}$. Thus, as $x_B \in \mathbb{K}$ we know from ordinary

analysis that $e^{x_B} \neq 0$ therefore $b(e^x) \neq 0$ yielding that e^x is invertible.

Next, investigate how the product of exponentials behaves. Let $x = x_c + x_a$ where $x_c \in \mathbb{K}_c$ and $x_a \in \mathbb{K}_a$. Notice that since $x_a^2 = 0$ we find that $e^{x_a} = 1 + x_a$. Also if $x \in \mathbb{K}_c$ then it follows $e^x \in \mathbb{K}_c$. The mixed case is more interesting, let $x \in \Lambda(\mathbb{K})$,

$$\begin{aligned} e^x &= e^{x_c+x_a} \\ &= 1 + (x_c + x_a) + \tfrac{1}{2!}(x_c + x_a)^2 + \tfrac{1}{3!}(x_c + x_a)^3 + \dots \\ &= 1 + x_c + x_a + \tfrac{1}{2!}(x_c^2 + 2x_a x_c) + \tfrac{1}{3!}(x_c^3 + 3x_a x_c^2) + \dots) \\ &= (1 + x_c + \tfrac{1}{2!}x_c^2 + \tfrac{1}{3!}x_c^3 + \dots) + x_a(1 + x_c + \tfrac{1}{2!}x_c^2 + \tfrac{1}{3!}x_c^3 + \dots) \\ &= e^{x_c} + x_a e^{x_c} \\ &= e^{x_c}(1 + x_a) \\ &= e^{x_c} e^{x_a}. \end{aligned} \tag{2.53}$$

Example 2.9.2. *Suppose that $z = x + \theta$ where $x \in \mathbb{K}_c$ and $\theta \in \mathbb{K}_a$ then if we define $F(x, \theta) = e^{x+\theta} = e^x + \theta e^x$. This is a toy example of a component field expansion. We say the superfield F has component fields a and b if $F = a + \theta b$. For $F(x, \theta) = e^x + \theta e^x$. we have component fields $a = b = e^x$. Later on x will play the role of physical space.*

Next take $v, w \in \Lambda(\mathbb{K})$,

$$\begin{aligned} e^v e^w &= e^{v_c+v_a} e^{w_c+w_a} \\ &= e^{v_c} e^{v_a} e^{w_c} e^{w_a} \\ &= e^{v_c} e^{w_c}(1 + v_a)(1 + w_a) \\ &= e^{v_c+w_c}(1 + v_a + w_a + v_a w_a) \\ &= e^{v_c+w_c}(e^{v_a+w_a} + v_a w_a) \\ &= e^{v+w} + v_a w_a e^{v_c+w_c} \end{aligned} \tag{2.54}$$

Notice that in the case that either v or w is in $\mathbb{C}_c$ the term $v_a w_a = 0$. Only in the case where both of the supernumbers have anticommuting components do we find a departure from the usual behavior of the exponential function. We may recall that the exponential function provides an isomorphism between the group of numbers under addition and the group of non-zero numbers under multiplication. It would seem we will not be able to provide an analogous isomorphism of $\Lambda(\mathbb{K})$ under addition and non-zero $\Lambda(\mathbb{K})$ under multiplication. The unusual nature of the anticommuting numbers will spoil it for the general case. However, we will be able to argue the same isomorphism for the commuting supernumbers $\mathbb{K}_c$. As it turns out we will observe that this is a typical pattern for the exponential mapping. Later when we discuss super Lie groups and algebras we will find that exponentiation works on the even part of the super Lie algebra (which is analogous to $\mathbb{K}_c$ here)

2.10 Super Conjugation

Let $\mathbb{K} = \mathbb{C}$ in this section. We will follow DeWitt [39] and define conjugation as follows:

$$\begin{aligned}(\zeta^i)^* &= \zeta^i \\ (zw)^* &= w^* z^* \\ (z+w)^* &= z^* + w^*\end{aligned} \tag{2.55}$$

The body of a supernumber is commutating so we recover ordinary complex conjugation on the body. This definition of super conjugation differs from the conventions given by Alice Rogers in [102]. We discuss this distinction in Section 2.10.2.

Definition 2.10.1. *A supernumber z is real if $z^* = z$. A supernumber z is imaginary if $z^* = -z$. Generally supernumbers are neither real nor imaginary, but we can always write*

$$z = \frac{1}{2}(z + z^*) + \frac{1}{2}(z - z^*) \stackrel{def}{=} Re\{z\} + iIm\{z\} \tag{2.56}$$

Notice that because we have defined the Grassmann generators to be real, we arrive at the interesting identity:

$$(\zeta^{i_1}\zeta^{i_2}\dots\zeta^{i_k})^* = \zeta^{i_k}\dots\zeta^{i_2}\zeta^{i_1} = (-1)^{\frac{1}{2}k(k-1)}\zeta^{i_1}\zeta^{i_2}\dots\zeta^{i_k}. \tag{2.57}$$

Thus if $z = z_B + \sum_{k=1}^{\infty}\frac{1}{k!}z_{i_1 i_2\dots i_k}\zeta^{i_1}\zeta^{i_2}\dots\zeta^{i_k}$ then

$$z^* = z_B^* + \sum_{k=1}^{\infty}\frac{1}{k!}z^*_{i_1 i_2\dots i_k}(-1)^{\frac{1}{2}k(k-1)}\zeta^{i_1}\zeta^{i_2}\dots\zeta^{i_k} \tag{2.58}$$

provided the mapping $z \mapsto z^*$ is continuous so that conjugation is additive over infinite sums. Observe that z and z^* have the same Grassmann coefficients up to a factor of ± 1. This observation motivates the proposition that follows.

Proposition 2.10.2. *Let z^* denote the super conjugate of z then*

1. *if $z \in \Lambda(\mathbb{C})$ then $||z|| = ||z^*||$ and,*
2. *if $z \in \mathbb{C}_c \cup \mathbb{C}_a$ then $\epsilon(z^*) = \epsilon(z)$.*

In other words, super conjugation does not change the norm or parity of a super number. So the definitions that follow are unambiguous.

Definition 2.10.3. *Define the sets of all commuting or anticommuting real supernumbers by,*

$$\begin{aligned}\mathbb{R}_c = \mathbb{R}_c(\mathbb{C}) &= \{z \in \mathbb{C}_c \mid z^* = z\} \\ \mathbb{R}_a = \mathbb{R}_a(\mathbb{C}) &= \{z \in \mathbb{C}_a \mid z^* = z\}\end{aligned} \tag{2.59}$$

where the notations $\mathbb{R}_c(\mathbb{C})$, $\mathbb{R}_a(\mathbb{C})$ draw our attention to the fact that the Grassmann coefficients are complex numbers. Also define the set of all real supernumbers

$\Lambda_{\mathbb{R}} = \{z \in \Lambda(\mathbb{C}) \mid z^* = z\}$ *and the set of all imaginary supernumbers* $i\Lambda_{\mathbb{R}} = \{iz \mid \in \Lambda_{\mathbb{R}}\}$.

Thus, to summarize, $\Lambda = \mathbb{C}_c \oplus \mathbb{C}_a = \mathbb{R}_c \oplus i\mathbb{R}_c \oplus \mathbb{R}_a \oplus i\mathbb{R}_a$ and $\Lambda = \Lambda_{\mathbb{R}} \oplus i\Lambda_{\mathbb{R}}$.

Definition 2.10.4. *Define* $\mathbb{R}_c(\mathbb{R}) = {}^0\Lambda(\mathbb{R})$ *and* $\mathbb{R}_a(\mathbb{R}) = {}^1\Lambda(\mathbb{R})$.

In this work we insist that if real Grassmann coefficients are used, then the $\mathbb{R}$ must appear explicitly. Thus $\mathbb{R}_c$ will always denote $\mathbb{R}_c(\mathbb{C})$.

2.10.1 Why $\mathbb{R} = \mathbb{K}$ is a Strange Choice

Suppose we try $\mathbb{K} = \mathbb{R}$ and otherwise follow the same definitions as given in the preceding section. What then would we find about the components of real or imaginary supernumbers? Consider that $(\zeta^i\zeta^j)^* = (\zeta^j)^*(\zeta^i)^* = \zeta^j\zeta^i = -\zeta^i\zeta^j$. Thus if $z = z^*$ then

$$z = z_o + z_i\zeta^i + \frac{1}{2}z_{ij}\zeta^i\zeta^j + \cdots = z_o^* + z_i^*\zeta^i - \frac{1}{2}z_{ij}^*\zeta^i\zeta^j + \cdots .$$

Consequently, z-real implies $z_o = z_o^*$, $z_i = z_i^*$ whereas $z_{ij} = -z_{ij}^*$. Suppose you wanted a supernumber which was even, real and bodiless. If you worked in B_L with $L < 4$, then your only choice would be zero. Essentially the quirk is that although the generators ζ^i are real, products of generators $\zeta^{i_1}\zeta^{i_2}\ldots\zeta^{i_k}$ need not be (see Equation 2.57).

Generally, we find that if $z = z^*$, then z would have certain homogeneous components real and the rest pure imaginary. A similar comment applies to $z = -z^*$ so there is a rather peculiar connection between the reality condition of the supernumber and the reality conditions for the Grassmann coefficients (see Equation 2.58). Given our choice in this section of $\mathbb{K} = \mathbb{R}$ we would find that the pure imaginary Grassmann coefficients were forced to be zero. We find this an odd restriction on supernumbers. If we employ an operation of super conjugation, then we will, from this point onward, take $\mathbb{K} = \mathbb{C}$. Now it is still true that there is a somewhat complicated relation between the reality conditions of the Grassmann components of a supernumber and the reality condition of supernumber itself. However, our concepts of a *super real number* or a *super imaginary number* are rather natural and mesh well with the physics literature.

2.10.2 Rogers' Super Conjugation

If one wishes to define super conjugation (different than the one we have already introduced) so that real super numbers have real Grassmann coefficients, then one is led to define super conjugation of the Grassmann generators as follows (see [102])

$$(\beta^j)^* = i\beta^j. \tag{2.60}$$

Other than this condition the Rogers-conjugation shares the same algebraic properties as our super conjugation, most notably $(zw)^* = w^* z^*$. However, the definition of a *real* supernumber differs significantly from our convention; Rogers says that C is real iff $C^* = i^{\epsilon(C)} C$.

Notice this definition makes the generators β^j real. Also products of two generators are real under this definition,

$$(\beta^i \beta^j)^* = (\beta^j)^* (\beta^i)^* = i^2 \beta^j \beta^i = -(-\beta^i \beta^j) = i^0 \beta^i \beta^j.$$

It follows that higher products of Rogers' Grassmann generators are real as well.

Certainly this is an interesting convention, but it is undesirable for our purposes because the reality conditions for super numbers are *parity dependent*. This would result in a fair amount of clutter later on so we content ourselves to use DeWitt's super conjugation. We admit our conventions are strange in that there is something odd about a real super number having real and imaginary Grassmann components.

Chapter 3

Super Linear Algebra

Linear algebra over $\mathbb{R}$ or $\mathbb{C}$ lies at the heart of much of modern mathematics. In supermathematics we find a similar story. We will loosely follow the work of Rogers, Jadczyk and Pilch, DeWitt and Buchbinder and Kunzenko. The definitions we present in this chapter are taken in part from all the authors above.

We begin by defining spaces which are analogous to $\mathbb{R}^n$ or $\mathbb{C}^n$, namely $\Lambda^n(\mathbb{K})$, $\Lambda^{p+q}(\mathbb{K})$, $\mathbb{S}^{p|q}(\mathbb{K})$, $\mathbb{S}^{\bar{p}|\bar{q}}(\mathbb{K})$ where $\mathbb{S} = \mathbb{R}$ or $\mathbb{C}$. These are all Banach spaces built from taking various Cartesian products of $\Lambda(\mathbb{K})$, $\mathbb{S}_c(\mathbb{K})$ and $\mathbb{S}_a(\mathbb{K})$. For most applications we take Grassmann coefficients from $\mathbb{K} = \mathbb{C}$ in which case we may drop the notation indicating the field $\mathbb{K}$. We follow the notations in [29] namely $\mathbb{R}^{p|q}(\mathbb{C}) = \mathbb{R}^{p|q}$ and $\mathbb{C}^{p|q}(\mathbb{C}) = \mathbb{C}^{p|q}$.

Next we discuss DeWitt's definition of a super vector space as well as Jadczyk and Pilch's more refined ideas about $\mathbb{K}_c$-supervector spaces. In addition Alice Rogers' various super modules are defined. Just as in the ordinary case we can construct real or complex supervector spaces. What is new is that we can also restrict our attention to commuting super numbers ($\mathbb{C}_c, \mathbb{R}_c, {}^0\Lambda(\mathbb{R})$). Sometimes the commuting supernumbers are the desirable scalars since multiplication by commuting super numbers does not change parity.

Another wrinkle in the super case is that there is a distinction between left and right linear operators. Left linear operators allow us to pull out scalars to the right without any extra signs. Right linear operators allow us to pull out scalars to the left without any extra signs. Unfortunately there are exceptions to this language in the literature. For example, see [70]. Generally, when we pull out scalars from a linear operator we must take care to generate signs via a Koszul sign convention. We note that just as in ordinary linear algebra we may represent a linear operator via matrix multiplication relative to a choice of basis. We explain our conventions concerning matrices in detail.

Multi-linear mappings play an important role in superanalysis since the iterated

Frechet derivative is a symmetric multi-linear mapping. We discuss our definitions from [37] and relate them to those given in [68].

In the last part of the chapter we show that a left-linear mapping on a (p, q) dimensional supervector space is uniquely defined by its action on the even subspace the supervector space. We discuss how (p, q) and $(p|q)$ dimensional supervector spaces are related. In later chapters the left-linear extension of a map from the even subspace to the total space is a convenient construction. Banach theory typically tells us something about the even part, then we use the algebra at the end of this chapter to extend the map to the total space.

3.1 Algebraic Preliminaries

3.1.1 $\mathbb{Z}_2$-Graded Algebraic Concepts

We collect here a few basic definitions which are used broadly beyond the particular type of supermathematics we consider in this work. We also use these structures, but typically we have to replace the complex scalars with supernumbers. This is a nontrivial step in general since it takes us from the realm of finite dimensional mathematics to that of infinite dimensional Banach spaces. In any case we settle these common ideas for future reference.

Definition 3.1.1. *A $\mathbb{Z}_2$-graded vector space U over a field $\mathbb{K}$ is a vector space over $\mathbb{K}$ with subspaces U_0 and U_1 such that $U = U_0 \oplus U_1$. Vectors in U_0 are called even and have parity $\epsilon(U_0) = 0$ whereas vectors in U_1 are called odd and have parity $\epsilon(U_1) = 1$. Suppose $dim(U_0) = p$ and $dim(U_1) = q$, then we say U has graded dimension (p, q).*

Algebraists often refer to such spaces as superspaces. However, we will reserve that term for spaces built over supernumbers. Graded will always refer to $\mathbb{Z}_2$-grading in this chapter.

Definition 3.1.2. *A $\mathbb{Z}_2$-graded algebra $V = V_0 \oplus V_1$ is a graded vector space with a multiplication such that $1 \in V_0$ and $V_r V_s \subset V_{r+s}$ mod 2.*

Definition 3.1.3. *A graded-commutative algebra W is a $\mathbb{Z}_2$-graded algebra such that for all $v \in W_r$ and $w \in W_s$ we have $vw = (-1)^{rs} wv$ for $r, s = 0, 1$.*

3.1.2 Superalgebras and Modules

Let $\mathcal{S}$ denote any one of the following:

$$\mathcal{S} \in \{\ \Lambda(\mathbb{C}),\ \mathbb{C}_c,\ \Lambda_{\mathbb{R}},\ \mathbb{R}_c,\ \Lambda(\mathbb{R}),\ {}^0\Lambda(\mathbb{R})\ \}$$

Each of these is closed under multiplication. We are primarily interested in the cases $\mathcal{S} = \Lambda(\mathbb{C}), \mathbb{C}_c, \Lambda_{\mathbb{R}}$ and $\mathbb{R}_c$. When ${}^1\mathcal{S} = 0$ we say that the supernumbers $\mathcal{S}$ are *commuting*.

- If $\mathcal{S} = \mathbb{C}_c$ or $\mathbb{R}_c$ or ${}^0\Lambda(\mathbb{R})$ then ${}^1\mathcal{S} = 0$
- If $\mathcal{S} = \Lambda(\mathbb{C})$ then ${}^1\mathcal{S} = {}^1\Lambda(\mathbb{C}) = \mathbb{C}_a$.
- If $\mathcal{S} = \Lambda_{\mathbb{R}}$ then ${}^1\mathcal{S} = \mathbb{R}_a$.
- If $\mathcal{S} = \Lambda(\mathbb{R})$ then ${}^1\mathcal{S} = {}^1\Lambda(\mathbb{R}) = \mathbb{R}_a(\mathbb{R})$.

The definitions below are mostly due to Alice Rogers in [102].

Definition 3.1.4. *A graded left $\mathcal{S}$-module is a graded vector space $U = U_0 \oplus U_1$ which is also a left module which respects the parity structures of U and $\mathcal{S}$; that is ${}^0\mathcal{S}U_r \subset U_r$ and ${}^1\mathcal{S}U_r \subset U_{r+1}$ for $r \in \{0,1\} = \mathbb{Z}_2$. Similarly, a graded right $\mathcal{S}$-module is a graded vector space $U = U_0 \oplus U_1$ which is a right module which respects the parity structures of U and $\mathcal{S}$; that is $U_r{}^0\mathcal{S} \subset U_r$ and $U_r{}^1\mathcal{S} \subset U_{r+1}$ for $r \in \{0,1\} = \mathbb{Z}_2$. A $\mathcal{S}$-bimodule is a left-right $\mathcal{S}$-module U that satisfies an intertwining relations $(\alpha v)\beta) = \alpha(v\beta)$ for all $\alpha, \beta \in \mathcal{S}$, $v \in U$, and $\gamma w = (-1)^{\epsilon_\gamma \epsilon_w} w\gamma$ for all $w \in U_0 \cup U_1$ and $\gamma \in {}^0\mathcal{S} \cup {}^1\mathcal{S}$.*

There is a nice connection between left and right modules over $\mathcal{S}$

Proposition 3.1.5. *Notice that if U is a left $\mathcal{S}$-module, then it is also a right $\mathcal{S}$-module with action defined in such a manner as to respect the Koszul sign rule. Let $\alpha \in {}^0\mathcal{S} \cup {}^1\mathcal{S}$ and let $X \in U_0 \cup U_1$ then*

$$X\alpha = (-1)^{\epsilon(\alpha)\epsilon(X)} \alpha X$$

defines a right $\mathcal{S}$-module action on $U = U_0 \oplus U_1$. Moreover, U is a $\mathcal{S}$-bimodule with respect to these left and right actions. Conversely, given a right $\mathcal{S}$-module we can construct a left action that gives a bimodule structure.

In the discussion above when $\mathcal{S} = \mathbb{C}_c, \mathbb{R}_c$ or ${}^0\Lambda(\mathbb{R})$ we have ${}^1\mathcal{S} = 0$ so many of the relations are trivially satisfied. In what follows there is an important distinction between commuting supernumbers $\mathcal{S}$ (which have ${}^1\mathcal{S} = 0$) and mixed supernumbers $\mathcal{S}$ (which have ${}^1\mathcal{S} \neq 0$).

Definition 3.1.6. *Let $\mathcal{S} \in \{\Lambda(\mathbb{C}), \Lambda_{\mathbb{R}}(\mathbb{C}), \Lambda(\mathbb{R})\}$. Let V be a graded left $\mathcal{S}$ module and let $m = 1, 2, \ldots, p$, $\alpha = 1, 2, \ldots, q$ and $E_m \in V_0$, $\tilde{E}_\alpha \in V_1$, then we call $\{E_m, \tilde{E}_\alpha\}$ a pure basis of super dimension (p, q) if there exist $v^m, \tilde{v}^\alpha \in \mathcal{S}$ for each $v \in V$ such that*

$$v = \sum_{m=1}^{p} v^m E_m + \sum_{\alpha=1}^{q} \tilde{v}^\alpha \tilde{E}_\alpha = \sum_{M=1}^{p+q} v^M E_M$$

where we also denote $\{E_m, \tilde{E}_\alpha\} = \{E_M\}$ *with* $E_M = E_m$ *for* $M = m = 1, 2, \ldots p$ *and* $E_M = \tilde{E}_\alpha$ *for* $M = p + \alpha = p + 1, p + 2, \ldots p + q$. *For convenience denote* $\epsilon(E_M)$ *by* ϵ_M.

3.2 Definition of an Abstract Supervector Space

It should be noted that there are several popular uses of the term "supervector space". Among algebraists often when people speak of a super vector space they mean a $\mathbb{Z}_2$ graded vector space V over the complex numbers. That is, V is a vector space over $\mathbb{C}$ with subspaces V_0 and V_1 such that $V = V_0 \oplus V_1$. We will instead refer such as a graded vector space or a $\mathbb{C}$-supervector space. We reserve the term supervector space for a slightly more exotic object to be described in the next section. In short, graded vector spaces have a scalars in $\mathbb{K}$ whereas supervector spaces (for us) have scalars in $\Lambda(\mathbb{K})$. We follow Jadczyk and Pilch [68] in relaxing DeWitt's definition [39] to allow purely commuting super scalars, and we entertain the case of a real supervector space. Due to the variety of scalars available we will introduce some notation to treat them simultaneously.

Let $\mathcal{S}$ denote one of the following:

$$\mathcal{S} \in \{\ \Lambda(\mathbb{C}) = \mathbb{C}_c \oplus \mathbb{C}_a,\ \mathbb{C}_c,\ \Lambda_{\mathbb{R}}(\mathbb{C}) = \mathbb{R}_c \oplus \mathbb{R}_a,\ \mathbb{R}_c,\ \Lambda(\mathbb{R}),\ {}^0\Lambda(\mathbb{R})\ \}.$$

Each of these is closed under multiplication. We are primarily interested in $\mathcal{S} = \Lambda(\mathbb{C}), \mathbb{C}_c, \Lambda_{\mathbb{R}}$ and $\mathbb{R}_c$. An abstract vector space built over the supernumbers $\mathcal{S}$ is called a supervector space. When we take $\mathcal{S} = \Lambda(\mathbb{C})$ or $\mathcal{S} = \mathbb{C}_c$ we will say we have *complex superscalars* and a *complex supervector space*. Likewise when we take $\mathcal{S} = \mathbb{R}_c \oplus \mathbb{R}_a = \Lambda_{\mathbb{R}}$ or $\mathcal{S} = \mathbb{R}_c$ we will say we are using *real superscalars* for a *real supervector space*.

In particular, a supervector space V is a set of vectors with a vector addition $+ : V \times V \longrightarrow V$ which is commutative, associative, and distributive, along with left and right scalar multiplications which respect the parity and (possibly) conjugation properties of $\mathcal{S}$. That is,

1. $X + Y = Y + X$ for each $X, Y \in V$,
2. $(\alpha + \beta)X = \alpha X + \beta X$ and $X(\alpha + \beta) = X\alpha + X\beta$ for each $\alpha, \beta \in \mathcal{S}$ and $X \in V$,
3. $\alpha(X + Y) = \alpha X + \alpha Y$ and $(X + Y)\alpha = X\alpha + Y\alpha$ for each $\alpha \in \mathcal{S}$ and $X, Y \in V$,
4. $(\alpha X)\beta = \alpha(X\beta)$ for each $\alpha, \beta \in \mathcal{S}$ and $X \in V$,

5. $\alpha_c X = X\alpha_c$ for each $\alpha_c \in {}^0\mathcal{S}$ and $X \in V$,

6. V is the direct sum of even and odd subspaces; $V = V_0 \oplus V_1$. Moreover, vectors in these subspaces are called pure; they have definite parity ϵ which is defined as follows
$$\begin{array}{l} V_0 = \{\ X \in V \mid \epsilon(X) = 0\ \} \\ V_1 = \{\ X \in V \mid \epsilon(X) = 1\ \}. \end{array} \tag{3.1}$$
If $X \in V$ then we denote $X = X_0 + X_1$ where $X_0 \in V_0$ and $X_1 \in V_1$.

7. When $\mathcal{S}$ has anticommuting scalars then we insist that the parities of $\mathcal{S}$ interact with those of $\mathcal{S}$ as follows: If $X = X_0 + X_1$ then
$$\alpha_a X_1 = -X_1 \alpha_a$$
for all $\alpha_a \in {}^1\mathcal{S}$. When ${}^1\mathcal{S} = 0$ then this requirement is trivially satisfied.

8. When $\mathcal{S}$ is complex ($\mathcal{S} = \Lambda(\mathbb{C})$ or $\mathcal{S} = \mathbb{C}_c$), then we insist that there is a conjugation of vectors which interacts with the conjugation of supernumbers at follows:

   - $X^{**} = X$
   - $(X + Y)^* = X^* + Y^*$
   - $(\alpha X)^* = X^* \alpha^*$
   - $(X\alpha)^* = \alpha^* X^*$

   A super vector X is *real* if $X^* = X$. A super vector X is *imaginary* if $X^* = -X$. Generally super vectors are neither real nor imaginary, but we can always write
$$X = \frac{1}{2}(X + X^*) + \frac{1}{2}(X - X^*) \stackrel{def}{=} Re\{X\} + iIm\{X\} \tag{3.2}$$
   Every complex super vector space V has a real subspace defined as follows,
$$V_{\mathbb{R}} = \{\ X \in V \mid X^* = X\ \} \tag{3.3}$$
   From which we can construct the imaginary subspace,
$$iV_{\mathbb{R}} = \{\ iX \mid X \in V_{\mathbb{R}}\ \ \} \tag{3.4}$$
   Thus, for any complex supervector space, $V = V_{\mathbb{R}} \oplus iV_{\mathbb{R}}$.

9. If we have all the requirements for a complex supervector space except conjugation then we will say that it is a "$\mathcal{S}$-bimodule" or a "supervector space without conjugation".

Table 3.1: Canonical Supervector Spaces

Type / Dimension	(p,q)	$(p\|q)$	$(\bar{p}\|\bar{q})$
Complex Supervector Spaces	$\Lambda(\mathbb{C})(p,q)$	$\mathbb{C}^{p\|q}$	$\mathbb{C}^{\bar{p}\|\bar{q}}$
Real Supervector Spaces	$\Lambda_{\mathbb{R}}(\mathbb{C})(p,q)$	$\mathbb{R}^{p\|q}$	$\mathbb{R}^{\bar{p}\|\bar{q}}$
Real Supervector Spaces	$\Lambda(\mathbb{R})(p,q)$	$\mathbb{R}^{p\|q}(\mathbb{R})$	$\mathbb{R}^{\bar{p}\|\bar{q}}(\mathbb{R})$

Remark 3.2.1. *Suppose $\mathcal{S}$ contains complex supernumbers, if we have a $\mathbb{Z}_2$-graded $\mathcal{S}$-bimodule which possesses a conjugation operation that respects the module structure then we obtain a supervector space over $\mathcal{S}$. If $\mathcal{S}$ contains only real supernumbers, then a $\mathbb{Z}_2$-graded $\mathcal{S}$-bimodule is a supervector space over $\mathcal{S}$.*

Proposition 3.2.2. *Let $X \in V$ then $\epsilon(X) = \epsilon(X^*)$.*

Proof. Assume $\mathcal{S} = \Lambda(\mathbb{C})$, let $\alpha \in \mathbb{C}_a$ and let $X \in V$ be pure, then

$$\alpha X = (-1)^{\epsilon(X)\epsilon(\alpha)} X\alpha. \tag{3.5}$$

Now conjugate both sides and use that $(\alpha X)^* = X^*\alpha^*$

$$\begin{aligned} X^*\alpha^* &= ((-1)^{\epsilon(X)\epsilon(\alpha)} X\alpha)^* \\ &= \alpha^* X^* ((-1)^{\epsilon(X)\epsilon(\alpha)})^* \\ &= (-1)^{\epsilon(X)\epsilon(\alpha^*)} \alpha^* X^* \end{aligned} \tag{3.6}$$

where we used the facts that $(\pm 1)^* = \pm 1 \in \mathbb{C}_c$, and $\epsilon(\alpha^*) = \epsilon(\alpha)$ for the last step. The equations above shows that X^* and X share the same parity. □

Thus, we can make a direct sum decomposition of the supervector space V

$$V = V_0 \oplus V_1 = (V_{0\mathbb{R}} \oplus iV_{0\mathbb{R}}) \oplus (V_{1\mathbb{R}} \oplus iV_{1\mathbb{R}}) \tag{3.7}$$

Now if $\mathcal{S} = \mathbb{C}_c$ a similar, but easier, argument holds.

3.3 Canonical Supervector Spaces

In this section we define the spaces in our context which are analogous to $\mathbb{R}^n$ or $\mathbb{C}^n$ in ordinary linear algebra. Each of the spaces in the table can appear as the target spaces for coordinate maps on an abstract supervector space. As the notation suggests we have two types of dimension to keep in mind. If we say our space has super dimension (p,q) that indicates we are working with mixed superscalars. On the other hand, if we have a space with superdimension $(p|q)$ or $(\bar{p}|\bar{q})$ then we must use commuting superscalars in order that we not spoil the grading. In the section that follows this one we will learn that any abstract finite super-dimensional supervector space is one of the examples given in this section up to isomorphism.

3.3.1 Canonical Real or Complex Supervector Spaces over $\mathbb{K}$

The field $\mathbb{K}$ which provides the Grassmann coefficients is either $\mathbb{R}$ or $\mathbb{C}$. If we take $\mathbb{K} = \mathbb{R}$, then we would only find it natural to discuss real supervector spaces. However, if we take $\mathbb{K} = \mathbb{C}$, then we may consider "real" or "complex" supervector spaces. The "real" or "complex" refers to the nature of the superscalars which we denoted generically by $\mathcal{S}$. In this section we give the results which hold for either choice of $\mathbb{K}$ and both real and complex supervector spaces. Throughout this section $\Lambda = \Lambda(\mathbb{K})$, we choose to emphasize the $\mathbb{K}$ on certain points.

The Cartesian product of Λ k-times is $\Lambda^k = \Lambda \times \Lambda \times \cdots \times \Lambda$. We can readily verify that Λ^k is an vector space over $\mathbb{K}$ although it has no natural multiplication like Λ.

Definition 3.3.1. *We define a norm on Λ^k as follows,*

$$||(z^1, z^2, \ldots, z^k)|| = \sum_{i=1}^{k} ||z^i||. \tag{3.8}$$

We leave the proof that this is a norm to the reader. The fact that Λ^k is complete can be induced easily from the fact that Λ is complete.

Following our paper [37] we introduce a slight modification of our original notation to indicate the Grassmann coefficient field we define:

Definition 3.3.2. *Suppose $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$ and $\mathbb{S}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathbb{S}^{p|q}(\mathbb{K})$ denote the set of all $(p+q)$-tuples $z = (x^1, \ldots, x^p, \theta^1, \ldots, \theta^q)$ where $x^m \in \mathbb{S}_c(\mathbb{K})$ for $m = 1, 2, \ldots, p$ and $\theta^\alpha \in \mathbb{S}_a(\mathbb{K})$ for $\alpha = 1, 2, \ldots, q$. In a more compact notation we also write $z = (z^M)$ for $M = 1, 2, \ldots, p+q$. The norm on $\mathbb{S}^{p|q}(\mathbb{K})$ is induced from the norm on $\mathbb{S}_c(\mathbb{K}) \oplus \mathbb{S}_a(\mathbb{K})$,*

$$|| \ z \ || = \sum_{m=1}^{p} ||x^m|| + \sum_{\alpha=1}^{q} ||\theta^\alpha|| = \sum_{M=1}^{p+q} ||z^M||. \tag{3.9}$$

We define the dimension of $\mathbb{S}^{p|q}(\mathbb{K})$ to be $(p|q)$.

Observe that since $\mathbb{S}_c(\mathbb{K})$ and $\mathbb{S}_a(\mathbb{K})$ are complete, it follows that $\mathbb{S}^{p|q}(\mathbb{K})$ is complete and consequently $\mathbb{S}^{p|q}(\mathbb{K})$ is a Banach space. Moreover, following Jadzyck and Pilch [68] we can give $\Lambda(\mathbb{K})^k$ a grading.

Definition 3.3.3. *Let us denote $\mathbb{K}^{\bar{p}|\bar{q}} = {}^1\Lambda^p \times {}^0\Lambda^q$. If $k = p + q$, then we define $\Lambda(p, q) = \Lambda^k$ with the $\mathbb{Z}_2$-grading such that ${}^0\Lambda(p, q) = \mathbb{K}^{p|q} = {}^0\Lambda^p \times {}^1\Lambda^q$ and ${}^1\Lambda(p, q) = \mathbb{K}^{\bar{p}|\bar{q}} = {}^1\Lambda^p \times {}^0\Lambda^q$.*

Let us pause to examine the details of how $\Lambda(p, q)$ is identified with $\mathbb{C}^{p|q} \oplus \mathbb{C}^{\bar{p}|\bar{q}}$. Let $z = (z_1, \ldots, z_{p+q}) \in \Lambda^{p+q}(\mathbb{C})$ and further suppose that $z_M = x_M + \theta_M$, $x_M \in \mathbb{C}_c$,

$\theta_M \in \mathbb{C}_a$ for each $M = 1, 2, \ldots p+q$ then

$$\begin{array}{rl} z & = (z_1, \ldots, z_p, z_{p+1}, \ldots, z_{p+q}) \\ & = (x_1 + \theta_1, \ldots, x_p + \theta_p, x_{p+1} + \theta_{p+1}, \ldots, x_{p+q} + \theta_{p+q}) \\ & = (x_1, \ldots, x_p, \theta_{p+1}, \ldots, \theta_{p+q}) + (\theta_1, \ldots, \theta_p, x_{p+1}, \ldots, x_{p+q}) \end{array} \tag{3.10}$$

Observation 3.3.4. *In [68], the notation for $\Lambda(p,q)$ would be Λ^{p+q}. We have avoided this subtle notation because in their notation generally $\Lambda^{p+q} \neq \Lambda^{q+p}$.*

If $k = p + q$, then as point sets $\Lambda^k = \Lambda(p,q)$. For a particular $k \in \mathbb{N}$, there are many possible gradings we could give to split Λ^k into even and odd parts. For each possible splitting the norm is given as in 3.3.2. Let $M \in \{1, 2, \ldots k\}$, if $z^M \in \Lambda(\mathbb{C})$, then there exist $x^M \in \mathbb{C}_c$, $\theta^M \in \mathbb{C}_a$ such that $z^M = x^M + \theta^M$. Because the Grassmann components of even and odd supernumbers are non-overlapping, it follows that $||z^M|| = ||x^M|| + ||\theta^M||$. Thus,

$$\begin{array}{rl} || \ z \ || & = \sum_{M=1}^{k} ||z^M|| \\ & = \sum_{M=1}^{k} (\ ||x^M|| + ||\theta^M|| \ ) \\ & = \sum_{M=1}^{k} ||x^M|| + \sum_{M=1}^{k} ||\theta^M|| \\ & = \sum_{M=1}^{p} ||x^M|| + \sum_{M=1}^{p} ||\theta^M|| + \sum_{M=p+1}^{p+q} ||x^M|| + \sum_{M=p+1}^{p+q} ||\theta^M||. \end{array} \tag{3.11}$$

The equation above shows that $||z||$ is independent of the grading given to Λ^k. The comments given above for $\Lambda(\mathbb{C})(p,q)$ also apply to $\Lambda_{\mathbb{R}}(p,q)$ or $\Lambda(\mathbb{R})(p,q)$.

3.3.2 Canonical Complex Supervector Spaces

Logically the definitions that follow are implicit within the last section, but we wish to make the notation clear.

Definition 3.3.5. *The set of <u>even</u> vectors in $\Lambda(\mathbb{C})(p,q)$ are*

$$\mathbb{C}^{p|q} = \{ \ (y^1, y^2, \ldots, y^p, \theta^1, \theta^1, \ldots, \theta^q) \mid y^m \in {}^0\Lambda(\mathbb{C}) \text{ and } \theta^\mu \in {}^1\Lambda(\mathbb{C}) \ \}. \tag{3.12}$$

The set of <u>odd</u> vectors in $\Lambda(\mathbb{C})(p,q)$ are denoted by $\mathbb{C}^{\bar{p}|\bar{q}}$ (see [68]) and

$$\mathbb{C}^{\bar{p}|\bar{q}} = \{ \ (\theta^1, \theta^1, \ldots, \theta^p, y^1, y^2, \ldots, y^q,) \mid y^m \in {}^0\Lambda(\mathbb{C}) \text{ and } \theta^\mu \in {}^1\Lambda(\mathbb{C}) \ \}. \tag{3.13}$$

The <u>parity</u> of a vector $v \in \Lambda(\mathbb{C})(p,q)$ is $\epsilon(v)$ where $\epsilon(\mathbb{K}^{p|q}) = 0$ and $\epsilon(\mathbb{K}^{\bar{p}|\bar{q}}) = 1$.

3.3.3 Canonical Real Supervector Spaces over $\mathbb{K} = \mathbb{C}$

Conceptually we regard the $\mathbb{R}$ that appears in $\mathbb{R}^{p|q}$ to refer to *super conjugation*. It does not indicate that we generate supernumbers using $\mathbb{R}$-valued Grassmann coefficients. As we discussed previously this leads to undesirable peculiarities under our

conventions for super conjugation. We should comment that the notation here differs slightly from [37]. We have replaced $\mathbb{K}^{p|q}$ with the more descriptive $\mathbb{S}^{p|q}(\mathbb{K})$. With this change of notation, we are now free to consider $\mathbb{R}^{p|q}(\mathbb{C})$ (in [37] we did not study super conjugation so this was not an issue).

Definition 3.3.6. *The set of <u>even real</u> vectors in $\Lambda(\mathbb{C})(p,q)$ are*

$$\mathbb{R}^{p|q} = \{\ (y^1, y^2, \ldots, y^p, \theta^1, \theta^1, \ldots, \theta^q) \mid y^m \in \mathbb{R}_c(\mathbb{C}) \text{ and } \theta^\mu \in \mathbb{R}_a(\mathbb{C})\ \}. \tag{3.14}$$

The set of <u>odd real</u> vectors in $\Lambda(\mathbb{C})(p,q)$ are denoted by $\mathbb{R}^{\bar{p}|\bar{q}}$ and

$$\mathbb{R}^{\bar{p}|\bar{q}} = \{\ (\theta^1, \theta^1, \ldots, \theta^p, y^1, y^2, \ldots, y^q,) \mid y^m \in \mathbb{R}_c(\mathbb{C}) \text{ and } \theta^\mu \in \mathbb{R}_a(\mathbb{C})\ \}. \tag{3.15}$$

Let $\Lambda(p,q)_{\mathbb{R}} = \mathbb{R}^{p|q} \oplus \mathbb{R}^{\bar{p}|\bar{q}}$ where parity is defined as before, $\epsilon(\mathbb{R}^{p|q}) = 0$ and $\epsilon(\mathbb{R}^{\bar{p}|\bar{q}}) = 1$.

3.3.4 Canonical Supervector Spaces over $\mathbb{R}$

We will not have much occasion to use the objects in this section, but just to draw attention to the issue let us define real super vectors over the real numbers.

Definition 3.3.7. *The set of <u>even real</u> vectors in $\Lambda(\mathbb{R})(p,q)$ are*

$$\mathbb{R}^{p|q}(\mathbb{R}) = \{\ (y^1, y^2, \ldots, y^p, \theta^1, \theta^1, \ldots, \theta^q) \mid y^m \in \mathbb{R}_c(\mathbb{R}) \text{ and } \theta^\mu \in \mathbb{R}_a(\mathbb{R})\ \}. \tag{3.16}$$

The set of <u>odd real</u> vectors in $\Lambda(\mathbb{R})(p,q)$ are denoted $\mathbb{R}^{\bar{p}|\bar{q}}(\mathbb{R})$,

$$\mathbb{R}^{\bar{p}|\bar{q}}(\mathbb{R}) = \{\ (\theta^1, \theta^1, \ldots, \theta^p, y^1, y^2, \ldots, y^q,) \mid y^m \in \mathbb{R}_c(\mathbb{R}) \text{ and } \theta^\mu \in \mathbb{R}_a(\mathbb{R})\ \}. \tag{3.17}$$

If we were to work with $\Lambda(\mathbb{R})(p,q)$, then it would be more natural to adopt the super conjugation described in [102].

3.4 Linear Independence, Spanning and Total Dimension

In contrast to ordinary linear algebra we need to distinguish between right and left linear independence.

Definition 3.4.1. *Suppose that V is a $\mathcal{S}$-supervector space. Let $e_A \in V$ for all $A \in \mathcal{J}$, where $\mathcal{J}$ is a possibly infinite indexing set. We define a set of supervectors $\{e_A\}_{A\in\mathcal{J}}$ to be a left-linearly independent set of supervectors if and only if for each finite set $J \subseteq \mathcal{J}$*

$$\textstyle\sum_{m\in J} e_m X^m_+ = 0 \implies X^m_+ = 0\ \forall m \in J. \tag{3.18}$$

Similarly, a set of supervectors $\{e_A\}_{A\in\mathcal{J}}$ *is said to be a right-linearly independent set of supervectors if and only if for each finite set* $J \subseteq \mathcal{J}$

$$\sum_{m\in J} X^m_- e_m = 0 \implies X^m_- = 0 \ \forall m \in J. \tag{3.19}$$

We say the set of supervectors $\{e_A\}_{A\in\mathcal{J}}$ *is a linearly independent set of supervectors iff it is both a left and right linearly independent set of supervectors.*

Likewise we must make a distinction between right and left spanning sets.

Definition 3.4.2. *Suppose that* $W \subset V$ *where* V *is a* $\mathcal{S}$*-supervector space. Let* $e_A \in V$ *for all* $A \in \mathcal{J}$*, where* $\mathcal{J}$ *is a possibly infinite indexing set. We define a set of supervectors* $\{e_A\}_{A\in\mathcal{J}}$ *to be a left-spanning set for* W *iff for each* $w \in W$ *there exist finitely many superscalars* $X^m_+ \in \mathcal{S}$*,* $m \in J \subseteq \mathcal{J}$ *such that*

$$w = \sum\nolimits_{m\in J} e_m X^m_+.$$

Likewise, $\{e_A\}_{A\in\mathcal{J}}$ *is said to be a right-spanning set for* W *iff for each* $w \in W$ *there exist finitely many superscalars* $X^m_- \in \mathcal{S}$*,* $m \in J \subseteq \mathcal{J}$ *such that*

$$w = \sum\nolimits_{m\in J} X^m_- e_m.$$

Finally we say that a set of supervectors $\{e_A\}_{A\in\mathcal{J}}$ *is a spanning set for* W *iff it is both a left and right spanning set for* W*.*

We say that a spanning set is *minimal* iff there is no smaller set which also spans the space. We say a linearly independent set is maximal iff whenever we enlarge the set it becomes linearly dependent.

Remark 3.4.3. *Suppose* $S = \{e_A\}_{A\in\mathcal{J}}$ *is a set of pure supervectors, then we can show that* S *is a left linearly independent iff* S *is right linearly indpendent. Also* S *is a left spanning set for* W *iff* S *is a right spanning set. Fortunately, we will find that there always exists a pure basis for a finite super-dimensional supervector space. Once we have such a basis it will suffice to consider just left or right spans.*

Definition 3.4.4. *A basis for a* $\mathcal{S}$*-supervector space* V *is a linearly independent spanning set. We say that* V *is finite super-dimensional iff there exists a finite maximal linearly independent spanning set. Suppose that* $\{e_A\}^d_{A=1}$ *is a finite maximal linearly independent spanning set for* V*. We call that set a* <u>*basis*</u> *of* V *and define the number of vectors in that basis to be the* <u>*total dimension*</u> *"d" of* V*. With respect to the basis we define the left and right components of the vector* $X \in V$*,*

$$\begin{aligned} X = e_M X^M_+ \qquad & X^M_+ = \textit{left components of } X \\ X = X^M_- e_M \qquad & X^M_- = \textit{right components of } X \end{aligned}$$

The theorem that follows assures us that the dimension is well-defined.

Theorem 3.4.5. *If V is a supervector space which has finite basis, then every basis is finite. Moreover any two bases have the same number of elements.*

Proof. Assume that $\{e_M|1 \leq M \leq d\}$ is a finite basis of V and that $\{f_A|A \in \mathcal{A}\}$ is any basis. We show that $\mathcal{A}$ has no more than d elements. Suppose to the contrary that $A_1, A_2, \cdots, A_r, r > d$ are elements of $\mathcal{A}$ where r is large enough so that each of the elements e_M is a linear combination of the $\{f_{A_a}\}$

$$e_M = F_M^{\ a} f_a. \tag{3.20}$$

Such an r exists since, for each M, e_M is a finite linear combination of the basis $\{f_A|A \in \mathcal{A}\}$. To simplify notation, we denote the elements $\{f_{A_a}\}$ simply by $f_a = f_{A_a}, 1 \leq a \leq r$. Observe that we also have

$$f_a = G_a^{\ M} e_M \tag{3.21}$$

Consequently, $f_a = G_a^{\ M} F_M^{\ b} f_b$ and $e_M = F_M^{\ a} G_a^{\ N} e_N$ and it follows from linear independence that $G_a^{\ M} F_M^{\ b} = \delta_a^b$ and $F_M^{\ a} G_a^{\ N} = \delta_M^N$. Thus as matrices $GF = I_r, FG = I_d$ and $b(G)b(F) = I_r, b(F)b(G) = I_d$ where, as usual $b(F), b(G)$ denote the bodies of the matrices F, G. Finally, $d = tr(I_d) = tr(b(F)b(G)) = tr(b(G)b(F)) = tr(I_r) = r$. This contradiction then implies that $r \leq d$. Once we know $\mathcal{A}$ is finite with no more than d elements we can reverse the roles of the two bases in the proof and doing so implies that both $r \leq d$ and $d \leq r$. The theorem follows. □

3.4.1 Pure Basis

Definition 3.4.6. *Suppose that $\mathcal{S} = {}^0\mathcal{S} \oplus {}^1\mathcal{S}$ with ${}^0\mathcal{S} \neq 0$ and ${}^1\mathcal{S} \neq 0$, then a basis for a $\mathcal{S}$-supervector space $V = V_0 \oplus V_1$ is said to be a (p, q) dimensional pure-basis iff it is an ordered basis of supervectors such that the first p supervectors are in V_0 and the last q supervectors are in V_1. We say that V is a (p, q) dimensional supervector space.*

We should pause to note that if we have a $\mathcal{S}$-supervector space V where $\mathcal{S} = {}^0\mathcal{S}$, then unfortunately there may be no basis of V. For example, $\mathbb{C}^{p|q}$ contains no basis. We refer to such supervector spaces as *coordinated* supervector spaces, or $(p|q)$-dimensional supervector spaces.

Proposition 3.4.7. *Suppose that $\mathcal{S} = {}^0\mathcal{S} \oplus {}^1\mathcal{S}$ with ${}^0\mathcal{S} \neq 0$ and ${}^1\mathcal{S} \neq 0$. If V is a $\mathcal{S}$-supervector space of total dimension d then V is a (p, q)-dimensional supervector space for a unique pair of nonnegative integers p, q such that $p + q = d$. Moreover, there is a correspondence between V_0 and $\mathbb{K}^{p|q}$.*

Proof. Let $\{e_M\}$ be an arbitrary basis of a d dimensional supervector space V. For each index M, let

$$e_M = e^0_M + e^1_M. \tag{3.22}$$

where e^0_M is even and e^1_M is odd. Since $\{e_M\}$ is a basis there exist super matrices F and G such that

$$\begin{array}{ll} e^0_M & = e_N F^N{}_M \\ e^1_M & = e_N G^N{}_M \end{array} \tag{3.23}$$

hence,

$$e_M = e^0_M + e^1_M = e_N(F^N{}_M + G^N{}_M) \tag{3.24}$$

Let F denote the matrix $(F^N{}_M)$ and G the matrix $(G^N{}_M)$. Then F and G are $d \times d$ matrices over Λ such that $F + G = I_d$. Let $A = b(F)$ and $B = b(G)$ and consider the linear transformation from $\mathbb{C}^{2d}$ to $\mathbb{C}^d$ defined by $L((x,y))^t = [A|B](x,y)^t$ where $(x,y) \in \mathbb{C}^{2d}$ and the superscript t means transpose (to convert the row vector to a column vector). Now $L((x,y)) = xA+yB$ in row notation and $L((x,x)) = x(A+B) = x$ so L is surjective and the rank of the augmented matrix $[A|B]$ is d. The dimension of the column space is therefore d, and so there exists $d = p + q$ columns

$$b(F)_{M_1}, b(F)_{M_2}, \cdots, b(F)_{M_p}, b(G)_{N_1}, b(G)_{N_2}, \cdots, b(G)_{N_q}$$

of the matrix $[b(F)|b(G)]$ which are linearly independent and the matrix with these as its columns is invertible.

It follows that the submatrix $\mathcal{M}$ of $[F|G]$ defined by

$$\mathcal{M} = [F_{M_1}|F_{M_2}|\cdots, F_{M_p}|G_{N_1}|G_{N_2}|\cdots, G_{N_q}]$$

is invertible since its body is. Now F_{M_i} is the i-th column of $\mathcal{M}$, and G_{N_j} is the $(p+j)$-th column of $\mathcal{M}$; we use these to define

$$\tilde{e}^0_i = (e_1, e_2, \cdots, e_d)F_{M_i} \qquad \tilde{e}^1_j = (e_1, e_2, \cdots, e_d)G_{N_j}.$$

Since $\tilde{e}^0_i = e_M F^M{}_{N_i} = e^0_{M_i}$, we see that it is even. Similarly $\tilde{e}^1_j = e_N G^N{}_{M_j} = e^1_{N_j}$ is odd. Moreover,

$$(\tilde{e}^0_1, \tilde{e}^0_2, \cdots, \tilde{e}^0_p, \tilde{e}^1_1, \tilde{e}^1_2, \cdots, \tilde{e}^1_q) = (e_1, e_2, \cdots, e_{p+q})\mathcal{M}$$

where $\mathcal{M}$ is invertible. It is not difficult to prove that the vectors

$$\{\tilde{e}^0_1, \tilde{e}^0_2, \cdots, \tilde{e}^0_p, \tilde{e}^1_1, \tilde{e}^1_2, \cdots, \tilde{e}^1_q\}$$

are left and right linearly independent and that they form a left and right spanning set for the supervector space V. This follows from the fact that $\mathcal{M}$ is invertible. The theorem follows. □

Clearly $\Lambda(\mathbb{K})(p,q)$ is a supervector space of dimension (p,q) over the superscalars $\Lambda(\mathbb{K})$.

Definition 3.4.8. *The canonical basis for $\Lambda(\mathbb{K})(p,q)$ is the set $\{e_M\}_{M=1}^{p+q}$ of $(p+q)$-tuples*

$$e_1 = (1,0,\ldots,0), \quad e_2 = (0,1,\ldots,0), \quad e_{p+q} = (0,0,\ldots,1)$$

sometimes we will use the notation that $e_M = e_m$ for $M = m = 1,2,\ldots p$ while $e_M = e_\alpha$ for $M = \alpha + p$ for $\alpha = 1,2,\ldots q$. The type of index indicates the numbering employed. Capital indices like $M, N, \ldots$ typically run over all the indices whereas $m, n, \ldots$ run from 1 to p and $\alpha, \beta, \ldots$ run from 1 to q.

We continue to employ this notation for the arguments that follow in later sections. Given a supervector space V of dimension (p,q) we chose p even vectors and $q = d-p$ odd vectors and denote them as follows

$$\begin{array}{lll} E_m & m = 1,2,\ldots p & \text{even vectors (in } V_0) \\ E_\mu & \mu = 1,2,\ldots q & \text{odd vectors (in } V_1) \end{array}$$

these form a pure basis $\{E_m, E_\mu\}$ for V. Our convention is that repeated indices are summed over their values. Latin indices such as $m, n, \cdots = 1,2,\ldots p$ while Greek indices such as $\alpha, \beta, \mu, \cdots = 1,2,\ldots,q$.

Theorem 3.4.9. *Assume that V is a supervector space and that*

$$\{e_1^0, e_2^0, \cdots, e_p^0, e_1^1, e_2^1, \cdots, e_q^1\}$$

and

$$\{f_1^0, f_2^0, \cdots, f_r^0, f_1^1, f_2^1, \cdots, f_s^1\}$$

are pure bases of V. Then $p = r$ and $q = s$.

Proof. Write each basis in terms of the other as follows:

$$\begin{array}{rl} e_m^0 & = f_n^0 A^n{}_m + f_\beta^1 C^\beta{}_m \\ e_\alpha^1 & = f_k^0 B^k{}_\alpha + f_\beta^1 D^\beta{}_\alpha \end{array} \tag{3.25}$$

$$\begin{array}{rl} f_n^0 & = e_k^0 X^k{}_n + e_\beta^1 Z^\beta{}_n \\ f_\alpha^1 & = e_k^0 Y^k{}_\alpha + e_\beta^1 W^\beta{}_\alpha \end{array} \tag{3.26}$$

where the coefficients are in Λ. Consider the equation

$$e_m^0 = f_n^0 A^n{}_m + f_\beta^1 C^\beta{}_m.$$

If the matrices $A = (A^n{}_m)$ and $C = (C^\beta{}_m)$ are not pure, then write both of them as the sum of their even and odd components to obtain

$$e_m^0 = f_n^0({}^0A^n{}_m) + f_\beta^1({}^1C^\beta{}_m) + f_n^0({}^1A^n{}_m) + f_\beta^1({}^1C^\beta{}_m).$$

The term $f_n^0({}^1A^n{}_m) + f_\beta^1({}^0C^\beta{}_m)$ is odd, and since e_m^0 is even, it must be zero. It follows that $e_m^0 = f_n^0({}^0A^n{}_m) + f_\beta^1({}^0C^\beta{}_m)$, and in this way we see that we could have chosen the matrices A, C at the outset such that all the entries of A are even and all the entries of C are odd. Similar arguments show that one may choose the matrices D, X, W, Z such that all the entries of the matrices $D = (D^\beta{}_\alpha), X = (X^k{}_n)$, and $W = (W^\beta{}_\alpha)$ are even while all the entries of the matrices $B = (B^k{}_\alpha), Y = (Y^\beta{}_\alpha)$, and $Z = (Z^\beta{}_n)$ are odd. If

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.27}$$

$$N = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \tag{3.28}$$

then in an obvious notation

$$(\ \vec{e}^0,\ \vec{e}^1) = (\ \vec{f}^0,\ \vec{f}^1)M \quad \textit{and} \quad (\ \vec{f}^0,\ \vec{f}^1) = (\ \vec{e}^0,\ \vec{e}^1)N. \tag{3.29}$$

It follows that

$$(\ \vec{e}^0,\ \vec{e}^1) = (\ \vec{e}^0,\ \vec{e}^1)NM \quad \textit{and} \quad (\ \vec{f}^0,\ \vec{f}^1) = (\ \vec{f}^0,\ \vec{f}^1)MN \tag{3.30}$$

and consequently that

$$MN = \begin{pmatrix} I_r & 0 \\ 0 & I_s \end{pmatrix} \quad \textit{and} \quad NM = \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}. \tag{3.31}$$

Since the body mapping is a multiplicative homomorphism, we have

$$b(M)b(N) = \begin{pmatrix} I_r & 0 \\ 0 & I_s \end{pmatrix} \quad \textit{and} \quad b(N)b(M) = \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}. \tag{3.32}$$

But

$$b(M) = \begin{pmatrix} b(A) & 0 \\ 0 & b(D) \end{pmatrix} \textit{ and } b(N) = \begin{pmatrix} b(X) & 0 \\ 0 & b(W) \end{pmatrix} \tag{3.33}$$

and consequently

$$b(M)b(N) = \begin{pmatrix} b(A)b(X) & 0 \\ 0 & b(D)b(W) \end{pmatrix} \quad b(N)b(M) = \begin{pmatrix} b(X)b(A) & 0 \\ 0 & b(W)b(D) \end{pmatrix}.$$

It follows that $b(A)b(X) = I_r, b(D)b(W) = I_s$ and $b(X)b(A) = I_p, b(W)b(D) = I_q$. Thus $r = tr(b(A)b(X)) = tr(b(X)b(A)) = p$ and $s = tr(b(D)b(W)) = tr(b(W)b(D)) = q$. The theorem follows. □

It follows that (p, q) is an invariant of the supervector space V with total dimension $d = p + q$, thus the (p, q) dimensionality of V is well-defined.

Corollary 3.4.10. *Suppose that $\mathcal{S} = {}^0\mathcal{S} \oplus {}^1\mathcal{S}$ with ${}^0\mathcal{S} \neq 0$ and ${}^1\mathcal{S} \neq 0$. If V is a $\mathcal{S}$-supervectorspace which is (p,q)-dimensional, then there is a ${}^0\mathcal{S}$-linear isomorphism of V_0 and $\mathbb{K}^{p|q}$ and an ${}^0\mathcal{S}$-linear isomorphism of V_1 and $\mathbb{K}^{\bar{p}|\bar{q}}$.*

Proof. We know there exists a pure basis of well defined superdimension (p,q) from theorem 3.4.9. Let $X \in V_0$ then with respect to the pure basis we find,

$$X = y^m E_m + \theta^\mu E_\mu \tag{3.34}$$

where $y^m \in \mathbb{K}_c$ and $\theta^\mu \in \mathbb{K}_a$ since $\epsilon(\alpha Y) = \epsilon(\alpha) + \epsilon(Y)$. Observe the mapping

$$y^m E_m + \theta^\mu E_\mu \longrightarrow (y^1, y^2, \ldots, y^p, \theta^1, \theta^1, \ldots, \theta^q)$$

provides the isomorphism $V_0 = \mathbb{K}^{p|q}$. The proof that V_1 and $\mathbb{K}^{\bar{p}|\bar{q}}$ are ${}^0\mathcal{S}$-linear isomorphic is similar. □

3.4.2 Pure-Real Basis

Theorem 3.4.11. *If V is a super vector space of dimension (p,q), then there exists a pure basis $\{F_m, F_\mu\}$ of V such that $F_M^* = F_M$ for $M = m$ or $M = \mu$. Such a basis will be called a pure-real basis.*

Proof. The proof is analogous to the previous argument for constructing a pure basis. First recall that there exists a pure basis $\{E_m, E_\mu\}$ for V and notice that each pure basis vector can be broken into a real and imaginary part,

$$E_M = \tfrac{1}{2}(E_M + E_M^*) + \tfrac{1}{2}(E_M - E_M^*) = R_M + J_M \tag{3.35}$$

where $R_M = \frac{1}{2}(E_M + E_M^*)$ and $J_M = iI_M = \frac{1}{2}(E_M - E_M^*), 1 \leq M \leq d$. It is straightforward to verify that $R_M^* = R_M$ and $J_M^* = -J_m$, thus $I_M^* = I_M$. Let S denote the set $\{R_M, J_M\}$ containing 2d vectors. Note that because $R_M, J_M \in V$ which has basis $\{E_M\}$, there must exist super matrices F and G such that,

$$\begin{aligned} R_M &= F_M{}^N E_N \\ J_M &= G_M{}^N E_N \end{aligned} \tag{3.36}$$

hence,

$$\begin{aligned} E_M &= F_M{}^N E_N + G_M{}^N E_N \\ &= (F_M{}^N + G_M{}^N) E_N \end{aligned} \tag{3.37}$$

It should be clear from arguments analogous to those in the proof of the last theorem that the matrix $[b(F)|b(G)]$ has rank d. As before we can choose d linearly independent columns from $[F|G]$ to form an invertible $d \times d$ matrix $\mathcal{B}$. In particular, denoting the i^{th} column in F by F_i, we construct $\mathcal{B}$ as follows

$$\mathcal{B} = [F_{i_1} \mid F_{i_2} \mid \ldots \mid G_{i_{r+1}} \mid G_{i_{r+2}} \mid \ldots \mid G_{i_d}]. \tag{3.38}$$

We can use this matrix to change our basis to a new basis $\{\tilde{E}_M\}$ which is partly real and partly imaginary, keeping the ordering as in our construction of $\mathcal{B}(\mathcal{B}_1 = F_{i_1}$ and so on...). Define

$$\begin{array}{ll} \tilde{E}_M &= \mathcal{B}_M{}^N E_N \qquad M = 1, 2, \ldots r \\ \tilde{E}_M &= \mathcal{B}_M{}^N E_N \qquad M = r+1, r+2, \ldots d \end{array} \tag{3.39}$$

then by the very definitions of the matrices F and G we can verify that the first r vectors are real and the last $d - r$ vectors above are imaginary,

$$\begin{array}{ll} \tilde{E}_M &= \mathcal{B}_M{}^N E_N = F_{i_M}{}^N E_N = R_{i_M} \qquad M = 1, 2, \ldots r \\ \tilde{E}_M &= \mathcal{B}_M{}^N E_N = G_{i_M}{}^N E_N = J_{i_M} \qquad M = r+1, r+2, \ldots d. \end{array} \tag{3.40}$$

Finally we construct the pure real basis as follows,

$$\begin{array}{ll} F_M = \tilde{E}_M = R_{i_M} \qquad M = 1, 2, \ldots r \\ F_M = i\tilde{E}_M = I_{i_M} \qquad M = r+1, r+2, \ldots d \end{array} \tag{3.41}$$

The manner in which we constructed this basis guarantees that it is pure as well as real. However we may have altered the canonical ordering of the pure basis. We like to put the p-even basis vectors first, then the q-odd vectors last in the ordering. This poses no real difficulty because at the end of the proof above we can simply permute the ordering to the standard order and obtain a canonically ordered pure, real basis. Notice that our choice of d linearly independent vectors had to respect the dimension (p, q) of V since we previously proved that any pure basis must have the same number of even and odd vectors. Thus we have shown that we can always choose a pure basis which is also real for any finite dimensional super vector space V. □

3.4.3 Standard Basis

Pure-real bases are useful,but they have some curious properties. A (p, q)-dimensional pure-real basis $\{E_m, E_\mu\}$ has $E_M^* = E_M$ and $\epsilon(E_M) = \epsilon_M$. We expand $X \in V_{0\mathbb{R}}$ with respect to the basis as follows (sum over $m = 1, 2, \ldots, p$ and $\mu = 1, 2, \ldots, q$)

$$X = X^m E_m + X^\mu E_\mu \tag{3.42}$$

Clearly $X^m \in \mathbb{C}_c$ and $X^\mu \in \mathbb{C}_a$ as X is even. Observe,

$$\begin{array}{ll} X^* &= (X^m E_m)^* + (X^\mu E_\mu)^* \\ &= E_m (X^m)^* + E_\mu (X^\mu)^* \\ &= E_m X^m - X^\mu E_\mu \end{array} \tag{3.43}$$

Suppose X is real, then $X^* = X = X^m E_m + X^\mu E_\mu$. We find, with respect to a pure-real basis, the odd components of an even-real supervector are pure imaginary:

$$(X^m)^* = X^m \qquad (X^\mu)^* = -X^\mu. \tag{3.44}$$

Definition 3.4.12. *Given a pure-real basis $\{E_m, E_\mu\}$, introduce a corresponding standard basis $\{\mathcal{E}_m, \mathcal{E}_\mu\}$ defined by $\mathcal{E}_m = E_m$ and $\mathcal{E}_\mu = iE_\mu$.*

3.4.4 Complex Supervector Spaces

Proposition 3.4.13. *Let $V = V_0 \oplus V_1$ be a supervector space over $\mathcal{S} = \Lambda(\mathbb{C})$ with total dimension d then there exist bijections that establish the following correspondences,*

$$V_0 \mapsto \mathbb{C}^{p|q} \qquad V_{0\mathbb{R}} \mapsto \mathbb{R}^{p|q} \qquad V_{\mathbb{R}} \mapsto \Lambda_{\mathbb{R}}(p,q)$$

Proof. Our proof makes use of the various special bases we have defined in the preceding sections. Notice we already established that $V_0 \mapsto \mathbb{C}^{p|q}$ in Corollary 3.4.10.

Let $X \in V_{0\mathbb{R}}$, then with respect to the standard basis we find,

$$X = x^m \mathcal{E}_m + \theta^\mu \mathcal{E}_\mu \tag{3.45}$$

X is even so we know that $x^m \in \mathbb{C}_c$ and $\theta^\mu \in \mathbb{C}_a$. Also, note $\mathcal{E}_m^* = E_m^* = E_m$ and $\mathcal{E}_\mu^* = (iE_\mu)^* = -iE_\mu = -\mathcal{E}_\mu$. Consider that

$$\begin{aligned} X^* &= (x^m \mathcal{E}_m)^* + (\theta^\mu \mathcal{E}_\mu)^* \\ &= (\mathcal{E}_m)^*(x^m)^* + (\mathcal{E}_\mu)^*(\theta^\mu)^* \\ &= \mathcal{E}_m (x^m)^* - \mathcal{E}_\mu (\theta^\mu)^* \\ &= (x^m)^* \mathcal{E}_m + (\theta^\mu)^* \mathcal{E}_\mu. \end{aligned} \tag{3.46}$$

If $X^* = X$, then $(x^m)^* \mathcal{E}_m + (\theta^\mu)^* \mathcal{E}_\mu = x^m \mathcal{E}_m + \theta^\mu \mathcal{E}_\mu$. We find that all the components of an even real vector are real with respect to the standard basis; that is $x^m \in \mathbb{R}_c$ and $\theta^\mu \in \mathbb{R}_a$. The correspondence is given via the standard basis,

$$x^m \mathcal{E}_m + \theta^\mu \mathcal{E}_\mu \longrightarrow (x^1, x^2, \ldots, x^p, \theta^1, \theta^1, \ldots, \theta^q).$$

The proof that $V_{\mathbb{R}} \mapsto \Lambda_{\mathbb{R}}(p,q) = \mathbb{R}^{p|q} \oplus \mathbb{R}^{\bar{p}|\bar{q}}$ follows by almost the same argument. □

We now state another useful correspondence for a complex supervector space.

Proposition 3.4.14. *Let V be a (p,q)-dimensional complex supervector space, then V is also a $(2p, 2q)$ dimensional real supervector space over $\Lambda_{\mathbb{R}} = \mathbb{R}_c \oplus \mathbb{R}_a$.*

Proof. Theorem 3.4.11 shows there exists a (p,q)-dimensional pure real basis $\{E_m, E_\mu\}$ for V. Let $Z \in V$ then there exist $Z^m, Z^\mu \in \Lambda(\mathbb{C})$ such that $Z = Z^m E_m + Z^\mu E_\mu$. Furthermore, there exist $x^m, y^m, \theta^\mu, \phi^\mu \in \Lambda_{\mathbb{R}}$ such that $Z^m = x^m + iy^m$ and $Z^\mu = \theta^\mu + i\phi^\mu$.

Observe that

$$\begin{aligned} Z &= (x^m + iy^m)E_m + (\theta^\mu + i\phi^\mu)E_\mu \\ &= x^m E_m + y^m iE_m + \theta^\mu E_\mu + \phi^\mu iE_\mu \end{aligned} \tag{3.47}$$

Thus the $\Lambda_{\mathbb{R}}$-span of the supervectors $\{E_m, iE_m, E_\mu, iE_\mu\}$ generates V. Moreover, E_M and iE_M are linearly independent with respect to $\Lambda_{\mathbb{R}}$. This shows that V is a real supervector space. For each $m = 1, 2, \ldots, p$ both E_m and iE_m are even while for each $\mu = 1, 2, \ldots, q$ both E_μ and iE_μ are odd. Thus as a super-real supervector space we find V is $(2p, 2q)$-dimensional. □

3.4.5 Super Dimension and Coordinates

When the supervector space V has mixed superscalars $\mathcal{S}$ (${}^1\mathcal{S} \neq 0$) and finite total dimension d then we can find a pure basis of V so the usual idea of dimension makes good sense. We defined total dimension d to be the number of vectors in a basis for V. Moreover, we saw that there exists a pure basis with p-even vectors and q-odd vectors such that $d = p + q$. So we say such a supervector space is (p, q)-dimensional. We define a *coordinate map* for V to be an $\mathcal{S}$-isomorphism of V and $\mathcal{S}^{p+q}$.

A common source of trouble is the vexing fact that $V = \mathbb{K}^{p|q}$ has no pure basis. As a supervector space the grading on $\mathbb{K}^{p|q}$ is somewhat trivial; $V = \mathbb{K}^{p|q}$ with $V_0 = \mathbb{K}^{p|q}$ and $V_1 = 0$. The superscalars for V are $\mathbb{K}_c$ so there is no way to obtain vectors such as $(0, \ldots, \theta^1, \ldots, \theta^q)$ from a finite minimal spanning set. We note that it is possible to have a supervector space with commuting superscalars and a nontrivial odd part, but we will not make use of such spaces.

To remedy the missing basis shortcoming [68] chose to define dimension of supervector spaces in terms of the coordinate maps. If the coordinate maps go to $\mathbb{K}^{p|q}$ then the dimension is $(p|q)$. If the coordinate maps go to $\mathbb{K}^{\bar{p}|\bar{q}}$ then the dimension is $(\bar{p}|\bar{q})$.

Definition 3.4.15. *Let V be a $\mathcal{S}$-supervector space with $\mathcal{S} = {}^0\mathcal{S}$ then*

1. *V is $(p|q)$-dimensional iff there is an ${}^0\mathcal{S}$-isomorphism from V to $({}^0\mathcal{S})^p \times ({}^1\mathcal{S})^q$.*
2. *V is $(\bar{p}|\bar{q})$-dimensional iff there is an ${}^0\mathcal{S}$-isomorphism from V to $({}^1\mathcal{S})^p \times ({}^0\mathcal{S})^q$.*

In each case we say that such isomorphisms are coordinate maps and that the supervector space is either $(p|q)$ or $(\bar{p}|\bar{q})$ dimensional. Moreover, we insist that all the coordinate maps for a particular supervector space share the same super dimension.

The distinction between $(p|q)$ and $(\bar{q}|\bar{p})$ is just one of ordering. We refer to $(\bar{p}|\bar{q})$-dimensional supervector spaces as *bizarro supervector spaces* since the usual ordering is backwards.

Definition 3.4.16. *Let V be a $\mathcal{S}$-supervector space. Suppose $v \in V$ and Ψ is a coordinate mapping on V, then $\Psi(v)$ are the coordinates of v. In particular,*

1. *If V is $(p|q)$ dimensional then $\Psi(v) = (v^m, v^\alpha) \in \mathbb{K}^{p|q} = ({}^0\mathcal{S})^p \times ({}^1\mathcal{S})^q$,*
2. *If V is $(\bar{p}|\bar{q})$ dimensional then $\Psi(v) = (v^\alpha, v^m) \in \mathbb{K}^{\bar{p}|\bar{q}} = ({}^1\mathcal{S})^p \times ({}^0\mathcal{S})^q$,*
3. *If V is (p, q) dimensional then $\Psi(v) = (v^1, \ldots, v^{p+q}) \in \mathcal{S}^{p+q}$.*

One should notice that $\mathbb{K}^{p|q}$ is not a left $\Lambda(\mathbb{K})$-module, multiplication by $\mathbb{K}_a$ distorts the structure of $\mathbb{K}^{p|q}$. It is important to distinguish the difference between dimension (p, q) and dimension $(p|q)$. It is fairly obvious that we can obtain left Λ modules from graded vector spaces by simply tensoring with Λ. However not all left Λ-modules have such structure (see Example 4.2a in [98]). Hence the class of left Λ modules is larger than that of graded vector spaces.

Proposition 3.4.17. *Let $\mathcal{S} \in \{\Lambda(\mathbb{C}), \Lambda_\mathbb{R}, \Lambda(\mathbb{R})\}$ and suppose $V = V_0 \oplus V_1$ is a (p, q) dimensional $\mathcal{S}$-supervector space. It follows that V_0 is a $(p|q)$ dimensional ${}^0\mathcal{S}$-supervector space.*

Proof. Clearly V_0 is closed under ${}^0\mathcal{S}$ superscalar multiplication. Pick a pure basis $\{E_m, E_\alpha\}$ in V then observe $v \in V_0$ has the basis expansion $v = E_m v^m + E_\alpha v^\alpha$ with $(v^m, v^\alpha) \in ({}^0\mathcal{S})^p \times ({}^1\mathcal{S})^q$ thus we define the obvious coordinate map on V_0; $\psi(v) = (v^m, v^\alpha)$. Thus V_0 is a $(p|q)$-dimensional ${}^0\mathcal{S}$-supervector space. □

On the other hand if we are given a supervector space over commuting supernumbers, we are free to enlarge it to a supervector space over mixed supernumbers of the same type.

Proposition 3.4.18. *Let ${}^0\mathcal{S} \in \{{}^0\Lambda(\mathbb{C}), {}^0\Lambda_\mathbb{R}, {}^0\Lambda(\mathbb{R})\}$ and suppose V_0 is a $(p|q)$ dimensional ${}^0\mathcal{S}$-supervector space. Then there exists a (p, q) dimensional $\mathcal{S}$-supervector space $V = V_0 \oplus V_1$.*

Proof. Let $V = V_0 \oplus (({}^1\mathcal{S})^p \times ({}^0\mathcal{S})^q)$ where we define $V_1 = ({}^1\mathcal{S})^p \times ({}^0\mathcal{S})^q$. We have by assumption a coordinate map $\psi_0 : V_0 \to ({}^0\mathcal{S})^p \times ({}^1\mathcal{S})^q$. Define $\psi = \psi_0 + id_{({}^1\mathcal{S})^p \times ({}^0\mathcal{S})^q}$. Then $\psi : V \to \mathcal{S}^{p+q}$. Construct a pure basis of dimension (p, q) via the inverse images of the canonical basis in $\mathcal{S}^{p+q}$, $\{\psi^{-1}(e_m), \psi^{-1}(e_\alpha)\}$. It follows V is a (p, q)-dimensional $\mathcal{S}$-supervector space. □

Remark 3.4.19. *Throughout this section we denoted the points as row vectors. It will sometimes be the case that we mean for these spaces to be constructed with column vectors. The meaning should be clear from the context.*

3.5 Linear Operators

In this section we study the basic types of linear operators on supervector spaces. A linear operator with respect to $\mathbb{K}$ which is also a Λ-module morphism is called a left or right linear operator. Mappings on supervector spaces may preserve or distort the parity of their inputs. Parity preserving maps are called even while parity changing maps are called odd. We show how any left-linear map can be written as the sum of an even and odd linear operator. The parity map $\mathcal{P}$ is an involution which is neither a left nor a right linear operator. Finally, we examine how the endomorphisms of a supervector space form an associative superalgebra.

3.5.1 Left and Right Linearity

We have observed there is a distinction between left and right linear independence, spanning and scalar multiplication. Not surprisingly there is also a distinction between left and right linear operators.

Definition 3.5.1. *Given bimodules or supervector spaces V, W over $\mathcal{S}$ we say that L is a left-$\mathcal{S}$-linear operator if $L : V \longrightarrow W$ satisfies*

1. $L(X + Y) = L(X) + L(Y)$
2. $L(X\alpha) = L(X)\alpha$

for all $X, Y \in V$ and $\alpha \in \mathcal{S}$. Likewise, $R : V \longrightarrow W$ is a right-$\mathcal{S}$-linear operator if it satisfies

1. $(X + Y)R = (X)R + (Y)R$
2. $(\alpha X)R = \alpha(X)R$

for all $X, Y \in V$ and $\alpha \in \mathcal{S}$. The notation $(X)R$ may be replaced with $R(X)$ in which case we have $R(\alpha X) = \alpha R(X)$. The set of all left-$\mathcal{S}$-linear operators from V to W is denoted by $L^+(V, W)$. Likewise, the set of all right-$\mathcal{S}$-linear operators from V to W is $L^-(V, W)$. Right and left linear mappings defined above may also be referred to as $\mathcal{S}$-bimodule homomorphisms. Left endomorphisms on V are denoted $L^+(V, V) = End^+(V)$ and right endomorphisms are denoted by $End^-(V)$.

Notice that right operators act to the left while left operators act to their right. Also left linear operators allow $\mathcal{S}$-scalars to pull out on the right without any extra signs, whereas right linear operators allow $\mathcal{S}$-scalars to pull out on the left without any extra signs. We are following the notation in [29] where Buchbinder and Kuzenko use "+" for left and "-" for right.

Definition 3.5.2. *The usual $\mathbb{K}$-linear operators from V to W will be denoted $L(V, W)$.*

3.5.2 Parity of Linear Mappings

Up to now we have assigned parity to particular super numbers and super vectors. We now discuss how to assign a parity to particular types of left linear operators. As was the case for super numbers and vectors we will also find that every left linear operator can be decomposed into an even and an odd left linear operator.

Definition 3.5.3. *The parity mapping $\mathcal{P} \in L(V,V)$ as follows,*

$$\mathcal{P}(X) = \mathcal{P}({}^0X + {}^1X) = {}^0X - {}^1X \tag{3.48}$$

Clearly the parity map is $\mathbb{K}$-linear and it also has the following nice properties, assuming $\alpha \in \Lambda$ is pure,

$$\begin{array}{l} \mathcal{P}^2(X) = X \\ \mathcal{P}(\alpha X) = (-1)^{\epsilon(\alpha)}\alpha\mathcal{P}(X) \\ \mathcal{P}(X\alpha) = (-1)^{\epsilon(\alpha)}\mathcal{P}(X)\alpha \end{array} \tag{3.49}$$

Thus we see that the parity mapping is neither left nor a right mapping.

Let $L \in L^+(V,W)$ and observe that (we use the same symbol $\mathcal{P}$ for both V and W)

$$L \ = \tfrac{1}{2}(L + \mathcal{P}L\mathcal{P}) + \tfrac{1}{2}(L - \mathcal{P}L\mathcal{P}) \tag{3.50}$$

Defining,

$$\begin{array}{rl} {}^0L & = \frac{1}{2}(L + \mathcal{P}L\mathcal{P}) \\ {}^1L & = \frac{1}{2}(L - \mathcal{P}L\mathcal{P}) \end{array} \tag{3.51}$$

We claim that ${}^0L$ preserves the parity of the vectors on which it acts, whereas ${}^1L$ changes the parity of the vectors on which it acts; that is,

$$\begin{array}{lll} a.) & {}^0L & : {}^0V \longrightarrow {}^0W \\ b.) & {}^0L & : {}^1V \longrightarrow {}^1W \\ c.) & {}^1L & : {}^0V \longrightarrow {}^1W \\ d.) & {}^1L & : {}^1V \longrightarrow {}^0W \end{array} \tag{3.52}$$

Let us prove d. Let $X \in {}^1V$, consider then

$$\begin{array}{rl} {}^1L(X) & = \frac{1}{2}(\ L(X) - \mathcal{P}(L(\mathcal{P}(X)))\ ) \\ & = \frac{1}{2}(\ L(X) - \mathcal{P}(L({}^0X - {}^1X))\ ) \\ & = \frac{1}{2}(\ L({}^1X) + \mathcal{P}(L({}^1X))\ ) \\ & = \frac{1}{2}(\ L(X) + {}^0(L(X)) - {}^1(L(X))\ ) \\ & = \frac{1}{2}(\ {}^0(L(X)) + {}^1(L(X)) + {}^0(L(X)) - {}^1(L(X))\ ) \\ & = {}^0(L(X)) \end{array} \tag{3.53}$$

This establishes that ${}^1L(X) \in {}^0V$ which is precisely what we set out to prove. The other cases follow from very similar arguments which we leave to the reader. We thus define the parity of a linear mapping to be zero if it preserves the parity of pure

vectors or one if it changes even to odd and odd to even. In short $\epsilon({}^0L) = 0$ while $\epsilon({}^1L) = 1$ so we can summarize,

Definition 3.5.4. *If $L \in L^+(V,W)$ and $L(V_r) \subseteq V_s$ then we say L is a pure left linear operator. The parity of a pure operator L is denoted $\epsilon(L)$ and we have that*

$$\epsilon(L(X)) = \epsilon(L) + \epsilon(X) \tag{3.54}$$

for all pure super vectors $X \in V_0 \cup V_1$. For future reference, we may also refer to even as commuting or bosonic and odd as anticommuting or fermionic.

Remark 3.5.5. *The parity of a mapping is usually defined according to one of two rules,*

1. *Even mappings map even elements to even elements and odd to odd. Odd mappings map even elements to odd elements and odd to even. Here we must assume that the domain and range have a $\mathbb{Z}_2$-grading.*

2. *Even mappings map <u>all</u> elements to even elements. Odd mappings map <u>all</u> elements to odd elements. Here we need not assume the domain has any $\mathbb{Z}_2$-grading. Typically this idea is used when the domain is not $\mathbb{Z}_2$-graded.*

We will encounter the second grading in a later chapter.

3.5.3 Endomorphisms $End^+(V)$ Form a Super Algebra

$End^+(V)$ is itself a super vector space. We define addition and scalar multiplication pointwise, let $L, L_1, L_2 \in End^+(V)$

$$\begin{array}{lll} a.\ (L_1 + L_2)(X) = L_1(X) + L_2(X) & \forall\ X \in V & \\ b.\ (\alpha L)(X) = \alpha L(X) & \forall\ X \in V,\ \alpha \in \Lambda & \\ c.\ (L\alpha)(X) = L(\alpha X) & \forall\ X \in V,\ \alpha \in \Lambda & \end{array} \tag{3.55}$$

Clearly, the definitions above insure that $L_1 + L_2, \alpha L, L\alpha \in End^+(V)$. Next we define a multiplication on $End^+(V)$ by composition, let $L_1, L_2 \in End^+(V)$,

$$(L_1 L_2)(X) = L_1(L_2(X)) \qquad \forall\ X \in V \tag{3.56}$$

A vector space with a multiplication is an algebra, a super vector space with a $\mathbb{Z}_2$-graded multiplication is a super algebra. The parity of the composite follows the rule

$$\epsilon(L_1 \circ L_2) = \epsilon(L_1) + \epsilon(L_2).$$

Hence, $End^+(V)$ is a superalgebra. By similar arguments the right endomorphisms $End^-(V)$ form a superalgebra.

3.6 Matrix Calculations for Supervector Spaces

Matrix calculations are an essential tool in super linear algebra. We begin by briefly reminding the reader of the block matrix construction of $gl(p,q,\mathbb{C})$ which is a $\mathbb{Z}_2$-graded algebra over $\mathbb{C}$. Next $gl(m\times n,\Lambda)$ is defined to be matrices of supernumbers. These allow a different $\mathbb{Z}_2$-grading which is independent of the block structure of the matrix. We discuss matrix multiplication, addition, superscalar multiplication and super conjugation for $gl(m\times n,\Lambda)$. We find that the traditional definitions naturally endow $gl(m\times n,\Lambda)$ with the structure of a supervector space over Λ. The generalization of $gl(p,q,\mathbb{C})$ is $gl(p,q,\Lambda)$. As point sets $gl(p,q,\Lambda)=gl(p\times q,\Lambda)$, however they have distinct $\mathbb{Z}_2$-gradings so we must take care to distinguish them. We give definitions of superscalar multiplication and conjugation which give $gl(p,q,\Lambda)$ the structure of a supervector space. Moreover, we find that the matrices in $gl(p,q,\Lambda)$ are naturally induced from left linear operators with respect to a pure basis on a (p,q)-dimensional Λ-supervector space. Nonsingular supermatrices are studied. We find that a supermatrix is invertible iff it has an invertible body. Finally we discuss how even supermatrices in $gl(p,q,\Lambda)$ induce ${}^0\Lambda$-linear mappings on $(p|q)$ dimensional supervector spaces. In short, all the various types of linear mappings have matrix representations. This is interesting given that the $(p|q)$-dimensional supervector spaces have no basis. Fortunately, $(p|q)$-dimensional supervector spaces do possess coordinate maps, and we make use of those to bypass the basis concept.

3.6.1 Graded complex matrices, $gl(p,q,\mathbb{C})$

As a set $gl(p,q,\mathbb{C})$ is simply all $(p+q)\times(p+q)$ matrices with complex entries. We give $gl(p,q,\mathbb{C})$ a $\mathbb{Z}_2$-grading as follows: let $M\in gl(p,q,\mathbb{C})$ then

$$M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\qquad {}^0M=\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}\qquad {}^1M=\begin{pmatrix}0 & B\\ C & 0\end{pmatrix} \tag{3.57}$$

Where $A\in gl(p^2,\mathbb{C}),\ B\in gl(p\times q,\mathbb{C}),\ C\in gl(q\times p,\mathbb{C}),\ D\in gl(q^2,\mathbb{C})$. Clearly, for each $M\in gl(p,q,\mathbb{C})$ we have $M={}^0M+{}^1M$. This is enough to show that the vector space $gl(p,q,\mathbb{C})$ is a graded vector space. Define $gl(p,q,\mathbb{C})^0$ to be all even matrices in $gl(p,q,\mathbb{C})$ and $gl(p,q,\mathbb{C})^1$ to be all odd matrices in $gl(p,q,\mathbb{C})$. A short calculation will reveal,

$$gl(p,q,\mathbb{C})^r gl(p,q,\mathbb{C})^s\subset gl(p,q,\mathbb{C})^{r+s}\qquad mod(2). \tag{3.58}$$

Consequently, $gl(p,q,\mathbb{C})$ is an associative graded algebra over $\mathbb{C}$. It is not hard to endow these matrices with a non-associative multiplication analogous to the commutator bracket.

$$[M,N]=MN-(-1)^{\epsilon(M)\epsilon(N)}NM. \tag{3.59}$$

This gives an anticommutator when both matrices are odd ($\epsilon(M)=\epsilon(N)=1$), or it gives a commutator when either of the matrices is even ($\epsilon(M)=0$ or $\epsilon(N)=0$).

Again a short calculation will demonstrate,

$$[gl(p,q,\mathbb{C})^r, gl(p,q,\mathbb{C})^s] \subset gl(p,q,\mathbb{C})^{r+s} \qquad mod(2). \tag{3.60}$$

Often such an algebra with such a $\mathbb{Z}_2$-graded bracket over $\mathbb{C}$ is called a Lie superalgebra. All the finite dimensional semisimple Lie superalgebras in this sense have been classified (see [69] or [43]). We prefer to call these algebras $\mathbb{Z}_2$-graded Lie algebras over $\mathbb{C}$, or simply graded Lie algebras, since we will use the term Lie superalgebra to mean something quite different from this concept.

3.6.2 Super Matrices, $gl(m \times n, \Lambda)$

We will denote the set of all $m \times n$ arrays of supernumbers by $gl(m \times n, \Lambda)$. The set of all $n \times n$ square super matrices will be denoted by $gl(n, \Lambda)$. Addition and scalar multiplication of super matrices in $gl(m \times n, \Lambda)$ are defined as in the usual case,

$$\begin{array}{ll} 1.)\ (A+B)_{ij} & = A_{ij} + B_{ij} \\ 2.)\ (\alpha A)_{ij} & = \alpha A_{ij}. \\ 3.)\ (A\alpha)_{ij} & = A_{ij}\alpha. \end{array} \tag{3.61}$$

In the case that $m = n$ we define the product AB as usual by

$$(AB)_{ij} = A_{ik}B_{kj} \qquad sum\ over\ k \tag{3.62}$$

A matrix $A \in gl(n, \Lambda)$ is pure if and only if all of its entries are pure and share the same parity, in which case the parity of a pure supermatrix A in $gl(n, \Lambda)$ is defined by,

$$\epsilon(A) = \epsilon(A_{ij}). \tag{3.63}$$

Notice that every super matrix can be split into an even and odd part by splitting the components into anticommuting and commuting pieces, $A = {}^0A + {}^1A$ where if $A_{ij} = (A_{ij})_c + (A_{ij})_a$ with $(A_{ij})_c \in {}^0\Lambda$ and $(A_{ij})_a \in {}^1\Lambda$ for all i, j, then

$$({}^0A)_{ij} = (A_{ij})_c \qquad \text{and} \qquad ({}^1A)_{ij} = (A_{ij})_a. \tag{3.64}$$

Conjugation is also defined in a natural way for $gl(n, \Lambda)$; $(A^*)_{ij} = (A_{ij})^*$. Notice that,

$$\begin{array}{ll} (zA)^*{}_{ij} & = ((zA)_{ij})^* \\ & = (zA_{ij})^* \\ & = (A_{ij})^* z^* \\ & = (A^*)_{ij} z^* \\ & = (A^* z^*)_{ij} \implies (zA)^* = A^* z^* \quad (anti - involution). \end{array} \tag{3.65}$$

It is not difficult to show that $gl(m\times n,\Lambda)$ is a supervector space. In fact $gl(m\times n,\Lambda)$ is isomorphic to Λ^{mn} as a Λ-bimodule under the mapping ϕ defined by,

$$\phi(A)=(A_{11},A_{12},\ldots,A_{1n},A_{21},\cdots,A_{2n},\cdots,A_{mn}). \tag{3.66}$$

Indeed, $\phi(\lambda A\mu)=(\lambda A_{11}\mu,\lambda A_{12}\mu,\ldots,\lambda A_{mn}\mu)=\lambda(A_{11},A_{12},\ldots,A_{mn})\mu=\lambda\phi(A)\mu$, for $A\in gl(m\times n,\Lambda)$ and $\lambda,\mu\in\Lambda$. Moreover, $\phi(A)^*=\phi(A^*)$ and in view of the isomorphism a norm may be induced on $gl(m\times n,\Lambda)$ via the equation,

$$||A||=||\phi(A)||. \tag{3.67}$$

With this definition of the norm of a supermatrix we can easily deduce that $gl(m\times n,\Lambda)$ is a Banach space. In fact, when $A\in gl(m\times p,\Lambda)$ and $B\in gl(p\times n,\Lambda)$ we can prove

$$||AB||\leq||A||\;||B||. \tag{3.68}$$

Notice that in the special case that $m=n$ we denote $gl(m\times n,\Lambda)$ by simply $gl(n,\Lambda)$ and in this case, $gl(n,\Lambda)$ is an associative graded algebra, i.e.,

$$gl(n,\Lambda)^r gl(n,\Lambda)^s\subset gl(n,\Lambda)^{r+s}\qquad mod(2). \tag{3.69}$$

where addition of r,s is modulo $\mathbb{Z}_2$. It is also a Lie superalgebra in the sense that

$$[gl(n,\Lambda)^r,gl(n,\Lambda)^s]\subset gl(n,\Lambda)^{r+s}\qquad mod(2). \tag{3.70}$$

Notational Warnings

Warning 1: Scalar multiplication is not always done as in 3.) of Eqn. 3.61; there is another notion of scalar multiplication which is tailored to match up with the matrix of a left-linear operator. We will use juxtaposition to indicate the standard scalar multiplication, and later on we will introduce $\cdot$ to alert the reader when the other type of scalar multiplication is used. Also the notion of parity employed in $gl(n,\Lambda)$ is not the only one possible; we will use the notation $gl(p,q|\Lambda)$ to draw attention to the (p,q) type matrix parity. The matrices in $gl(p,q|\Lambda)$ have a different operation of conjugation as well.

Warning 2: The grading defined on $gl(m\times n,\Lambda)$ has the property that a matrix is even iff every entry in the matrix is even, and a matrix is odd iff every entry in the matrix is odd. In the next subsection we define another grading distinct from this. To distinguish between the two, the set of $p\times q$ matrices over Λ will be denoted by $gl(p,q,\Lambda)$ when this new grading is used. If A is a matrix, one will have to know whether A belongs to $gl(m\times n,\Lambda)$ or whether it belongs to $gl(p,q,\Lambda)$ in order to determine the meaning of ${}^0A$ and ${}^1A$ as the notions of even and odd in the two spaces are different. On the other hand, when we write A_c we will ALWAYS mean by this

notation the matrix A whose entries are all even supernumbers, and when we write A_a we will ALWAYS mean by this notation the matrix A whose entries are all odd supernumbers. In the supervector space $gl(m \times n, \Lambda)$, ${}^0A = A_c$ and ${}^1A = A_a$ but this is not true in the supervector space $gl(p, q, \Lambda)$ described below.

3.6.3 (p,q) Graded Super Matrices, $gl(p, q, \Lambda)$

As a set $gl(p, q, \Lambda)$ is simply all $(p+q) \times (p+q)$ matrices with supernumbers as entries. We give $gl(p, q, \Lambda)$ a $\mathbb{Z}_2$-grading as follows: let $M \in gl(p, q, \Lambda)$, then

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad {}^0M = \begin{pmatrix} A_c & B_a \\ C_a & D_c \end{pmatrix} \qquad {}^1M = \begin{pmatrix} A_a & B_c \\ C_c & D_a \end{pmatrix} \tag{3.71}$$

Where $A \in gl(p^2, \Lambda),\ B \in gl(p \times q, \Lambda),\ C \in gl(q \times p, \Lambda),\ D \in gl(q^2, \Lambda)$. Clearly, for each $M \in gl(p, q, \Lambda)$ we have $M = {}^0M + {}^1M$. This is enough to show that $gl(p, q, \Lambda)$ forms a graded vector space. Define $gl(p, q, \Lambda)^0$ to be all even matrices in $gl(p, q, \Lambda)$, and $gl(p, q, \Lambda)^1$ to be all odd matrices in $gl(p, q, \Lambda)$. A short calculation will reveal,

$$gl(p, q, \Lambda)^r gl(p, q, \Lambda)^s \subset gl(p, q, \Lambda)^{r+s} \qquad mod(2). \tag{3.72}$$

This makes $gl(p, q, \Lambda)$ an associative graded algebra over $\mathbb{C}$. It is not hard to endow these matrices with a non-associative multiplication analogous to the commutator bracket.

$$[M, N] = MN - (-1)^{\epsilon(M)\epsilon(N)} NM. \tag{3.73}$$

This gives an anticommutator when both matrices are odd, i.e., when $\epsilon(M) = \epsilon(N) = 1$. It gives a commutator when either of the matrices is even, i.e., when $\epsilon(M) = 0$ or $\epsilon(N) = 0$. A short calculation will demonstrate,

$$[gl(p, q, \Lambda)^r, gl(p, q, \Lambda)^s] \subset gl(p, q, \Lambda)^{r+s} \qquad mod(2). \tag{3.74}$$

In fact, this is supervector space if we define the scalar multiplication by supernumbers in the appropriate manner. These definitions may look obtuse, but the reader should note that these definitions are made to insure that later these are the matrices of certain linear transformations. Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.75}$$

denote an arbitrary matrix in $gl(p, q, \Lambda)$. We define a scalar multiplication and a conjugation of M. First define left scalar multiplication of $z \in \mathbb{C}_c \cup \mathbb{C}_a$ and $M \in gl(p, q, \Lambda)$ by

$$z \cdot M = \begin{pmatrix} zA & zB \\ (-1)^{\epsilon(z)} zC & (-1)^{\epsilon(z)} zD. \end{pmatrix} \tag{3.76}$$

Next, define right scalar multiplication of $z \in \mathbb{C}_c \cup \mathbb{C}_a$ and $M \in gl(p,q,\Lambda)$ by

$$M \cdot z = \begin{pmatrix} Az & (-1)^{\epsilon(z)}Bz \\ Cz & (-1)^{\epsilon(z)}Dz. \end{pmatrix} \tag{3.77}$$

It is sufficient to give these definitions for pure supernumbers since we may extend the definition to the general case by requiring that for any super number z, $z \cdot M = z_c \cdot M + z_a \cdot M$ and $M \cdot z = M \cdot z_c + M \cdot z_a$. Since $z(Aw) = (zA)w$ for supernumbers z, w and appropriate matrices A, it follows that $z \cdot (M \cdot w) = (z \cdot M) \cdot w$.
Finally, define super conjugation for a pure super matrix M,

$$M^{s*} = \begin{pmatrix} A^* & (-1)^{\epsilon(M)+1}B^* \\ (-1)^{\epsilon(M)}C^* & D^*. \end{pmatrix} \tag{3.78}$$

Again we extend this definition linearly for general $M = {}^0M + {}^1M$. There is some work to do to verify that $gl(p,q,\Lambda)$, with scalar multiplication and superconjugation defined above, is a supervector space. In particular, let M be a pure super matrix and let z be a pure supernumber, we show $M^{s*} \cdot z^* = (z \cdot M)^{s*}$. Consider the l.h.s,

$$\begin{aligned} M^{s*} \cdot z^* &= \begin{pmatrix} A^* & (-1)^{\epsilon(M)+1}B^* \\ (-1)^{\epsilon(M)}C^* & D^* \end{pmatrix} \cdot z^* \\ &= \begin{pmatrix} A^*z^* & (-1)^{\epsilon(M)+1+\epsilon(z)}B^*z^* \\ (-1)^{\epsilon(M)}C^*z^* & (-1)^{\epsilon(z)}D^*z^* \end{pmatrix} \end{aligned} \tag{3.79}$$

Where we have used the fact $\epsilon(z^*) = \epsilon(z)$. Next, consider the r.h.s,

$$\begin{aligned} (z \cdot M)^{s*} &= \begin{pmatrix} zA & zB \\ (-1)^{\epsilon(z)}zC & (-1)^{\epsilon(z)}zD \end{pmatrix}^{s*} \\ &= \begin{pmatrix} (zA)^* & (-1)^{\epsilon(z\cdot M)+1}(zB)^* \\ (-1)^{\epsilon(z)+\epsilon(z\cdot M)}(zC)^* & (-1)^{\epsilon(z)}(zD)^* \end{pmatrix} \\ &= \begin{pmatrix} A^*z^* & (-1)^{\epsilon(M)+1+\epsilon(z)}B^*z^* \\ (-1)^{\epsilon(M)}C^*z^* & (-1)^{\epsilon(z)}D^*z^* \end{pmatrix}. \end{aligned} \tag{3.80}$$

In the last step we used that $\epsilon(z \cdot M) = \epsilon(z) + \epsilon(M)$. Thus we have shown that $M^{s*} \cdot z^* = (z \cdot M)^{s*}$. We leave it as an exercise to the reader to finish the verification that $gl(p,q,\Lambda)$ is indeed a supervector space.

The scalar multiplication and superconjugation introduced for (p,q)-graded supermatrices is not the only one possible. Our construction here will give us matrices which correspond to left-linear operators relative to a pure basis. Alternatively, one can define another kind of conjugation and scalar multiplication that give matrices correspondant to right-linear operators relative to a pure basis. We content ourselves to focus on left-linear operators. These correspond more naturally to our traditional ideas about differentiation.

3.6.4 Invertible Supermatrices, $GL(n, \Lambda)$

The set of all invertible $n \times n$ matrices of supernumbers is denoted $GL(n, \Lambda)$. It is in fact a multiplicative super group. The body of a supermatrix is the matrix of the bodies, for $A \in gl(n, \Lambda)$

$$(b(A))_{ij} = b(A_{ij}) \tag{3.81}$$

Then it follows that

$$(b(AC))_{ij} = b((AC)_{ij}) = b(A_{ik}C_{kj}) = (b(A))_{ik}(b(C))_{kj} = (b(A)b(C))_{ij} \tag{3.82}$$

in matrix notation, $b(AC) = b(A)b(C)$. We often find it convenient to denote the body of A by A_B. Let $X \in GL(n, \Lambda)$ then there exists X^{-1} such that $XX^{-1} = 1 = X^{-1}X$. Note,

$$b(XX^{-1}) = b(X)b(X^{-1}) = b(1) = 1. \tag{3.83}$$

Therefore, if $X \in GL(n, \Lambda)$, then $b(X) \in GL(n, \mathbb{C})$. The converse is also true. Given $b(X) \in GL(n, \mathbb{C})$ we can construct $X \in GL(n, \Lambda)$ such its body is $b(X)$. Towards that construction, note that F_S is formed by taking the soul of each element in F. Then,

$$F = F_B + F_S. \tag{3.84}$$

Following the intuition of Section 2.8, construct the inverse of F as follows (assuming that F_B^{-1} exists),

$$F^{-1} = F_B^{-1} + \sum_{k=1}^{\infty}(-1)^k(F_B^{-1}F_S)^k F_B^{-1} \tag{3.85}$$

Now notice that $F = F_B + F_S = F_B(1 + F_B^{-1}F_S)$ hence,

$$\begin{aligned}
FF^{-1} &= F_B(1 + F_B^{-1}F_S)(F_B^{-1} + \textstyle\sum_{k=1}^{\infty}(-1)^k(F_B^{-1}F_S)^k F_B^{-1}) \\
&= F_B F_B^{-1} + F_B \textstyle\sum_{k=1}^{\infty}(-1)^k(F_B^{-1}F_S)^k F_B^{-1} \\
&\quad + F_B F_B^{-1} F_S F_b^{-1} + F_B F_B^{-1} F_S \textstyle\sum_{k=1}^{\infty}(-1)^k(F_B^{-1}F_S)^k F_B^{-1} \\
&= 1 + \textstyle\sum_{k=1}^{\infty}(-1)^k F_B(F_B^{-1}F_S)^k F_B^{-1} + F_B F_B^{-1} F_S F_b^{-1} \\
&\quad - \textstyle\sum_{k=1}^{\infty}(-1)^{k+1} F_B(F_B^{-1}F_S)^{k+1} F_B^{-1} \\
&= 1 + \textstyle\sum_{k=1}^{\infty}(-1)^k F_B(F_B^{-1}F_S)^k F_B^{-1} + F_B F_B^{-1} F_S F_b^{-1} \\
&\quad - \textstyle\sum_{k=2}^{\infty}(-1)^k F_B(F_B^{-1}F_S)^k F_B^{-1} \\
&= 1 + (-1)^{-1} F_B(F_B^{-1}F_S)^1 F_b^{-1} + F_B F_B^{-1} F_S F_b^{-1} \\
&= 1.
\end{aligned} \tag{3.86}$$

The question arises; does the calculation above make sense? In particular, does the series above converge to a supermatrix with finite norm? Notice that $gl(n,\Lambda)$ has implicit within its definition the requirement that the norms of the supermatrices are finite. We need this restriction to insure that we have an honest Banach space. The proof of convergence follows from purely Banach theoretic arguments (see [82]). Bruzzo and Cianci also give an interesting proof in [24] pages 215-216.

3.6.5 The Matrix of a Left Linear Operator

Given a pure basis $\{e_M\} = \{e_m, e_\mu\}$ for V, we can capture the action of any left linear operator on V by multiplication of some supermatrix. That is given $L \in End^+(V)$ we can find a supermatrix F such that,

$$L(e_N) = e_M F^M{}_N \tag{3.87}$$

We make no particular restriction on the components of the supermatrix. However we do introduce notation for the blocks of the matrix which respects the parity of the pure basis in use,

$$F^M{}_N = \begin{pmatrix} A^m{}_n & B^m{}_\nu \\ C^\mu{}_n & D^\mu{}_\nu \end{pmatrix} \tag{3.88}$$

The set of all such matrices is $gl(p,q,\Lambda)$. Let $X = e_M X^M_+ \in V$ and $L \in End^+(V)$

$$\begin{aligned} L(X) &= L(e_N X^N_+) \\ &= L(e_N) X^N_+ \\ &= e_M F^M{}_N X^N_+ \end{aligned} \tag{3.89}$$

Letting $X' = L(X)$ we can read off the left components of X' from the above,

$$X'^M_+ = F^M{}_N X^N_+ \tag{3.90}$$

3.6.6 Matrices of Left Linear Operators are in $gl(p,q,\Lambda)$

The matrix of $L_1 + L_2$ is found as follows. Suppose that F is the matrix of $L_1 \in End^+(V)$ and G is the matrix of $L_2 \in End^+(V)$, then consider,

$$\begin{aligned} (L_1 + L_2)(e_N) &= L_1(e_N) + L_2(e_N) \\ &= e_M F^M{}_N + e_M G^M{}_N \\ &= e_M (F^M{}_N + G^M{}_N) \end{aligned} \tag{3.91}$$

Therefore, $F + G$ is the matrix of $L_1 + L_2$.

The matrix of zL and Lz are found as follows. Suppose that $L \in End^+(V)$ and

let z be a pure supernumber,

$$\begin{aligned}(zL)(e_N) &= zL(e_N) \\ &= ze_M F^M{}_N \\ &= e_M(-1)^{\epsilon(z)\epsilon_M} zF^M{}_N\end{aligned} \tag{3.92}$$

Where we have introduced the shorthand $\epsilon_M = \epsilon(e_M)$. Thus, assuming that F has the same block decomposition as before in eq. 3.88

$$(zL)^M{}_N = \begin{pmatrix} zA^m{}_n & zB^m{}_\nu \\ (-1)^{\epsilon(z)} zC^\mu{}_n & (-1)^{\epsilon(z)} zD^\mu{}_\nu \end{pmatrix} = z \cdot (F^M{}_N) \tag{3.93}$$

where $z \cdot (L)^M_N$ refers to the left multiplication in $gl(p, q, \Lambda)$. Notice for an impure supernumber we simply break it up into its pure parts and apply the formula above to each part and add those together. Now we consider how the right multiplication works. Again suppose that $L \in End^+(V)$ and let z be a pure supernumber,

$$\begin{aligned}(Lz)(e_N) &= L(ze_N) \\ &= L((-1)^{\epsilon(z)\epsilon_N} e_N z) \\ &= L(e_N)(-1)^{\epsilon(z)\epsilon_N} z \\ &= e_M F^M{}_N (-1)^{\epsilon(z)\epsilon_N} z \\ &= e_M (-1)^{\epsilon(z)\epsilon_N} F^M{}_N z\end{aligned} \tag{3.94}$$

Thus, again assuming F has the same block decomposition as before in eq. 3.88,

$$(Lz)^M{}_N = \begin{pmatrix} A^m{}_n z & (-1)^{\epsilon(z)} B^m{}_\nu z \\ C^\mu{}_n z & (-1)^{\epsilon(z)} D^\mu{}_\nu z \end{pmatrix} = (F^M{}_N) \cdot z \tag{3.95}$$

We then define right multiplication in $gl(p, q, \Lambda)$ by the formula above.

3.6.7 Matrix of a Pure Left Linear Operator on V

We now consider what additional conditions are placed on the matrix of a linear operator if it is pure. Let $L \in End^+(V)$ and choose a pure basis $\{E_m, E_\mu\}$ then there exists a supermatrix F defined by $L(E_N) = E_M F^M{}_N$ which has block structure as given below,

$$F^M{}_N = \begin{pmatrix} A^m{}_n & B^m{}_\nu \\ C^\mu{}_n & D^\mu{}_\nu \end{pmatrix} \tag{3.96}$$

Now if the operator L is pure, it follows that the parity of F must obey,

$$\epsilon(F^M{}_N) = \epsilon(L) + \epsilon(M) + \epsilon(N) \tag{3.97}$$

The formula above can be verified case by case without much difficulty. Let us examine how this works together with the other parities we have discussed; let L be

a pure operator and X a pure vector,

$$\begin{aligned}\epsilon(L(X)) &= \epsilon(E_M F^M{}_N X^N_+) \\ &= \epsilon(E_M) + \epsilon(F^M{}_N) + \epsilon(X^N_+) \\ &= \epsilon(E_M) + \epsilon(L) + \epsilon(M) + \epsilon(N) + \epsilon(X^N_+) \\ &= \epsilon(L) + \epsilon(N) + \epsilon(X^N_+) \\ &= \epsilon(L) + \epsilon(X)\end{aligned} \tag{3.98}$$

Notice this is in good accord with eq. 3.54.

3.6.8 Matrix Calculations on $(p|q)$ Dimensional Supervector Spaces

To begin we describe matrix calculations on $\mathbb{K}^{p|q}$. Let L be a ${}^0\Lambda$-linear mapping $L : \mathbb{K}^{p|q} \to \mathbb{K}^{p|q}$ then there exists $M \in gl(p, q|\Lambda)_0$ such that

$$L(x, \theta) = (x, \theta)M.$$

for all $(x, \theta) \in \mathbb{K}^{p|q}$. In particular, $A \in gl(p \times p, \Lambda)_0, D \in gl(q \times q, \Lambda)_0$ while $B \in gl(p \times q, \Lambda)_1, C \in gl(q \times p, \Lambda)_1$ and

$$L(x, \theta) = (x, \theta)M = \begin{pmatrix} x & \theta \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} xA + \theta C & xB + \theta D \end{pmatrix}.$$

The calculation makes good sense as xA and θC are length p even entried row vectors whereas xB and θD are length q odd entried row vectors.

Suppose that $V = \mathbb{K}^{p|q}$ is the canonical $(p|q)$-dimensional with coordinate map $\Psi : V \to \mathbb{K}^{p|q}$. Let $L : V \to V$ be a ${}^0\Lambda$-linear map, then there exists a matrix $M \in gl(p, q|\Lambda)_0$ such that

$$L(v) = \Psi^{-1}(\Psi(v)M).$$

Clearly we can just as well make these calculations for column vectors or for other types of ${}^0\Lambda$-linear mappings with domain a $(p|q)$-dimensional supervector and range an $(r|s)$ dimensional or (r, s) dimensional supervector space. We point these facts out since it may concern the reader that one might not be able to make such calculations as there is no basis for $(p|q)$ dimensional supervector spaces. Fortunately this is no problem since we still have coordinates for $(p|q)$ dimensional supervector spaces, and that is enough to perform matrix calculations.

3.7 Extensions of Linear Operators

Our goal in this section is to show that if we have an operator defined on the even part of some supervector space with superscalars $\mathcal{S}$ such that ${}^1\mathcal{S} \neq 0$, then there exists a unique extension to the whole supervector space.

Proposition 3.7.1. *A left-$\mathcal{S}$-linear operator on a (p,q) dimensional $\mathcal{S}$-supervector space is determined by its action on pure basis. Let V a (p,q) dimensional vector space over $\mathcal{S}$ with pure basis $\{E_m, E_\alpha\} = \{E_M\}$ and suppose L_1, L_2 are left-$\mathcal{S}$-linear operators on V. If $L_1(E_M) = L_2(E_M)$ for all M, then $L_1 = L_2$.*

Proof. Observe, $L_1(E_M) = L_2(E_M)$ implies $L_1(E_M)v^M = L_2(E_M)v^M$ thus $L_1(E_M v^M) = L_2(E_M v^M)$. We find $L_1(v) = L_2(v)$ for all $v \in V$. Therefore $L_1 = L_2$. □

Proposition 3.7.2. *Let V a (p,q) dimensional vector space over $\mathcal{S}$ with pure basis $\{E_m, E_\alpha\} = \{E_M\}$ and suppose W is a $\mathcal{S}$-supervector space. Given a set of $p+q$ supervectors $\{w_N\}_{N=1}^{p+q}$ in W we can define a unique left $\mathcal{S}$-linear operator $L : V \to W$ such that $L(E_N) = w_N$. In other words, if L is defined on a pure basis of V, then there exists a unique left-$\mathcal{S}$-linear extension of L to all of V. We will refer to this process as "left-$\mathcal{S}$-linearly extending L".*

Proof. The formula below suffices to define L on all of V, let $X = E_M X^M \in V$,

$$L(X) = w_M X^M = L(E_M)X^M.$$

Let $X, Y \in V$ and let $c \in \mathcal{S}$,

$$\begin{aligned} L(X+Y) &= L(E_M(X^M + Y^M)) \\ &= L(E_M)(X^M + Y^M) \\ &= L(E_M)X^M + L(E_M)Y^M \\ &= L(X) + L(Y) \end{aligned} \tag{3.99}$$

Let $X \in V$ and let $c \in \mathcal{S}$,

$$\begin{aligned} L(Xc) &= L(E_M(Xc)^M) \\ &= L(E_M)(Xc)^M \\ &= L(E_M)(X)^M c \\ &= L(X)c. \end{aligned} \tag{3.100}$$

Finally, Proposition 3.7.1 gives us uniqueness of this extension. □

Let us recall an important fact from Chapter 2.

Proposition 3.7.3. *Cancellation property for $\mathcal{S}$ with ${}^1\mathcal{S} \neq 0$: Suppose that $x\zeta = y\zeta$ for all $\zeta \in {}^1\mathcal{S}$ then $x = y$.*

This generalizes to supervector spaces which have a pure basis.

Proposition 3.7.4. *Cancellation property for V a (p,q) dimensional vector space over $\mathcal{S}$: Let $v, w \in V$ if $v\zeta = w\zeta$ for all $\zeta \in {}^1\mathcal{S}$ then $v = w$.*

Proof. Since V is (p,q) dimensional, there exists a pure basis $\{E_M\}_{M=1}^{p+q}$. Thus there exist supernumbers $v^M, w^M \in \mathcal{S}$ such that $v = E_M v^M$ and $w = E_M w^M$. Let $\zeta \in {}^1\mathcal{S}$ and note that

$$v\zeta = w\zeta \quad \Longrightarrow \quad E_M v^M \zeta = E_M w^M \zeta$$

and deduce from the linear independence of the pure basis that for each M $v^M\zeta = w^M\zeta$ and for all $\zeta \in {}^1\mathcal{S}$. Thus by the cancellation property for supernumbers $v^M = w^M$ for each M. The proposition follows. □

Proposition 3.7.5. *Let V_0 be the even part of a canonical supervector space V of dimension (p,q) and suppose $L : V_0 \to W$ is a ${}^0\mathcal{S}$-linear mapping from V_0 to a (r,s)-dimensional supervector space W. There exists a left-$\mathcal{S}$-linear mapping $\hat{L}$ from $V = V_0 \oplus V_1$ to W such that $\hat{L}|(V_0) = L$.*

Proof. It suffices to define $\hat{L}$ on the canonical basis $\{e_m, e_\alpha\}$. Since $e_m \in V_0$ for each $1 \leq m \leq p$ we simply define $\hat{L}(e_m) = L(e_m)$. The definition of $\hat{L}$ on the odd-sector is more subtle. We argue that the equation below will motivate a definition for $\hat{L}(e_\alpha)$ for each $1 \leq \alpha \leq q$. For each $\zeta \in {}^1\mathcal{S}$ let $\hat{L}(e_\alpha)$ be the supervector in W such that,

$$\hat{L}(e_\alpha)\zeta = L(e_\alpha\zeta).$$

Notice the r.h.s is well-defined since $e_\alpha\zeta \in V_0$. Suppose towards the purpose of constructing $\hat{L}$ that it is an operator and let us express $\hat{L}$ and L above in terms of the pure basis $\{f_N\}_{N=1}^{r+s}$ for W, there should exist supermatrices T and S such that

$$\hat{L}(e_\alpha) = f_N T^N{}_\alpha \qquad L(e_\alpha\zeta) = f_N S^N{}_M (e_\alpha\zeta)^M = f_N S^N{}_\alpha \zeta$$

The operator equation $\hat{L}(e_\alpha)\zeta = L(e_\alpha\zeta)$ becomes the following matrix equation with respect to the pure basis $\{f_N\}_{N=1}^{r+s}$ for W,

$$T^N{}_\alpha \zeta = S^N{}_\alpha \zeta$$

We insist this equation holds for each $1 \leq N \leq r+s$ and for all $\zeta \in {}^1\mathcal{S}$. Thus by the cancellation property for supernumbers we find that the supermatrix T must satisfy

$$T^N{}_\alpha = S^N{}_\alpha$$

for all $N = 1, 2, \ldots, r+s$ and $\alpha = 1, 2, \ldots, q$. Notice that the supermatrix S is given by assumption. Now that we have collected a few observations we give the proof.

Let $S^N{}_M$ be the matrix of L with respect to the canonical basis $\{e_m, e_\alpha\}$ and the pure basis $\{f_N\}_{N=1}^{r+s}$ for W; $L(X) = L(e_M X^M) = f_N S^N{}_M X^M$ for all $(X^M) \in \mathbb{K}^{p|q}$. The

linear operator $\hat{L}$ on V is constructed as follows,

$$\hat{L}(e_M) = f_N S^N{}_M$$

then we extend left $\mathcal{S}$-linearly, and the proposition follows. □

Proposition 3.7.6. *Let V be a (p,q) dimensional supervector space over $\mathcal{S}$. If L_1 and L_2 are left-$\mathcal{S}$-linear mappings from V to V such that $L_1(v) = L_2(v)$ for all $v \in V_0$, then $L_1 = L_2$.*

Proof. V is a (p,q) dimensional supervector space so there exists a pure basis $\{E_m, E_\alpha\}$. Let ζ^α be odd superscalars for $\alpha = 1, 2, \ldots, q$. Notice that $\zeta^\alpha E_\alpha$ is an even supervector (we intend a summation over $\alpha = 1, 2, \ldots, q$). Since L_1 and L_2 agree on even vectors, we have

$$L_1(E_\alpha \zeta^\alpha) = L_2(E_\alpha \zeta^\alpha)$$

and by left $\mathcal{S}$-linearity

$$L_1(E_\alpha)\zeta^\alpha = L_2(E_\alpha)\zeta^\alpha.$$

It is instructive to consider the case $\zeta^\alpha = 0$ for all α except $\zeta^\beta = \zeta \neq 0$ for a fixed β with $1 \leq \beta \leq q$. Hence,

$$L_1(E_\beta)\zeta = L_2(E_\beta)\zeta.$$

The equation above holds for all ζ thus $L_1(E_\beta) = L_2(E_\beta)$ by Proposition 3.7.4. Finally since β was arbitrary we have that $L_1(E_M) = L_2(E_M)$ for all basis vectors E_M thus by Proposition 3.7.1 we have $L_1 = L_2$. □

3.8 Multi-linear Operators and Graded Symmetry

In our paper [37] we gave the following definition of multi-linearity.

Definition 3.8.1. *Let $\mathfrak{g}$ be a supervector space with basis $\{e_B\}$ and let $\beta : \mathfrak{g}^k \to \Lambda$. We say that β is multi-linear over $\mathfrak{g}^0$ iff for some pure basis $\{e_B\}$ of $\mathfrak{g}$,*

$$\beta(v_1, v_2, \cdots, v_k) = v_1^{A_1} v_2{}^{A_2} \cdots v_k^{A_k} \beta(e_{A_k}, \cdots, e_{A_2}, e_{A_1})$$

for $v_1, v_2, \cdots, v_k \in \mathfrak{g}^0$.

The primary goal of this section is to discover how the above arises as a special case within the definitions given in [68]. We should mention that [68] gives these definitions for arbitrary graded Banach spaces and an abstract Banach Grassmann algebra Q. We replace Q with $\mathcal{S}$ and consider bimodules or supervector spaces with $\mathcal{S}$-superscalars. This is not much of a restriction since to our knowledge the supernumbers listed in our usual choices for $\mathcal{S}$ are the only infinitely generated Banach Grassmann algebras popular in the literature (with the exception of Pestov's creative nonstandard analysis examples [94]).

Definition 3.8.2. *Let V, W be $\mathcal{S}$-bimodules or supervector spaces. A p-linear map $f : V \times \cdots \times V \mapsto W$ is graded symmetric or graded antisymmetric iff for all $v^{k_i} \in V_0 \cup V_1$, $1 \leq i \leq p$*

$$f(v^1, \ldots, v^k, v^{k+1}, \ldots, v^p) = (-1)^{\epsilon(v^k)\epsilon(v^{k+1})+\epsilon(f)} f(v^1, \ldots, v^{k+1}, v^k, \ldots, v^p)$$

where, by definition, $\epsilon(f) = 0$ for f graded-symmetric and $\epsilon(f) = 1$ for f graded-antisymmetric.

Finally we may find the following generalization of left-linearity to multi-linear maps useful

Definition 3.8.3. *Let $V^1, V^2, \ldots, V^p, W$ be $\mathcal{S}$-bimodules or supervector spaces. Suppose that $f : V^1 \times V^2 \times \cdots \times V^p \mapsto W$ is a p-linear map, then f is called left-p-$\mathcal{S}$-linear iff for all $v^{k_i} \in V_0 \cup V_1$, $1 \leq i \leq p$ and $\alpha \in {}^0\mathcal{S} \cup {}^1\mathcal{S}$*

$$f(v^1, \ldots, \alpha v^k, \ldots, v^p) = (-1)^{(\epsilon(v^1)+\epsilon(v^2)+\cdots+\epsilon(v^{k-1}))\epsilon(\alpha)} \alpha f(v^1, \ldots, v^k, \ldots, v^p).$$

This definition (borrowed from [68]) follows our definition for left-linear maps. If we pull out pure scalars to the left, we must generate signs by the Koszul sign conventions.

Observation 3.8.4. *A short calculation will reveal that our "p-multi-linear map over $\mathfrak{g}^0$" is, in the language of [68], a* **graded symmetric left p-Λ-linear mapping on $\mathfrak{g}^p$.** *That is, if $\beta : \mathfrak{g} \times \mathfrak{g} \times \cdots \times \mathfrak{g}$ is p-multi-linear then $\beta \in L_L(\mathfrak{g}, \mathfrak{g}, \ldots, \mathfrak{g}; \Lambda)$ (using notation of [68]).*

Remark 3.8.5. *The space of p-linear maps can be given the structure of a $\mathcal{S}$-bimodule. Moreover, p-linear maps on the Cartesian product become linear mappings on the corresponding tensor product, and p-left-linear maps become left-linear maps. We refer the interested reader to [68] for details.*

The definitions given for left mappings can likewise be given for right mappings. However, if we consider supervector spaces or bi-modules over **commuting superscalars** $\mathcal{S} = {}^0\mathcal{S}$, then the distinction between left and right linearity is removed and we simply speak of ${}^0\mathcal{S}$-linear or ${}^0\Lambda$-linear mappings. For example, we will find that supersmooth functions possess ${}^0\Lambda$-linear Frechet differentials.

3.9 Super Lie Algebras

Lie algebras play an important role in manifold theory. We will find that super Lie algebras play a similar role. We establish some foundational algebraic results for super Lie algebras in this section.

Definition 3.9.1. *A graded Lie algebra is a graded vector space $U = U_0 \oplus U_1$ over $\mathbb{K}$ with a bilinear bracket $[\ ,\ ] : U \times U \to U$ which is graded $[U_r, U_s] \subset U_{r+s}$ for $r, s = 0, 1$, and for all $a, b, c \in U_0 \cup U_1$ with parities $\epsilon_a, \epsilon_b, \epsilon_c$ satisfies the graded Jacobi indentity*

$$(-1)^{\epsilon_a\epsilon_c}[a,[b,c]] + (-1)^{\epsilon_b\epsilon_a}[b,[c,a]] + (-1)^{\epsilon_c\epsilon_b}[c,[a,b]] = 0$$

and the graded skewsymmetry condition,

$$[a,b] = -(-1)^{\epsilon_a\epsilon_b}[b,a].$$

Algebraists often refer to such graded Lie algebras as superalgebras. However we will reserve that term for algebras built over Λ. An associative graded algebra can be given the structure of a graded Lie algebra by defining the bracket to be

$$[a,b] = ab - (-1)^{\epsilon_a\epsilon_b}ba.$$

This is the supercommutator which functions both as a commutator and an anti-commutator depending on the inputs. A graded commutative algebra has a trivial supercommutator and will be called a graded-Abelian Lie algebra. Many things are known about graded Lie algebras over $\mathbb{C}$, see [69] for the classification of all finite dimensional semisimple graded Lie algebras and a rehashing of much of classical Lie theory in the graded case.

Definition 3.9.2. *A graded Lie left Λ-module is a graded Lie algebra W over $\mathbb{K}$ which is a left Λ-module such that*

$$[\alpha X, Y] = \alpha[X, Y]$$

for all $\alpha \in \Lambda$ and $X, Y \in W$.

Definition 3.9.3. *A graded Lie right Λ-module is a graded Lie algebra W over $\mathbb{K}$ which is a right Λ-module such that*

$$[X, Y\alpha] = [X, Y]\alpha$$

for all $\alpha \in \Lambda$ and $X, Y \in W$.

Proposition 3.9.4. *Given a left Λ-module V we can construct a right Λ-module according to the rule*

$$X\alpha \equiv (-1)^{\epsilon(X)\epsilon(\alpha)}\alpha X. \tag{3.101}$$

for all $X \in V_0 \cup V_1$ and $\alpha \in {}^0\Lambda \cup {}^1\Lambda$. Likewise, a graded Lie left Λ-module W is given a natural graded right Lie Λ-module under the same rule.

It is trivial to verify that V has a right Λ-module structure as defined in the proposition (see Proposition 3.1.5). Consider the following to see that if W is a left Lie

Λ-module, then it is a right Lie Λ-module,

$$\begin{aligned}
[X, Y\alpha] &= (-1)^{\epsilon(X)\epsilon(Y\alpha)+1}[Y\alpha, X] \\
&= (-1)^{\epsilon(X)(\epsilon(Y)+\epsilon(\alpha))+1}[(-1)^{\epsilon(Y)\epsilon(\alpha)}\alpha Y, X] \\
&= (-1)^{\epsilon(X)\epsilon(Y)+\epsilon(\alpha)(\epsilon(X)+\epsilon(Y))+1}\alpha[Y, X] \\
&= (-1)^{\epsilon(X)\epsilon(Y)+\epsilon(\alpha)(\epsilon(X)+\epsilon(Y))+1}(-1)^{\epsilon(\alpha)\epsilon([Y,X])}[Y, X]\alpha \\
&= (-1)^{\epsilon(X)\epsilon(Y)+\epsilon(\alpha)(\epsilon(X)+\epsilon(Y))+1}(-1)^{\epsilon(\alpha)(\epsilon(Y)+\epsilon(X))}(-1)^{\epsilon(X)\epsilon(Y)+1}[X, Y]\alpha \\
&= [X, Y]\alpha
\end{aligned} \tag{3.102}$$

we have employed the useful relations $\epsilon(Y\alpha) = \epsilon(Y)+\epsilon(\alpha)$ and $\epsilon([Y, X]) = \epsilon(Y)+\epsilon(X)$ for all pure $X, Y \in W$ and pure $\alpha \in \Lambda$ to make the needed cancellations. This calculation shows that we can always induce a right Lie-Λ-module structure on W given that W is a left Lie-Λ-module.

Definition 3.9.5. *A $\mathcal{S}$-supervector space which is also a left Lie $\mathcal{S}$-module is called a super Lie algebra.*

As we discussed previously when $\mathcal{S}$ are complex superscalars, we assume that the supervector space possesses a conjugation. Let us recall the definition of a *pure basis*

Definition 3.9.6. *Let V be a graded left Λ module and let $m = 1, 2, \ldots p$, $\alpha = 1, 2, \ldots q$ and $E_m \in V_0$, $\tilde{E}_\alpha \in V_1$ then we call $\{E_m, \tilde{E}_\alpha\}$ a pure basis of graded dimension (p, q) if there exist $v^m, \tilde{v}^\alpha \in \Lambda$ for each $v \in V$ such that*

$$v = \sum_{m=1}^{p} v^m E_m + \sum_{\alpha=1}^{q} \tilde{v}^\alpha \tilde{E}_\alpha = \sum_{M=1}^{p+q} v^M E_M$$

where we also denote $\{E_m, \tilde{E}_\alpha\} = \{E_M\}$ with $E_M = E_m$ for $M = m = 1, 2, \ldots p$ and $E_M = \tilde{E}_\alpha$ for $M = p + \alpha = p + 1, p + 2, \ldots p + q$. For convenience denote $\epsilon(E_M)$ by ϵ_M.

Definition 3.9.7. *Given a Lie left Λ-module V of graded dimension (p, q) with pure basis $\{E_M\}$, $M = 1, 2, \ldots, p + q$, there exist structure constants $f^K_{MN} \in \Lambda$ such that $[E_M, E_N] = \sum_{K=1}^{p+q} f^K_{MN} E_K$ for all $M, N = 1, 2, \ldots, p + q$. If V possesses a pure basis for which $s(f^K_{MN}) = 0$ for all M, N, K, then we say that V is a conventional Lie left Λ-module, otherwise we say V is unconventional.*

In fact, conventional Lie left Λ-modules correspond to graded Lie algebras. Our treatment of super Lie groups in Chapter 7 will include unconventional Lie left Λ-modules which to our knowledge are not fully classified at this time.

Chapter 4

Super Differentiation

4.1 Multivariate Grassmann Analysis

Our goal in this Chapter is to introduce the notion of *superdifferentiation*. The superderivative of a function is the best linear approximation of the function which respects the $\mathbb{Z}_2$-grading of the superspace. If we ignore the $\mathbb{Z}_2$-grading, then we would have the Frechet derivative. Every superdifferentiable function is also differentiable. However, the converse is not true. There are C^∞ functions which are not G^∞.

The properties proven here are often taken as the starting point for many students of theoretical physics. One could view these properties as formal definitions, just positing that Grassmann variables can be differentiated as explained in detail below. This viewpoint is not without merit as it allows the student to pursue more exotic questions in a shorter time of preparation. However, one should ask if the formal approach can be replaced with a more traditional one. The answer is, in fact, yes. We can view superderivatives as arising from a limiting process.

We should comment briefly how our work here relates to earlier treatments. We follow Rogers' construction of the so called G^∞ or *supersmooth* functions. Unlike Rogers' original treatment we will work over Grassmann variables built over the complex numbers. Much of the discussion from Rogers' will transfer over to our case here. However some features will be new. For example, we will examine how Jadyzck and Pilch simplified the concept of supersmoothness [68].

Generally, the discussion in this chapter will mirror the standard discussion of how to do calculus on $\mathbb{R}^n$. It should be noted that this is surprising given that we are working with an infinite dimensional Banach space. The fact that the supernumbers also form a Banach algebra with $|| \; ab \; || \leq ||a|| \; ||b||$ allows us to borrow many proofs directly from the standard case. The existence of the p-even and q-odd coordinates give the theory the resemblance of the finite dimensional theory.

Lastly, the analysis and calculus developed in this chapter will provide us the background to define a supermanifold. A supermanifold will be endowed with precisely the structure needed to allow us to take superderivatives locally much as we do in this chapter. Thus, it is important to build a firm understanding of calculus on $\mathbb{K}^{p|q}$ so it can be lifted up to the supermanifold in Chapter 6.

4.2 Superdifferentiable and Supersmooth Functions on $\mathbb{K}^{p|q}$

In Section 3.3 we defined a norm on $\mathbb{K}^{p|q}$. We define a function to be continuous on a subset $U \subseteq \mathbb{K}^{p|q}$ iff it is continuous with respect to the topology generated by that norm. The following definition is due to Alice Rogers in [98].

Definition 4.2.1. *Let U be open in $\mathbb{K}^{p|q}$ and let $f : U \to \Lambda$. Then*

1. *f is said to be G^0 on U if f is continuous on U.*

2. *f is said to be G^1 on U if there exist $p+q$ continuous functions $G_M f : U \to \Lambda$, $M = 1, 2, \ldots, p+q$ and a function $\eta : \mathbb{K}^{p|q} \to \Lambda$ such that, if $(a, b), (a+h, b+k) \in U$*

$$f(a+h, b+k) = f(a, b) + \sum_{m=1}^{p} h^m (G_m f)(a, b) + \sum_{\alpha=1}^{q} k^\alpha (G_{p+\alpha} f)(a, b) + ||(h, k)|| \eta(h, k)$$

where $||\eta(h, k)|| \to 0$ as $||(h, k)|| \to 0$. We say f is superdifferentiable in this case.

3. *for each positive integer s, f is said to be G^s on U if f is G^1 on U, and it is possible to choose $G_M f : U \to \Lambda$, $M = 1, 2, \ldots, p+q$ which satisfy 2. and which are G^{s-1} on U.*

4. *f is said to be G^∞ on U if f is G^s for every positive integer s. We say f is supersmooth in this case.*

5. *for each positive integer s, let $g : U \to \Lambda^s$, where Λ^s is the Cartesian product of s-copies of Λ, and let $\Pi_M : \Lambda^s \to \Lambda$ be the projection onto the M-th factor ($\Pi_M(c^1, c^2, \ldots, c^{p+q}) = c^M$). Then g is said to be G^s or G^∞ if and only if each component function $g^M = \Pi_M \circ g$, $1 \leq M \leq p+q$ is G^s or G^∞.*

We also denote $G_M f = \partial_M f = \partial f / \partial z^M$.

There are ambiguities that arise in choosing the functions $G_M f$ in the case that the underlying Grassmann algebra has only finitely many generators. That ambiguity

has been dealt with in various ways by different authors. To deal with this difficulty, Rogers introduced the "z-mapping" in [98] and later the GH^∞ functions [101]; additionally Rothstein [106] suggested another solution, and Bruzzo [23] introduced the notion of a G-function. All of these are similar in spirit to Rogers' original definition which is of course inspired by classical analysis. We avoid the ambiguity by focusing on the case of infinitely many Grassmann generators. In the infinite case the ambiguity is not present. As a consequence we are forced to use infinite dimensional Banach manifolds in our treatment.

Definition 4.2.2. *Let U be an open subset of $\mathbb{K}^{p|q}$ and let $f : U \to \Lambda$, then f is (Frechet) differentiable at $z \in U$ if there exists a continuous $\mathbb{K}$-linear function $d_z f : \mathbb{K}^{p|q} \to \Lambda$ such that*

$$\boxed{\lim_{H\to 0} \frac{f(z+H) - f(z) - d_z f(H)}{|| \ H \ ||} = 0.} \tag{4.1}$$

If f is differentiable at each $z \in U$ and if the mapping $z \mapsto d_z f$ is continuous (with respect to the sup norm on $L(\mathbb{K}^{p|q}, \Lambda)$), then we say $f \in C^1(U, \Lambda)$. If the mapping $z \mapsto d_z f$ is continuously differentiable for each $z \in U$, then we say $f \in C^2(U, \Lambda)$. The definitions of $C^k(U, \Lambda)$ and $C^\infty(U, \Lambda)$ are made iteratively, and we refer the reader to Serge Lang's text [80] for details.

This is the usual definition of the Frechet derivative for functions on finite dimensional normed linear spaces, and it is also a good definition here. The main difference between Frechet differentiation on a Banach space and superdifferentiation is that in Frechet differentiation there is no consideration of the grading of the space. The Frechet derivative is insensitive to the parity properties of superspace.

Example 4.2.3. *The Frechet derivative of a linear function is simply the function itself. With that in mind we can define the following function. Let $f : \Lambda(\mathbb{C}) \to \Lambda(\mathbb{C})$ be defined by*

$$f(x) = x_o + x_1\zeta^1$$

for each $x = x_0 + x_i\zeta^i + x_{ij}\zeta^i\zeta^j + \cdots$. Let $c \in \mathbb{C}$, then note

$$f(cx) = f(cx_0 + cx_1\zeta^1 + \cdots) = cx_0 + cx_1\zeta^1 = cf(x)$$

so we have $\mathbb{C}$-linearity, and in fact it is clear that f is smooth. In contrast consider $\alpha = \zeta^1\zeta^2 \in \mathbb{K}_a$,

$$f(\alpha x) = f(\zeta^1\zeta^2(x_0 + x_i\zeta^i + x_{ij}\zeta^i\zeta^j + \cdots)) = f(x_0\zeta^1\zeta^2 + 0 + 0 + x_3\zeta^1\zeta^2\zeta^3 + \cdots) = 0.$$

We observe that f is not $\mathbb{C}_c$-linear. Thus the Frechet differential is not $\mathbb{C}_c$-linear. Once this example is understood it is easy to find many other examples of functions

which are smooth but not supersmooth. See Example 5.4.3 for another way a smooth function can fail to be supersmooth.

Proposition 4.2.4. *If U is open in $\mathbb{K}^{p|q}$ and $f \in G^\infty(U)$, then $f \in C^\infty(U, \Lambda)$ the space of all C^∞ maps of U into Λ. In particular, for $s = 1, 2, \ldots,$ the s-th total derivative of f is a continuous multi-linear transformation from $(\mathbb{K}^{p|q})^s$ to Λ such that,*

$$d_c^s f(H_1, \ldots, H_s) = [d^s f(c)][H_1, \ldots, H_s] = \sum\nolimits_{M_1,\ldots,M_s=1}^{p+q} H_1^{M_1} \cdots H_s^{M_s} (G_{M_s} \cdots G_{M_1} f)(c)$$

for all $c \in U$, $(H_1, \ldots, H_s) \in (\mathbb{K}^{p|q})^s$ and $G_{M_s} \cdots G_{M_1} f$ continuous on U.

This is Proposition 2.8 of [98]. The d^s is an iterated Frechet derivative and is explained in [80] for the infinite dimensional case.

Observation 4.2.5. *If the following equation for $d_c^s f$ holds*

$$d_c^s f(H_1, \ldots, H_s) = [d^s f(c)][H_1, \ldots, H_s] = \sum\nolimits_{M_1,\ldots,M_s=1}^{p+q} H_1^{M_1} \cdots H_s^{M_s} (G_{M_s} \cdots G_{M_1} f)(c)$$

then it is clear that $d_c^s f$ is ${}^0\Lambda$-linear in each of its s inputs; $d_c^s f(H_1, \ldots, \alpha H_k, \ldots) = \alpha d_c^s f(H_1, \ldots, H_k, \ldots)$ for all $\alpha \in {}^0\Lambda$ and $(H_1, \ldots, H_s) \in (\mathbb{K}^{p|q})^s$. Thus in the situation considered in Proposition 4.2.4 we find $d_c^s f \in L((\mathbb{K}^{p|q})^s, \Lambda)$ the set of continuous ${}^0\Lambda$-s-linear mappings on the Cartesian product of $\mathbb{K}^{p|q}$ s-times.

Remark 4.2.6. *The proof of Proposition 2.8 in [98] is appropriate for the various choices for $\mathcal{S}$ we have considered. Alice Rogers' proof was given for $\Lambda(\mathbb{R})$ and the associated $\mathbb{R}^{p|q}(\mathbb{R})$, but it applies equally well to $\Lambda(\mathbb{C})$ and our associated $\mathbb{R}^{p|q}$. The underlying Banach theory for $\mathbb{R}^{p|q}$ is not one of genuine complex derivatives since at the Grassmann level we actually have either pure imaginary or real Grassmann coefficient functions. A more interesting question from our perspective is how complex super derivatives are connected to real super derivatives. We will explore this question in Chapter 5.*

Proposition 4.2.7. *The converse of Proposition 4.2.4 is true. If $f \in C^\infty(U, \Lambda)$ is such that the iterated Frechet differential satisfies*

$$d_c^s f(H_1, \ldots, H_s) = [d^s f(c)][H_1, \ldots, H_s] = \sum\nolimits_{M_1,\ldots,M_s=1}^{p+q} H_1^{M_1} \cdots H_s^{M_s} (B_{M_s \ldots M_1})(c)$$

for all $c \in U$, $(H_1, \ldots, H_s) \in (\mathbb{K}^{p|q})^s$ where $B_{M_s \ldots M_1}$ are continuous functions on U then $f \in G^\infty(U)$ and the super partial derivatives of f are given by the $B_{M_s \ldots M_1}$ functions,

$$\begin{aligned} B_{M_1} &= G_{M_1} f \\ B_{M_1 M_2} &= G_{M_1}(G_{M_2} f) \\ B_{M_1 \ldots M_s} &= G_{M_1} \cdots G_{M_s} f. \end{aligned} \tag{4.2}$$

Proof. We are given that there exist coefficients $(B_{M_s\dots M_1})(c)$ such that

$$d_c^s f(H_1,\dots,H_s) = [d^s f(c)][H_1,\dots,H_s] = \sum\nolimits_{M_1,\dots,M_s=1}^{p+q} H_1^{M_1}\cdots H_s^{M_s}(B_{M_s\dots M_1})(c).$$

for each $c \in U$. We choose the obvious candidate for the superdifferentials of f,

$$G_{M_s}\cdots G_{M_1} f = B_{M_s\dots M_1}.$$

Continuity is given and the required limiting condition follows from the assumption that f is smooth as well as the given equation for the iterated Frechet derivative. Finally, Proposition 5.2 in [68] shows that the superdifferentials satisfy the needed iterative conditions,

$$G_{M_s}\cdots G_{M_{k+1}}G_{M_k}\cdots G_{M_1} f = (-1)^{\epsilon_{k+1}\epsilon_k} G_{M_s}\cdots G_{M_k}G_{M_{k+1}}\cdots G_{M_1} f$$

□

Remark 4.2.8. *We assumed that $G_{M_s}\cdots G_{M_1} f$ were continuous on U. It may be the case that if $f \in C^\infty(U,\Lambda)$, then the continuity of $G_{M_s}\cdots G_{M_1} f$ follows by virtue of the continuity of $d^s f$, and in fact [68] claim this to be true.*

The following proposition summarizes most of this chapter.

Proposition 4.2.9. *Let U be open in $\mathbb{K}^{p|q}$, $f, g \in G^\infty(U)$, $a \in {}^0\Lambda \cup {}^1\Lambda$, and $\lambda \in \mathbb{K}$. Then*

1. *$f+g \in G^\infty(U)$ and $G_M(f+g) = G_M f + G_M g$ for $1 \le M \le p+q$*
2. *$\lambda f \in G^\infty(U)$ and $G_M(\lambda f) = \lambda G_M f$ for $1 \le M \le p+q$*
3. *If Π_c and Π_a represent projection maps of Λ onto ${}^0\Lambda$ and ${}^1\Lambda$, respectively, then $\Pi_c \circ f$ and $\Pi_a \circ f$ are in $G^\infty(U)$. Moreover $G^\infty(U)$ is a graded vector space with*

   $$G^\infty(U)_0 = \{f \in G^\infty(U) \mid \Pi_c \circ f = f\} \qquad G^\infty(U)_1 = \{f \in G^\infty(U) \mid \Pi_a \circ f = f\}$$

   We define $\epsilon(G^\infty(U)_r) = r$ for $r = 0, 1$ as usual.
4. *$f \in G^\infty(U)_0 \cup G^\infty(U)_1$ then $af \in G^\infty(U)$ with $G_M(af) = (-1)^{\epsilon(a)\epsilon_M} a G_M f$*
5. *$f, g \in G^\infty(U)_0 \cup G^\infty(U)_1$ then $fg \in G^\infty(U)$ with $G_M(fg) = (G_M f)g + (-1)^{\epsilon(f)\epsilon_M} f G_M g$.*
6. *V open in $\mathbb{K}^{r|s}$ and $h \in G^\infty(V, \mathbb{K}^{p|q})$ then $f \circ h \in G^\infty[h^{-1}(U) \cap V]$ with*

   $$G_M(f\circ h)(a) = \sum\nolimits_{N=1}^{p+q} (G_M H^K)(a)(G_K f)[h(a)]$$

   for $H^M = \Pi_M \circ H$ for $1 \le M \le p+q$ and for all $a \in h^{-1}(U) \cap V$, $K = 1, 2, \dots, r+s$.

7. *If the interval I is open in $\mathbb{R}$ and $\tilde{h} \in C^\infty(I, \mathbb{K}^{p|q})$ then $f \circ \tilde{h} \in C^\infty[\tilde{h}^{-1}(U) \cap I, \mathbb{K}^{p|q}]$ and*

$$\frac{\partial}{\partial t}(f \circ \tilde{h}) = \sum\nolimits_{M=1}^{p+q} \frac{\partial \tilde{h}^M(t)}{\partial t}(G_M f)[\tilde{h}(t)]$$

for $t \in I$.

This is Proposition 2.12 of [98]. Parts (4.) and (5.) of the proposition above are easily extended by linearity to objects which are not pure. We supply our own proofs in this chapter.

4.3 Superdifferentiability Implies Frechet Differentiability

Theorem 4.3.1. *If $f : U \to \Lambda$ is superdifferentiable on the open subset $U \in \mathbb{K}^{p|q}$, then it is also continuously differentiable (in the Frechet sense). Moreover, the Frechet derivative is given in terms of the derivatives of f with respect to supercoordinates $\frac{\partial f}{\partial z^M} = \partial_M f$ which are continuous functions on U. That is, for each $(a, b) \in U$ and $H = (h, k) \in \mathbb{K}^{p|q}$,*

$$d_{(a,b)}f(h,k) = \sum_{m=1}^{p} h^m \partial_m f(a,b) + \sum_{\alpha=1}^{q} k^\alpha \partial_\alpha f(a,b). \tag{4.3}$$

where $\partial_m f$ and $\partial_\alpha f$, the superpartial derivatives of f, are continuous functions on U.

Proof. Since f is G^1 on U we know that there exist η and continuous partials $\frac{\partial f}{\partial z^M}$ such that

$$f(a+h, b+k) = f(a,b) + \sum_{m=1}^{p} h^m \partial_m f(a,b) + \sum_{\alpha=1}^{q} k^\alpha \partial_\alpha f(a,b) + ||(h,k)||\eta(h,k).$$

Let us define our candidate for the Frechet derivative,

$$d_{(a,b)}f(h,k) = \sum_{m=1}^{p} h^m \partial_m f(a,b) + \sum_{\alpha=1}^{q} k^\alpha \partial_\alpha f(a,b). \tag{4.4}$$

Note that $d_{(a,b)}f(h,k)$ is $\mathbb{K}$-linear in the (h,k) argument. Thus, $d_{(a,b)}f : \mathbb{K}^{p|q} \to \Lambda$ is $\mathbb{K}$-linear. Then consider, as our notation is that $H = (h,k)$ and $z = (a,b)$,

$$\begin{aligned} \lim_{H\to 0} \tfrac{f(z+H)-f(z)-d_z f(H)}{||\ H\ ||} &= \lim_{H\to 0} \tfrac{||(h,k)||\eta(h,k)}{||\ H\ ||} \\ &= \lim_{H\to 0} \eta(h,k) \\ &= 0. \end{aligned} \tag{4.5}$$

Thus $d_{(a,b)}f$ constructed as above is indeed the Frechet derivative.

We seek to show that the mapping $z \mapsto d_z f$ is continuous. Note $d_z f$ is an operator so its norm is defined as follows. Recall that if $L : V \mapsto W$ is a mapping on normed spaces V, W, then the norm of the operator L is given by $||L|| = sup\{||L(H)|| \mid ||H|| = 1\}$.

Let $\epsilon > 0$, then by the (assumed in definition of G^1) continuity of partial super derivatives at z_o, there exists $\delta_N > 0$ such that for $||z - z_o|| < \delta_N$ we have $||\partial_N f(z) - \partial_N f(z_o)|| < \epsilon/(p+q)$. Let $\delta = min\{\delta_N \mid 1 \leq N \leq p+q\}$ and suppose that $||z-z_o|| < \delta$. Observe that

$$\begin{aligned} ||d_z f - d_{z_o} f|| &= sup\{||(d_z f - d_{z_o})(h)|| \mid ||h|| = 1\} \\ &\leq sup\{\textstyle\sum_{N=1}^{p+q} ||h^N||\, ||\partial_N f(z) - \partial_N f(z_o)|| \mid ||h|| = 1\} \\ &\leq sup\{\textstyle\sum_{N=1}^{p+q} ||\partial_N f(z) - \partial_N f(z_o)|| \mid ||h|| = 1\} \\ &\leq sup\{\textstyle\sum_{N=1}^{p+q} \epsilon/(p+q) \mid ||h|| = 1\} \\ &\leq \epsilon. \end{aligned} \tag{4.6}$$

Thus the mapping $z \mapsto d_z f$ is continuous. □

Next we offer a converse to the preceding theorem. We explain now what additional conditions beyond differentiability will insure superdifferentiability.

Theorem 4.3.2. *Let $f : U \to \Lambda$ where U is open in $\mathbb{K}^{p|q}$. If f is differentiable on U and if there exist continuous functions B_M on U for $M = 1, 2, \ldots p+q$ such that for each $H \in \mathbb{K}^{p|q}$ and $z \in U$*

$$d_z f(H) = \sum_{M=1}^{p+q} H^M B_M(z) \tag{4.7}$$

then f is superdifferentiable on U and we can choose $\frac{\partial f}{\partial z^M} = B_M$.

Proof. By assumption of differentiability on U, for each $z \in U$, we find,

$$\lim_{H \to 0} \frac{f(z+H) - f(z) - d_z f(H)}{||\ H\ ||} = 0. \tag{4.8}$$

This suggests that we define the nonlinear part of f according to the formula

$$\eta(H) = \frac{f(z+H) - f(z) - d_z f(H)}{||\ H\ ||}. \tag{4.9}$$

Clearly, $\eta(H) \to 0$ as $H \to 0$. Moreover,

$$\begin{aligned} f(z+H) &= f(z) + d_z f(H) + \eta(H)||H|| \\ &= f(z) + \textstyle\sum_{M=1}^{p+q} H^M P_M + \eta(H)||H||. \end{aligned} \tag{4.10}$$

Therefore, f is superdifferentiable at z. We identify the partial derivatives of f with the continuous functions B_M on U, that is $\partial_m f = B_m$ for $m = 1, 2, \ldots p$ and $\partial_\alpha f = B_{p+\alpha}$ for $\alpha = 1, 2, \ldots q$. □

Theorems 4.3.2 and 4.3.1 show how $C^1(U, \Lambda)$ and $G^1(U)$ are related. The theorem that follows was unknown to us until the completion of [37]. Essentially it says that if we have a function which is in both $C^\infty(U, \Lambda)$ and $G^1(U)$, then the function is also in $G^2(U)$. Moreover it is also in $G^3(U), G^4(U), \ldots$ without any additional assumptions. This theorem motivates the definition for supersmoothness used in [68].

Theorem 4.3.3. *Let $f : U \to \Lambda$ where U is open in $\mathbb{K}^{p|q}$. If f is smooth on U and if there exist continuous functions B_M on U for $M = 1, 2, \ldots p+q$ such that for each $H \in \mathbb{K}^{p|q}$ and $z \in U$,*

$$d_z f(H) = \sum_{M=1}^{p+q} H^M B_M(z) \tag{4.11}$$

then f is supersmooth on U and we can choose $B_M = \frac{\partial f}{\partial z^M}$. In other words if a function is in $G^1(U)$ and $C^\infty(U, \Lambda)$ then it is in $G^\infty(U)$. Conversely, if a function is in $G^\infty(U)$ then it is automatically in $C^\infty(U, \Lambda)$.

Proof. The converse is follows immediately from Proposition 4.2.4.

Suppose that $f : U \to \Lambda$ where U is open in $\mathbb{K}^{p|q}$ and $f \in C^\infty(U, \Lambda)$ such that there exist continuous functions B_M on U such that

$$d_z f(H) = \sum_{M=1}^{p+q} H^M B_M(z). \tag{4.12}$$

for each $H \in \mathbb{K}^{p|q}$ and $z \in U$. The following proof is based on the proof in [68] pages 380-381 (See Proposition 5.1 of [68]). This proposition claims that the p-th Frechet differential is in fact a p-${}^0\Lambda$-linear mapping. Moreover, the p-th Frechet differential is a symmetric multi-linear mapping. Theorem 4.3.2 shows that the $p = 1$ case holds true. Assume inductively that $f \in G^k(U)$ for some $k \geq 1$. Consider the $(k+1)$-th Frechet differential of f satisfies the iterative condition,

$$d_z^{k+1} f = d_z(d_z^k f).$$

Because $d_z^{k+1} f$ is a symmetric ${}^0\Lambda$-multi-linear mapping, we find that $d_z(\alpha d_z^k f) = \alpha d_z(d_z^k f)$. Hence by Theorem 4.3.2 $d_z^k f$ is G^1 for each $z \in U$. Therefore, $f \in G^{k+1}(U)$ and by induction the theorem follows. □

4.4 $G^\infty(U)$ is a Almost a Supervector Space

Recall that a differentiable function f has a continuous Frechet differential df which is $\mathbb{K}$-linear and is the best linear approximation of f as described by Equation 4.1.

Proposition 4.4.1. *Let $f, g : \mathbb{K}^{p|q} \to \Lambda$.*

1. *If f, g are differentiable, then so are $f \pm g$. Moreover, the Frechet differential of the sum $f + g$ is simply the sum of the Frechet differentials; $d(f + g) = df + dg$.*
2. *If $c \in \Lambda$ and f is differentiable, then cf is differentiable. Moreover, the Frechet differential of cf is the product of c and with the Frechet differential; $d(cf) = cdf$*

Observation 4.4.2. *Notice c is a supernumber in 2.). In ordinary abstract Banach theory we would only be able to state 2.) for $c \in \mathbb{K}$. We have assumed that our functions have their range in Λ so multiplication by supernumbers is sensible.*

Proof. Let us begin the proof of 1.). Let U be open in $\mathbb{K}^{p|q}$ and let $f : U \to \Lambda$ and $g : U \to \Lambda$ be differentiable on U, then we claim that $d_z(f + g) = d_z f + d_z g$. Define,

$$\eta_f(H) = \frac{f(z + H) - f(z) - d_z f(H)}{|| \ H \ ||}. \tag{4.13}$$

And likewise,

$$\eta_g(H) = \frac{g(z + H) - g(z) - d_z g(H)}{|| \ H \ ||}. \tag{4.14}$$

By assumption of differentiability we know η_g and η_f tend to zero as $H \to 0$. Note,

$$\begin{array}{ll} f(z + H) & = f(z) + d_z f(H) + \eta_f(H)||H|| \\ g(z + H) & = g(z) + d_z g(H) + \eta_g(H)||H||. \end{array} \tag{4.15}$$

Define $\eta_{f+g} = \eta_f + \eta_g$ and consider,

$$\begin{array}{ll} (f + g)(z + H) & = f(z + H) + g(z + H) \\ & = f(z) + d_z f(H) + \eta_f(H)||H|| + g(z) + d_z g(H) + \eta_g(H)||H|| \\ & = (f + g)(z) + (d_z f + d_z g)(H) + \eta_{f+g}||H||. \end{array}$$

By assumption of differentiability of f and g we also know that $d_z f$ and $d_z g$ are $\mathbb{K}$-linear maps on $\mathbb{K}^{p|q}$ hence $d_z f + d_z g$ is linear on $\mathbb{K}^{p|q}$. To summarize, $f + g$ is differentiable at z, and the Frechet derivative is $d_z(f + g) = d_z f + d_z g$. Observe that the mapping $z \mapsto d_z f + d_z g$ is continuous since it is the sum of continuous mappings. The proof of 1.) follows.

Now we prove 2.). We show that if $f : U \to \Lambda$ is differentiable, then for $c \in \Lambda$ the function $cf : U \to \Lambda$, defined pointwise by $(cf)(z) = cf(z)$, is differentiable. Since f

is differentiable, it follows,

$$f(z+H) \;\; = f(z) + d_z f(H) + \eta_f(H)||H|| \tag{4.16}$$

with η_f tending to zero as $H \to 0$. We propose that $d_z cf = c d_z f$ and $\eta_{cf}(H) = c\eta_f(H)$. Multiplying Equation 4.16 by c yields,

$$\begin{aligned}(cf)(z+H) = cf(z+H) \;\; &= cf(z) + cd_z f(H) + c\eta_f(H)||H|| \\ &= (cf)(z) + d_z cf(H) + \eta_{cf}(H)||H||\end{aligned} \tag{4.17}$$

and clearly $\eta_{cf} \equiv c\eta_f$ tends to zero as required and $d_z(cf)$ is $\mathbb{K}$-linear. One might worry that we would have to commute the supernumber c somewhere introducing some signs. However, that is not the case. Finally observe that $z \mapsto cd_z f$ is the product of continuous mappings hence $z \mapsto d_z(cf)$ is continuous, and the proof of 2.) follows. □

Let $U \subseteq \mathbb{K}^{p|q}$ be open. We have shown that if $f, g \in G^\infty(U)$ and $c \in \Lambda$, then $f+g, cf \in G^\infty(U)$. It follows that $G^\infty(U)$ is a Λ-bimodule. We cannot quite call it a supervector space because there is not always a natural idea of conjugation. See Chapter 5 for details as to why $G^\infty(U)$ lacks a conjugation for $U \subseteq \mathbb{C}^{p|q}$.

4.5 Linearity of Superderivatives on $\mathbb{K}^{p|q}$

Let U be open in $\mathbb{K}^{p|q}$ and suppose that $f, g \in G^1(U)$, then by Theorem 4.3.1

$$d_z f(H) = \sum_{M=1}^{p+q} H^M \partial_M f(z) \;\text{ and }\; d_z g(H) = \sum_{M=1}^{p+q} H^M \partial_M g(z) \tag{4.18}$$

for each $z \in U$ and $H \in \mathbb{K}^{p|q}$. Now apply Proposition 4.4.1,

$$\begin{aligned} d_z(f+g)(H) \;\; &= d_z f(H) + d_z g(H) \\ &= \textstyle\sum_{M=1}^{p+q} H^M \partial_M f(z) + \sum_{M=1}^{p+q} H^M \partial_M g(z) \\ &= \textstyle\sum_{M=1}^{p+q} H^m \partial_M (f+g)(z) \end{aligned}$$

Therefore, by Theorem 4.3.1, we find that $f+g$ is superdifferentiable with superderivatives $\partial_M(f+g) = \partial_M f + \partial_M g$, $M = 1, 2, \ldots p+q$. Explicitly, in terms of even(x^m) and odd(θ^α) coordinates,

$$\frac{\partial}{\partial x^m}(f+g) = \frac{\partial f}{\partial x^m} + \frac{\partial g}{\partial x^m} \qquad\qquad \frac{\partial}{\partial \theta^\alpha}(f+g) = \frac{\partial f}{\partial \theta^\alpha} + \frac{\partial g}{\partial \theta^\alpha} \tag{4.19}$$

Notice that this amount of detail is superfluous. There is no distinction with respect to parity here.

In contrast, parity will be important in the determination of the superderivative of cf. Let c be a pure supernumber, that is $c \in \mathbb{K}_c \cup \mathbb{K}_a$. Now assume that U is open and that f is superdifferentiable at $z \in U$. By Proposition 4.4.1,

$$\begin{aligned} d_z(cf)(h,k) &= c(d_z f)(h,k) \\ &= c(\textstyle\sum_{m=1}^{p} h^m \partial_m f(z) + \sum_{\alpha=1}^{q} k^\alpha \partial_\alpha f(z)) \\ &= \textstyle\sum_{m=1}^{p} ch^m \partial_m f(z) + \sum_{\alpha=1}^{q} ck^\alpha \partial_\alpha f(z). \\ &= \textstyle\sum_{m=1}^{p} h^m c\partial_m f(z) + \sum_{\alpha=1}^{q} k^\alpha (-1)^{\epsilon(c)} c\partial_\alpha f(z). \end{aligned} \tag{4.20}$$

Hence, using Theorem 4.3.1, we find that cf is superdifferentiable with superderivatives,

$$\frac{\partial}{\partial x^m}(cf) = c\frac{\partial f}{\partial x^m} \qquad \frac{\partial}{\partial \theta^\alpha}(cf) = (-1)^{\epsilon(c)}\left(c\frac{\partial f}{\partial \theta^\alpha}\right). \tag{4.21}$$

Let $b \in \Lambda$ (not necessarily pure) then $b = b_c + b_a$ and using the result above and linearity of the superderivatives,

$$\frac{\partial}{\partial x^m}(bf) = b\frac{\partial f}{\partial x^m} \qquad \frac{\partial}{\partial \theta^\alpha}(bf) = b_c\frac{\partial f}{\partial \theta^\alpha} - b_a\frac{\partial f}{\partial \theta^\alpha}. \tag{4.22}$$

4.6 Graded Leibniz Rule for Superderivatives

Borrowing arguments from the standard case it can be shown that if U is open in $\mathbb{K}^{p|q}$ and $f, g \in C^1(U, \Lambda)$, then the Leibniz rule holds for Frechet derivatives,

$$d_z(fg)(H) = f(z)d_z g(H) + d_z f(H)g(z). \tag{4.23}$$

Recall that we can write any function with range Λ as the sum of an even and odd function. Let $f : \mathbb{K}^{p|q} \to \Lambda$, then for each $z \in \mathbb{K}^{p|q}$

$$f(z) = {}^0f(z) + {}^1f(z) \tag{4.24}$$

where ${}^0f(z) \in \mathbb{K}_c$ and ${}^1f(z) \in \mathbb{K}_a$. Assume that f is a pure function, so either $f = {}^0f$ or $f = {}^1f$ with $\epsilon_f = 0$ or $\epsilon_f = 1$, respectively. Next, assume that f and g are superdifferentiable so that continuous superderivatives of f and g exist and, (suppressing the explicit z-dependence)

$$\begin{aligned} d(fg)(H) &= fdg(H) + df(H)g \\ &= f\textstyle\sum_{N=1}^{p+q} H^N \partial_N g + (\sum_{N=1}^{p+q} H^N \partial_N f)g \\ &= \textstyle\sum_{N=1}^{p+q} H^N((-1)^{\epsilon_f \epsilon_N} f\partial_N g + (\partial_N f)g). \end{aligned}$$

Thus we find that fg is superdifferentiable with superderivatives,

$$\frac{\partial}{\partial z^N}(fg) = \frac{\partial f}{\partial z^N}g + (-1)^{\epsilon_f \epsilon_N} f\frac{\partial g}{\partial z^N}. \tag{4.25}$$

Explicitly since $\epsilon_m = 0$ and $\epsilon_\alpha = 1$,

$$\frac{\partial}{\partial x^n}(fg) = \frac{\partial f}{\partial x^n}g + f\frac{\partial g}{\partial x^n} \qquad \frac{\partial}{\partial \theta^\alpha}(fg) = \frac{\partial f}{\partial \theta^\alpha}g + (-1)^{\epsilon_f} f\frac{\partial g}{\partial \theta^\alpha}. \tag{4.26}$$

4.7 Chain Rule for Superderivatives

A vector-valued function is differentiable when all of its components are differentiable. A similar definition is given for superderivatives of supervector-valued functions.

Definition 4.7.1. *Let U be open in $\mathbb{K}^{p|q}$ and let V be open in $\mathbb{K}^{r|s}$. Then consider the function $f : U \to V$. We define f to be superdifferentiable (G^1) if and only if each of the component functions of f is superdifferentiable (G^1). That is the Frechet derivative can be expanded in terms of the superderivatives of the component functions f^N, $N = 1, 2, \ldots r+s$,*

$$d_z f^N(H) = \sum_{M=1}^{p+q} H^M \frac{\partial f^N}{\partial z^M}(z). \tag{4.27}$$

When $r = s = 1$ we have $f \in G^1(U)$, otherwise we denote $f \in G^1(U, V)$. Likewise we define $f : U \to V$ to be G^l or G^∞ iff each of its component functions is G^l or G^∞ respective.

Let U be open in $\mathbb{K}^{p|q}$ and let V be open in $\mathbb{C}^{r|s}$. Then consider functions $f : U \to V$ and $g : V \to \Lambda$. If f is differentiable at $z \in U$ and g is differentiable at $f(z) \in V$, then

$$d_z(g \circ f) = d_{f(z)}g \circ d_z f. \tag{4.28}$$

The proof of this fact is just as straightforward as the standard argument for finite dimensional Banach spaces. Next, suppose that f is superdifferentiable at $z \in U$ and g is superdifferentiable at $f(z) \in V$. Then denoting w^M for the variables on $\mathbb{C}^{r|s}$,

$$\begin{aligned} d_z(g \circ f)(H) &= d_{f(z)}g(d_z f(H)) \\ &= \textstyle\sum_{N=1}^{r+s} d_z f^N(H) \frac{\partial g}{\partial w^N}(f(z)) \\ &= \textstyle\sum_{N=1}^{r+s} \sum_{M=1}^{p+q} H^M \frac{\partial f^N}{\partial z^M}(z) \frac{\partial g}{\partial w^N}(f(z)) \\ &= \textstyle\sum_{M=1}^{p+q} H^M \sum_{N=1}^{r+s} \frac{\partial f^N}{\partial z^M}(z) \frac{\partial g}{\partial w^N}(f(z)). \end{aligned} \tag{4.29}$$

Thus the composite $g \circ f$ is superdifferentiable with superderivatives,

$$\frac{\partial}{\partial z^N}(g \circ f) = \sum_{N=1}^{r+s} \frac{\partial f^N}{\partial z^M}\frac{\partial g}{\partial w^N}. \tag{4.30}$$

Where we have suppressed the explicit z dependence.

Chapter 5

Conjugate Variables

It is often claimed that one can replace a pair of real variables with a pair of complex conjugate variables. The idea that z and $\bar{z}$ are independent is difficult to reconcile with the simple observation for z a complex supernumber $z^* = \bar{z}$. However, conjugate variables are both meaningful and useful. Our goal in this chapter is to expose a true explicit meaning for conjugate variables in the super case. This is especially important to physical applications since the fermionic coordinates are usually "parametrized" by conjugate variables. We explain how conjugate variables and their derivatives are simply an efficient notation for a more basic real formalism. We also define the *derivatives with respect to chiral coordinates* in a similar fashion; a derivative with respect to a chiral coordinate is a notation for a complex linear combination of real super derivatives. This chapter is in large part a generalization of the work of Reinhold Remmert [97] to our G^∞ category.

5.1 Conjugation and Superdifferentiation

Definition 5.1.1. *Suppose $U \subseteq \mathbb{R}^{2p|2q}$ or $U \subseteq \mathbb{C}^{p|q}$ and let f be a function $f : U \to \Lambda$. We define the conjugate function f^* by the rule $f^*(z) = (f(z))^*$ for each $z \in U$.*

The conjugate of f is real superdifferentiable if f is real superdifferentiable.

Proposition 5.1.2. *Let $U \subset \mathbb{R}^{2p|2q}$, then we claim that if $f : U \to \Lambda$ is superdifferentiable at on U then f^* is superdifferentiable on U.*

Proof. Suppose $f : U \to \Lambda$ is superdifferentiable at $z \in U$ is pure ($\epsilon(f) = 0, 1$) then,

$$\begin{aligned} f^*(z+H) &= (f(z+H))^* \\ &= (f(z) + \textstyle\sum_{N=1}^{p+q} H^N(\partial_N f)(z) + ||H||\eta(H))^* \\ &= (f(z))^* + \textstyle\sum_{N=1}^{p+q}((\partial_N f)(z))^*(H^N)^* + ||H||(\eta(H))^* \\ &= f^*(z) + \textstyle\sum_{N=1}^{p+q} H^N((-1)^{(\epsilon_f+\epsilon_N)\epsilon_N}(\partial_N f)(z))^* + ||H||\eta^*(H). \end{aligned}$$

Here $H \in \mathbb{R}^{p|q}$ hence $(H^M)^* = H^M$ for $M = 1, 2, \ldots p+q$, also we have used that $\epsilon(H^M) = \epsilon_M$. Since f is superdifferentiable we know that $\eta(H) \to 0$ as $H \to 0$ from which it follows that $\eta^*(H) \to 0$ as $H \to 0$. Thus we have shown that f^* is superdifferentiable at z and we can identify the superderivatives of f^* are

$$\partial_N f^* = (-1)^{(\epsilon_f + \epsilon_N)\epsilon_N} (\partial_N f)^*. \tag{5.1}$$

Moreover, these are continuous functions on U since they are related to the superderivatives of f by continuous operations thus f^* is superdifferentiable on U. The superderivatives of f^* break down as follows if we split into even and odd cases,

$$\frac{\partial}{\partial x^m}\Big(f^*\Big) = \Big(\frac{\partial f}{\partial x^m}\Big)^* \qquad\qquad \frac{\partial}{\partial \theta^\alpha}\Big(f^*\Big) = (-1)^{\epsilon_f + 1}\Big(\frac{\partial f}{\partial \theta^\alpha}\Big)^*. \tag{5.2}$$

Finally, since we proved the claim for pure functions we may extend the result linearly to treat an arbitrary function which is a sum of even and odd functions. □

A few comments about the case $U \subset \mathbb{C}^{p|q}$ are in order. If $f : U \subset \mathbb{C}^{p|q} \to \Lambda$ is superdifferentiable, we say that it is *complex superdifferentiable*. When f is complex superdifferentiable, we find that f^* is not complex superdifferentiable. The reason is that it was crucial that $H^* = H$, and for an arbitrary $H \in \mathbb{C}^{p|q}$ we cannot make such a claim. Consider the following calculation

$$\begin{aligned} f^*(z+H) &= (f(z+H))^* \\ &= (f(z) + \textstyle\sum_{N=1}^{p+q} H^N(\partial_N f)(z) + ||H||\eta(H))^* \\ &= (f(z))^* + \textstyle\sum_{N=1}^{p+q}((\partial_N f)(z))^*(H^N)^* + ||H||(\eta(H))^* \\ &= f^*(z) + \textstyle\sum_{N=1}^{p+q}(H^N)^*((-1)^{(\epsilon_f+\epsilon_N)\epsilon_N}(\partial_N f)(z))^* + ||H||\eta^*(H). \end{aligned}$$

This is almost what we want, if we could just replace $(H^N)^*$ with H^N. However, we cannot. This is not surprising. In ordinary complex variables we learn that if f is complex differentiable on $U \subset \mathbb{C}$ and $f = f(z)$, then $f^* = f^*(\bar{z})$ is not complex differentiable. In this chapter we seek to visit some of the most elementary questions of this type in the supercase. We found that the treatment of elementary complex variables by Reinhold Remmert [97] to be an algebraically lucid and useful work which naturally fit our general prejudices. We follow his general logic throughout this chapter.

5.2 Complex Verses Real Linearity

In this section we work through a number of lemmas which expose important connections between real linearity of complex maps and complex linearity of induced real mappings. This algebra forms the logical core of the Cauchy Riemann equations which we will discuss in Section 5.3.

5.2.1 Lemma I (Remmert)

Definition 5.2.1. *Let $\Lambda = \Lambda(\mathbb{C})$. A mapping $T : {}^0\Lambda \to \Lambda$ is $\mathbb{R}_c$-linear iff $T(z+w) = T(z) + T(w)$ and $T(za) = T(z)a$ for all $a \in \mathbb{R}_c$ and $z, w \in {}^0\Lambda$.*

Lemma 5.2.2. *We denote $z = (x + iy) \in {}^0\Lambda$ and $\bar{z} = (x - iy) \in {}^0\Lambda$ with $x, y \in \mathbb{R}_c$ throughout this lemma. Given mapping $T : {}^0\Lambda \to \Lambda$, then the following are equivalent:*

1. *T is $\mathbb{R}_c$-linear*
2. *$T(x + iy) = T(1)x + T(i)y$.*
3. *If we define $\lambda \equiv \frac{1}{2}(T(1) - iT(i))$ and $\mu \equiv \frac{1}{2}(T(1) + iT(i))$ then $T(z) = \lambda z + \mu\bar{z}$*

Proof. (1.) iff (2.): Recall that by definition T is $\mathbb{R}_c$-linear iff $T(z+w) = T(z)+T(w)$ for all $z, w \in {}^0\Lambda$ and $T(za) = T(z)a$ for all $z \in {}^0\Lambda$ and $a \in \mathbb{R}_c$. Let $z = x + iy$ as in the lemma,

$$T(z) = T(x + iy) = T(1x) + T(iy) = T(1)x + T(i)y.$$

Given T as in the lemma, the following equation is identically true.

$$\begin{aligned} \lambda z + \mu\bar{z} &= \tfrac{1}{2}(T(1) - iT(i))(x + iy) + \tfrac{1}{2}(T(1) + iT(i))(x - iy) \\ &= T(1)x + T(i)y \end{aligned} \tag{5.3}$$

Observe $\lambda z + \mu\bar{z} = T(z)$ iff $T(1)x + T(i)y = T(z)$ thus (2.) is equivalent to (3.). □

Remark 5.2.3. *It would be nice to extend this lemma directly for $T : {}^1\Lambda \to \Lambda$, but then we face the usual dilemma of ${}^1\Lambda$ lacking a basis, in particular $T(1)$ and $T(i)$ are nonsense in this case. However, not all is lost. We intend to apply these results to the differential and as we have previously shown that even though f may be defined on $\mathbb{C}^{p|q}$ it is more appropriate to consider the extension of df to $\Lambda(p, q)$ as the primary object of interest.*

We extend the results just given for ${}^0\Lambda$ to Λ.

Definition 5.2.4. *Let $\Lambda = \Lambda(\mathbb{C})$. A mapping $T : \Lambda \to \Lambda$ is left-$\Lambda_{\mathbb{R}}$-linear iff $T(z + w) = T(z) + T(w)$ and $T(az) = T(z)a$ for all $a \in \Lambda_{\mathbb{R}}$ and $z, w \in \Lambda$.*

Lemma 5.2.5. *We denote $z = (x + iy) \in \Lambda$ and $\bar{z} = (x - iy) \in \Lambda$ with $x, y \in \Lambda_{\mathbb{R}}$ throughout this lemma. Given mapping $T : \Lambda \to \Lambda$ then the following are equivalent:*

1. *T is left-$\Lambda_{\mathbb{R}}$-linear*
2. *$T(x + iy) = T(1)x + T(i)y$.*
3. *If we define $\lambda \equiv \frac{1}{2}(T(1) - iT(i))$ and $\mu \equiv \frac{1}{2}(T(1) + iT(i))$ then $T(z) = \lambda z + \mu\bar{z}$*

Proof. Recall that by definition T is left-$\Lambda_{\mathbb{R}}$-linear iff $T(z+w) = T(z) + T(w)$ for all $z, w \in {}^0\Lambda$ and $T(za) = T(z)a$ for all $z \in \Lambda$ and $a \in \Lambda_{\mathbb{R}}$. Let $z = x + iy$ as in the lemma,

$$T(z) = T(x + iy) = T(1x) + T(iy) = T(1)x + T(i)y.$$

Next we show (3.) is equivalent to (2.)

$$\begin{aligned} \lambda z + \mu \bar{z} &= \tfrac{1}{2}(T(1) - iT(i))(x + iy) + \tfrac{1}{2}(T(1) + iT(i))(x - iy) \\ &= T(1)x + T(i)y \\ &= T(z). \end{aligned} \tag{5.4}$$

□

5.2.2 Lemma II (Remmert)

Definition 5.2.6. *Let $\Lambda = \Lambda(\mathbb{C})$. A mapping $T : {}^0\Lambda \to \Lambda$ is ${}^0\Lambda$-linear iff $T(z+w) = T(z) + T(w)$ and $T(zw) = T(z)w$ for all $z, w \in {}^0\Lambda$.*

Every ${}^0\Lambda$-linear map is also $\mathbb{R}_c$-linear. The converse is not true, but we can give a condition which will insure that a $\mathbb{R}_c$ linear mapping is also a ${}^0\Lambda$-linear mapping.

Lemma 5.2.7. *Given $T : {}^0\Lambda \to \Lambda$ is $\mathbb{R}_c$-linear then the following are equivalent:*

1. $T(i) = iT(1)$
2. $T(z) = T(1)z$ *for all* $z \in {}^0\Lambda$
3. T *is* ${}^0\Lambda$*-linear*

Proof. Suppose T is as the lemma states and (1.) is true. Let $z \in {}^0\Lambda$ so there exist $x, y \in \mathbb{R}_c$ such that $z = x + iy$,

$$\begin{aligned} T(z) &= T(x + iy) \\ &= T(x) + T(iy) \\ &= T(1)x + T(i)y \\ &= T(1)x + T(1)iy \\ &= T(1)(x + iy) \\ &= T(1)z. \end{aligned} \tag{5.5}$$

Thus (1.) $\implies$ (2.). But (2.) $\implies$ (1.) is obvious. Now suppose (2.) is true. Let $z, w \in {}^0\Lambda$,

$$\begin{aligned} T(zw) &= (T(1))zw \\ &= (T(1)z)w \\ &= T(z)w. \end{aligned} \tag{5.6}$$

Thus (2.) $\implies$ (3.) and (3.) $\implies$ (1.) is obvious. □

Again the definition and lemma generalize nicely to the Λ-case.

Definition 5.2.8. *Let $\Lambda = \Lambda(\mathbb{C})$. A mapping $T : \Lambda \to \Lambda$ is left-Λ-linear iff $T(z+w) = T(z) + T(w)$ and $T(zw) = T(z)w$ for all $z, w \in \Lambda$.*

Every left-Λ-linear map is also left-$\Lambda_{\mathbb{R}}$-linear. The converse is not true, but we can give a condition which will insure that a left-$\Lambda_{\mathbb{R}}$ linear mapping is also a left-Λ-linear mapping.

Lemma 5.2.9. *Given $T : \Lambda \to \Lambda$ is left-$\Lambda_{\mathbb{R}}$-linear then the following are equivalent:*

1. *$T(i) = T(1)i$*
2. *$T(z) = T(1)z$ for all $z \in \Lambda$*
3. *T is left-Λ-linear*

Proof. Suppose T is as the lemma states and (1.) is true. Let $z \in \Lambda$ so there exist $x, y \in \Lambda_{\mathbb{R}}$ such that $z = x + iy$,

$$\begin{aligned} T(z) &= T(x+iy) \\ &= T(x) + T(iy) \\ &= T(1)x + T(i)y \\ &= T(1)x + T(1)iy \\ &= T(1)(x+iy) \\ &= T(1)z. \end{aligned} \tag{5.7}$$

Thus (1.) $\implies$ (2.). To see (2.) $\implies$ (1.) simply take $z = i$ to obtain $T(z) = T(i) = T(1)i$. Now suppose (2.) is true. Let $z, w \in \Lambda$,

$$\begin{aligned} T(zw) &= (T(1))zw \\ &= (T(1)z)w \\ &= T(z)w. \end{aligned} \tag{5.8}$$

Thus (2.) $\implies$ (3.). To see that (3.) $\implies$ (2.) take $z = 1$ thus $T(zw) = T(w) = T(1)w$. The lemma follows. □

5.2.3 Lemma III (Remmert)

It is a well-known and useful fact that $\mathbb{C}$ and $\mathbb{R}^2$ can be identified as vector spaces over $\mathbb{R}$ through the correspondence $x + iy \mapsto (x, y)$. Let us discuss natural extensions of this correspondence to the super case. Here we find that $\Lambda = \mathbb{C}_c \oplus \mathbb{C}_a$, ${}^0\Lambda = \mathbb{C}_c$, ${}^1\Lambda = \mathbb{C}_a$ correspond naturally to $\Lambda_{\mathbb{R}}^2$, $\mathbb{R}^{2|0}$ and $\mathbb{R}^{0|2}$ respectively.

Proposition 5.2.10. *The following mappings are Banach space isometries if we give $\mathbb{C}$ and $\Lambda_{\mathbb{R}}^2$ the 1-norm:*

1. $\Psi : \Lambda \to \Lambda^2_{\mathbb{R}}$ *defined by* $\Psi(x+iy) = (x,y)^T$
2. $\Psi_c : \mathbb{C}_c \to \mathbb{R}^{2|0}$ *defined by* $\Psi_c(x+iy) = (x,y)^T$
3. $\Psi_a : \mathbb{C}_a \to \mathbb{R}^{0|2}$ *defined by* $\Psi_a(x+iy) = (x,y)^T$

where "T" is for transpose to get column vectors.

Proof. While generally $||z+w|| \leq ||z|| + ||w||$ when the Grassmann components of z and w are non-overlapping we find that $||z+w|| = ||z||+||w||$. The proof is formulated at the level of Grassmann components. The 1-norm on $\mathbb{C}$ is defined so that if $z \in \mathbb{C}$ then $|z| = |x+iy| = |x| + |iy| = |x| + |y|$ where $x, y \in \mathbb{R}$ and $|x|$ denotes the absolute value of x. Consider $z \in \Lambda(\mathbb{C})$ then in terms of the Grassmann generators

$$z = \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} z_I \zeta^I.$$

Notice the Grassmann coefficients $z_I \in \mathbb{C}$ can be written in terms of their real and imaginary components; $z_I = x_I + iy_I$ with $x_I, y_I \in \mathbb{R}$ for each multi-index I. Thus

$$z = \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} (x_I + iy_I)\zeta^I = \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} x_I \zeta^I + \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} iy_I \zeta^I$$

Consequently,

$$||z|| = ||\sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} x_I \zeta^I + \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} iy_I \zeta^I|| = \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} |x_I| + \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} |iy_I|$$

Let $(x,y) \in \Lambda^2_{\mathbb{R}}$, observe that the 1-norm of (x,y) is denoted $||(x,y)||$ and is induced from the norm on Λ which is also denoted $||\cdot||$, $||(x,y)|| = ||x|| + ||y||$. Real supernumbers x, y have Grassmann expansions,

$$x = \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} x_I \zeta^I. \qquad y = \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} y_I \zeta^I.$$

Hence,

$$||x|| = \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} |x_I|. \qquad ||y|| = \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} |y_I|.$$

and as $|iy_I| = |y_I|$ for each I we find the identity,

$$||x+iy|| = ||x|| + ||y|| = ||(x,y)||$$

where the norm on the left is for $\Lambda(\mathbb{C})$, and the norm on the right is for $\Lambda_{\mathbb{R}}^2$. Notice $||\Psi(x+iy)|| = ||(x,y)|| = ||x+iy|| = ||x|| + ||y||$. thus Ψ is an isometry. Similar arguments hold for Ψ_c and Ψ_a. □

Matrix multiplication for matrices of supernumbers follows the same pattern as with ordinary real or complex entried matrices, with the caveat that we must maintain the multiplicative ordering of the supernumbers. Following Section 3.6.3 we define $gl(p \times q, \Lambda_{\mathbb{R}})$ to be $p \times q$ matrices with entries in $\Lambda_{\mathbb{R}}$.

Definition 5.2.11. *Let $v \in \Lambda^2$ and suppose $A \in gl(2 \times 2, \Lambda_{\mathbb{R}})$ then L_A and R_A are left and right multiplications by A on Λ^2 defined by*

$$L_A(v) = Av \qquad \text{and} \qquad R_A(v) = vA$$

Lemma 5.2.12. *Suppose that $A \in gl(2 \times 2, \Lambda_{\mathbb{R}})$ then*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

induces a $\mathbb{R}_c$-linear mapping $T : \mathbb{C}_c \to \Lambda$ defined by $T(z) = (\Psi^{-1} \circ L_A \circ \Psi_c)(z)$ for all $z \in \mathbb{C}_c$.

Proof. Let $z \in \mathbb{C}_c$ then there exist $x, y \in \mathbb{R}_c$ such that $z = x + iy$. Observe,

$$\begin{aligned} T(z) &= \Psi^{-1}(A\Psi_c(x+iy)) \\ &= \Psi^{-1}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right) \\ &= \Psi^{-1}\left(\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}\right) \\ &= (ax+by) + i(cx+dy). \end{aligned} \tag{5.9}$$

Now calculate $T(1)$,

$$\begin{aligned} T(1) &= \Psi^{-1}(A\Psi_c(1+i0)) \\ &= \Psi^{-1}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \\ &= \Psi^{-1}\left(\begin{pmatrix} a \\ c \end{pmatrix}\right) \\ &= a + ic. \end{aligned} \tag{5.10}$$

Next we calculate $T(i)$,

$$\begin{aligned} T(i) &= \Psi^{-1}(A\Psi_c(0+i)) \\ &= \Psi^{-1}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ &= \Psi^{-1}\left(\begin{pmatrix} b \\ d \end{pmatrix}\right) \\ &= b + id. \end{aligned} \tag{5.11}$$

Notice $T(z) = (ax + by) + i(cx + dy) = (a + ic)x + (b + id)y = T(1)x + T(i)y$. Thus by Lemma 5.2.2 we find that T is $\mathbb{R}_c$-linear.

□

The lemma above was for $\mathbb{C}_c = {}^0\Lambda$ whereas the following lemma is for $\Lambda = \mathbb{C}_c \oplus \mathbb{C}_a$. Because multiplicative ordering is not modified in the proof above, we can prove the following lemma by nearly the same calculation.

Lemma 5.2.13. *Suppose that $A \in gl(2 \times 2, \Lambda_{\mathbb{R}})$, then*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

induces a left-$\Lambda_{\mathbb{R}}$-linear mapping $T : \Lambda \to \Lambda$ defined by $T(z) = (\Psi^{-1} \circ L_A \circ \Psi)(z)$ for all $z \in \Lambda$.

5.2.4 Lemma IV (Remmert)

This lemma will reveal the Cauchy Riemann equations for a commuting super variable.

Lemma 5.2.14. *Suppose we are given a matrix $A \in gl(2 \times 2, \Lambda_{\mathbb{R}})$ such that*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the following are equivalent:

1. *The induced mapping $T = \Psi^{-1} \circ L_A \circ \Psi_c$ is $\mathbb{C}_c$-linear.*
2. *The entries of A satisfy $c = -b$ and $d = a$.*

Proof. Let us begin with the formula for the induced mapping we found in the proof for lemma 5.2.12,

$$T(x + iy) = (ax + by) + i(cx + dy)$$

Next assume (1.) is true and make use of part (2.) of Lemma 5.2.7 and the formula for $T(1)$ from Lemma 5.2.12

$$T(x+iy) = T(1)(x+iy) = (a+ic)(x+iy) = ax - cy + i(ay+cx)$$

Thus, equating real and imaginary parts of $T(x+iy)$ we find $ax - cy = ax + by$ and $cx + dy = ay + cx$. We conclude that $b = -y$ and $d = a$ since the equations held for all $x, y \in \mathbb{R}_c$. □

This lemma will reveal the Cauchy Riemann equations for super variables.

Lemma 5.2.15. *Suppose we are given a matrix $A \in gl(2\times 2, \Lambda_{\mathbb{R}})$ such that*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the following are equivalent:

1. *The induced mapping $T = \Psi^{-1} \circ L_A \circ \Psi$ is left-Λ-linear.*
2. *The entries of A satisfy $c = -b$ and $d = a$.*

Proof. Simply follow the same calculation as was used in Lemma 5.2.14. We never had to change the multiplicative ordering of the supernumbers so the calculation still holds. □

5.2.5 Right Linearity Matches Frechet Derivative

We have given the definition of G^1 for functions of open subsets of $\mathbb{K}^{p|q}$. Our goal here is to connect the derivatives of functions defined on $\mathbb{C}^{p|q}$ with those defined on $\mathbb{R}^{2p|2q}$. Let us begin by making an observation about the Frechet derivatives in each case, suppose that f is superdifferentiable,

1. df is $\mathbb{C}_c = {}^0\Lambda$-linear for $f : \mathbb{C}^{p|q} \to \Lambda$
2. $d(f \circ \Psi_c^{-1})$ is $\mathbb{R}_c$-linear for $f : \mathbb{R}^{2p|2q} \to \Lambda$.

Moreover, we can express df in terms of the Jacobian matrix, we denote $f = u + iv$ where both u and v are $\Lambda_{\mathbb{R}}$-valued functions.

If we extend df to the total tangent spaces $\Lambda(p,q)$ and $\Lambda_{\mathbb{R}}(2p,2q) = \mathbb{R}^{2p|2q} \oplus \mathbb{R}^{\bar{2}p|\bar{2}q}$, then the differentials are right-Λ and right-$\Lambda_{\mathbb{R}}$ linear mappings. The right-linearity follows from $d_x f(H) = X_x(f) = \sum_{A=1}^{p+q} \overline{H^A \frac{\partial f}{\partial z^A}}(x)$ since

$$\begin{aligned} d_x f(bH) &= \textstyle\sum_{A=1}^{p+q} (bH)^A \frac{\partial f}{\partial z^A}(x) \\ &= b \textstyle\sum_{A=1}^{p+q} \overline{H^A \frac{\partial f}{\partial z^A}}(x) \\ &= b d_x f(H) \end{aligned} \tag{5.12}$$

demonstrates we can pull super scalars to the left. The Jacobian matrix of f is J_f, and in row notation we define J_f via $d_x f(H) = \vec{H} J_f$. Right linear mappings allow us to extract superscalars to the left without any extra signs. We observe it is natural to use row vectors for this task. All the lemmas we just found for left-linear maps have close analogies for right-linear mappings. Moreover, the definitions for right linearity are analogous to those for left-linearity.

Lemma 5.2.16. *We denote $z = (x + iy) \in \Lambda$ and $\bar{z} = (x - iy) \in \Lambda$ with $x, y \in \Lambda_{\mathbb{R}}$ throughout this lemma. Given mapping $T : \Lambda \to \Lambda$, then the following are equivalent:*

1. *T is right-$\Lambda_{\mathbb{R}}$-linear*
2. *$T(x + iy) = xT(1) + yT(i)$.*
3. *If we define $\lambda \equiv \frac{1}{2}(T(1) - iT(i))$ and $\mu \equiv \frac{1}{2}(T(1) + iT(i))$ then $T(z) = z\lambda + \bar{z}\mu$*

Lemma 5.2.17. *Given $T : \Lambda \to \Lambda$ is right-$\Lambda_{\mathbb{R}}$-linear, then the following are equivalent:*

1. *$T(i) = iT(1)$*
2. *$T(z) = zT(1)$ for all $z \in \Lambda$*
3. *T is right-Λ-linear*

Lemma 5.2.18. *A matrix $A \in gl(2 \times 2, \Lambda_{\mathbb{R}})$*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

induces a right-$\Lambda_{\mathbb{R}}$-linear mapping $T : \Lambda \to \Lambda$ defined by $T(z) = (\Psi^{-1} \circ R_A \circ \Psi)(z)$ for all $z \in \Lambda$. (we use Ψ_c to mean the mapping to row vectors in this context)

Lemma 5.2.19. *Suppose we are given a matrix $A \in gl(2 \times 2, \Lambda_{\mathbb{R}})$ such that*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the following are equivalent:

1. *The induced mapping $T = \Psi^{-1} \circ R_A \circ \Psi$ is Λ-linear.*
2. *The entries of A satisfy $c = -b$ and $d = a$.*

Theorem 5.2.20. *Introduce the notation*

$$\Psi_{p|q} = \Psi_c \times \cdots \times \Psi_c \times \Psi_a \times \cdots \times \Psi_a : \mathbb{C}^{p|q} \to \mathbb{R}^{2p|2q}.$$

If $f : \mathbb{C}^{p|q} \to \Lambda$ *is complex-super-differentiable on* U*, then* $f \circ \Psi_{p|q}^{-1} : \Psi_{p|q}(U) \subseteq \mathbb{R}^{2p|2q} \to \Lambda$ *is real-super-differentiable on* $\Psi_{p|q}(U)$ *and denoting* $f = u + iv$ *where* $Re(f \circ \Psi_{p|q}^{-1}) = u$ *and* $Im(f \circ \Psi_{p|q}^{-1}) = v$ *then we find the Cauchy Riemann equations*

$$\frac{\partial u}{\partial x^M} = \frac{\partial v}{\partial y^M} \qquad \frac{\partial v}{\partial x^M} = -\frac{\partial u}{\partial y^M}$$

hold for each $M = 1, 2, \ldots, p+q$ *and* $z^M = x^M + iy^M$ *where* $(z^M) \in U$*.*

Proof. The proof follows from the lemmas we have discussed. We give a detailed proof in the case of one supervariable in the section that follows. □

5.3 Cauchy Riemann Equations for One Super Complex Variable

We consider the simplest interesting cases in this discussion. There are two cases:

1. $f : \mathbb{C}_c \to \Lambda$ is (complex) superdifferentiable.
2. $f : \mathbb{C}_a \to \Lambda$ is (complex) superdifferentiable.

Both of these cases differ from the classic non-super case since df acts on the total tangent space which happens to be Λ for both cases. Our goal is to see what we can say about these functions once reinterpreted as functions of two real super variables. In particular, what can we say about the (real) superdifferentiability of $f \circ \Psi_c^{-1}$ and $f \circ \Psi_a^{-1}$?

Let us begin with case (1.). Suppose $f : \mathbb{C}_c \to \Lambda$ is (complex) superdifferentiable. Notice that $T\mathbb{C}_c = \Lambda$ and since $f \in G^1(\mathbb{C}_c)$, we can easily deduce df satisfies $df(bX) = bdf(X)$ for all $b \in \Lambda$ and $X \in T\mathbb{C}_c = \Lambda$. Thus $df : \Lambda \to \Lambda$ is a right-Λ-linear mapping. The induced mapping $\Psi \circ df \circ \Psi^{-1} : \Lambda_{\mathbb{R}}^2 \to \Lambda_{\mathbb{R}}^2$ is linear. Therefore, there exists a matrix representative with respect to the canonical basis $\Psi(1) = (1, 0)$ and $\Psi(i) = (0, 1)$,

$$(\Psi \circ df \circ \Psi^{-1})(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = vA$$

for all $v = (x, y) \in \Lambda_{\mathbb{R}}^2$ and $a, b, c, d \in \Lambda_{\mathbb{R}}$. Solving $(\Psi \circ df \circ \Psi^{-1})(v) = vA = R_A(v)$ for df yields $df = \Psi^{-1} \circ R_A \circ \Psi$. We have exactly the situation described in Lemma 5.2.19. df is a right-Λ-linear map which is induced from the 2x2 matrix A. Hence the matrix A has the form,

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

In fact, the matrix A is the Jacobian matrix for the function f in real notation.

Let us pause to discuss the Jacobian in this context. Generally if $g : \mathbb{R}^{2|0} \to \Lambda_{\mathbb{R}}^2$ then superdifferentiability gives us the existence of partial derivatives of all the component functions. Denoting $g = (g^1, g^2)$,

$$dg^1(h^1, h^2) = h^1 \frac{\partial g^1}{\partial x} + h^2 \frac{\partial g^1}{\partial y} \quad \text{and} \quad dg^2(h^1, h^2) = h^1 \frac{\partial g^2}{\partial x} + h^2 \frac{\partial g^2}{\partial y}$$

so in row notation, using $h = (h^1, h^2)$

$$dg(h) = \left(h^1 \frac{\partial g^1}{\partial x} + h^2 \frac{\partial g^1}{\partial y}, h^1 \frac{\partial g^2}{\partial x} + h^2 \frac{\partial g^2}{\partial y} \right) = \begin{pmatrix} h^1 & h^2 \end{pmatrix} \begin{pmatrix} \partial g^1/\partial x & \partial g^2/\partial x \\ \partial g^1/\partial y & \partial g^2/\partial y \end{pmatrix} = hJ_g$$

where we have introduced the Jacobian matrix J_g.

Let us apply the notation above to the case $g = \Psi \circ f \circ \Psi_c^{-1} = (u, v)$ where u, v are functions from $\mathbb{R}^{2|0}$ to $\Lambda_{\mathbb{R}}$ then,

$$J_{\Psi \circ f \circ \Psi_c^{-1}} = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}.$$

where we have used the notations

$$u_x = \frac{\partial u}{\partial x}, \qquad u_y = \frac{\partial u}{\partial y}, \qquad v_x = \frac{\partial v}{\partial x}, \qquad v_y = \frac{\partial v}{\partial y}$$

for the various (real) super derivatives.

Theorem 5.3.1. *If $f : U \subseteq \mathbb{C}_c \to \Lambda$ is complex superdifferentiable on U then the induced mapping $f \circ \Psi_c^{-1} = u + iv : \Psi_c(U) \subseteq \mathbb{R}^{2|0} \to \Lambda$ is real superdifferentiable on $\Psi_c(U)$ and the super Cauchy Riemann equations hold on $\Psi_c(U)$;*

$$u_x = v_y \quad \textit{and} \quad u_y = -v_x.$$

Moreover, suppressing the Ψ notations,

$$f'(z) = u_x + iv_x = v_y - iu_y$$

Proof. Complex superdifferentiability of $f : U \subseteq \mathbb{C}_c \to \Lambda$ implies

$$d_z f(h) = h \frac{\partial f}{\partial z}$$

thus $d_z f : \Lambda \to \Lambda$ is a right-Λ linear map. Use Lemma 5.2.18 to see there exists

$J_f(z) \in gl(2 \times 2, \Lambda_{\mathbb{R}})$ such that it induces $d_z f$ as follows,

$$d_z f(h) = (\Psi^{-1} \circ R_{J_f(z)} \circ \Psi)(h)$$

Real differentiability and the Cauchy Riemann equations follow from the algebra given above the theorem and Lemma 5.2.19. □

Let us discuss case (2.). Suppose $f : \mathbb{C}_a \to \Lambda$ is (complex) superdifferentiable. If we examine the arguments that established Theorem 5.3.1, then we will note that the algebra involved did not require us to commute elements anywhere. Thus the arguments will hold again in this context, and we can state an analogous theorem for a function of one odd complex variable $\theta = \phi^1 + i\phi^2$ (denoting odd real variables by ϕ^1 and ϕ^2).

Theorem 5.3.2. *If $f : \mathbb{C}_a \to \Lambda$ is (complex) superdifferentiable, then the induced mapping $f \circ \Psi_a^{-1} = u + iv : \mathbb{R}^{0|2} \to \Lambda$ is (real) superdifferentiable and the super Cauchy Riemann equations hold;*

$$\frac{\partial u}{\partial \phi_1} = \frac{\partial v}{\partial \phi_2} \quad \textit{and} \quad \frac{\partial u}{\partial \phi_2} = -\frac{\partial v}{\partial \phi_1}.$$

Moreover, suppressing the Ψ notations,

$$f'(\theta) = \frac{\partial u}{\partial \phi_1} + i\frac{\partial v}{\partial \phi_1} = \frac{\partial v}{\partial \phi_2} - i\frac{\partial u}{\partial \phi_2}$$

5.4 Formal Derivatives of Conjugate Variables

In this section we define partial derivatives with respect to complex supercoordinates z^M in terms of partial derivatives with respect to real super coordinates x^M and y^m where $z^M = x^M + iy^M$. These are formal derivatives since they are not generally understood in terms of a limiting process. In this section we will suppress the Ψ notation; we identify $x + iy$ and (x, y) hopefully without danger of confusion.

Definition 5.4.1. *Let (z^M) be complex coordinates in $\mathbb{C}^{p|q}$ and suppose $(x^M), (y^M)$ are real coordinates in $\mathbb{R}^{p|q}$ such that $z^M = x^M + iy^M$ for each $M = m = 1, 2, \ldots, p$ and $M = \alpha = 1, 2, \ldots, q$, then we define the formal symbols*

$$\frac{\partial}{\partial z^M} = \frac{1}{2}\left(\frac{\partial}{\partial x^M} - i\frac{\partial}{\partial y^M}\right) \qquad \frac{\partial}{\partial \bar{z}^M} = \frac{1}{2}\left(\frac{\partial}{\partial x^M} + i\frac{\partial}{\partial y^M}\right)$$

These act on $\Lambda(\mathbb{C})$-valued functions whose domain resides in $\mathbb{R}^{p|q}$. They are simply a notation to encode a complex-linear combination of real super derivatives.

5.4.1 Cauchy Riemann Equations and the $\partial/\partial z$-Notation

We will see how the formal derivatives of the last section match complex derivatives in the appropriate context. Observe, if the Cauchy Riemann equations $u_x = v_y$ and $u_y = -v_x$ hold, then

$$\begin{aligned} \tfrac{\partial}{\partial z}(f) &= \tfrac{1}{2}(\tfrac{\partial}{\partial x} - i\tfrac{\partial}{\partial y})(u+iv) \\ &= \tfrac{1}{2}[u_x - iu_y + iv_x - i^2 v_y] \\ &= u_x + iv_x \end{aligned} \tag{5.13}$$

Similarly, just changing the last step we find that $\frac{\partial}{\partial z}(f) = u_y - iv_y$. However, from the very definition of super derivatives in the last section we showed that $f'(z) = u_x + iv_x$. In constrast, the total derivative of f(z) with respect to $\bar{z}$ is not well-defined since the differential of f is not right-complex-linear in $\bar{z}$. We can say that $\frac{\partial}{\partial \bar{z}}$ is a well-defined operation on functions of z alone; we simply require that $\partial f/\partial \bar{z} = \frac{1}{2}(f_x - if_y)$ with the understanding that f(z) should be replaced with f(x,y) to make the differentiations sensible.

Remark 5.4.2. *Total derivatives of z and $\bar{z}$ only make sense for functions of just z or just $\bar{z}$. Partial derivatives with respect to z and $\bar{z}$ are just a notations for differentiations on the associated functions of two real variables. While the notation appears complex, it is in fact just notation for a theory of real variables. These remarks apply equally well to even or odd variables.*

Example 5.4.3. *A good example to illustrate the difference between real and complex differentiability is $f(z) = \bar{z} = x - iy$ where $z = x + iy \in {}^r\Lambda$. This function is not complex super differentiable since df is not right-Λ-linear. This is also seen by examining the failure of the Cauchy Riemann equations. We have that $u = x$ and $v = -y$ thus $u_x \neq v_y$. However, $f(x,y) = x - iy$ is clearly real super differentiable since df is right-$\Lambda_{\mathbb{R}}$-linear. In summary, f is not in $G^1({}^r\Lambda)$. However f is in $G^1(\mathbb{R}^{2|0})$ for $r = 0$ or in $G^1(\mathbb{R}^{0|2})$ for $r = 1$. In both cases f is smooth since it is a polynomial at the level of Grassmann coefficients.*

5.4.2 Properties of Formal Derivatives

Suppose that $z = x+iy \in \mathbb{C}_c$ and $\bar{z} = x-iy \in \mathbb{C}_c$ such that $x, y \in \mathbb{R}_c$ or $z = x+iy \in \mathbb{C}_a$ and $\bar{z} = x - iy \in \mathbb{C}_a$ with $x, y \in \mathbb{R}_a$, then we obtain the following properties for the formal derivatives with respect to z and $\bar{z}$:

1. $\partial_z(f+g) = \partial_z(f) + \partial_z(g)$
2. $\partial_{\bar{z}}(f+g) = \partial_{\bar{z}}(f) + \partial_{\bar{z}}(g)$
3. $\partial_z(fc) = \partial_z(f)c$ and $\partial_z(cf) = (-1)^{\epsilon(z)\epsilon(c)}c\partial_z(f)$

4. $\partial_{\bar{z}}(fc) = \partial_{\bar{z}}(f)c$ and $\partial_{\bar{z}}(cf) = (-1)^{\epsilon(\bar{z})\epsilon(c)}c\partial_{\bar{z}}(f)$
5. $\frac{\partial z}{\partial z} = 1$ and $\frac{\partial \bar{z}}{\partial \bar{z}} = 1$
6. $\frac{\partial z}{\partial \bar{z}} = 0$ and $\frac{\partial \bar{z}}{\partial z} = 0$
7. $\partial_z(fg) = (\partial_z f)g + (-1)^{\epsilon(f)\epsilon(z)}f(\partial_z g)$
8. $\partial_{\bar{z}}(fg) = (\partial_{\bar{z}} f)g + (-1)^{\epsilon(f)\epsilon(z)}f(\partial_{\bar{z}} g)$

Here f, g are functions of the real supervariables x, y, $c \in \Lambda$, and where appropriate we assume the functions or numbers are pure. These properties that reflect the fact that ∂_z and $\partial_{\bar{z}}$ are derivations on functions of two supervariables. The following equation is an interesting heuristic since $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{i}{2}(\bar{z} - z)$

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

where z could be a commuting or an anticommuting variable. A similar equation holds for $\bar{z}$. It is tempting to think of this as the chain rule corresponding to a coordinate change on the real superplane, but neither z nor $\bar{z}$ are real supervariables.

5.5 Algebra of Conjugate Variables for $\mathbb{R}^{4|4}$

Because we would like to use θ to refer to the reparametrized odd-coordinates of $\mathbb{R}^{4|4}$ we begin by denoting a typical point in $\mathbb{R}^{4|4}$ by (x^m, ϕ^k). These coordinates on $\mathbb{R}^{4|4}$ are natural from a mathematical view point; they satisfy the simple reality conditions $(x^m)^* = x^m$ and $(\phi^k)^* = \phi^k$. Unfortunately, it is not immediately obvious how to generalize a Lorentz covariance for the odd-coordinates. However, if we reparametrize the odd coordinates so that they form Weyl spinors over Minkowski space, then it is known how to transform such coordinates under a Lorentz transformation (or more accurately a corresponding $SL(2, \mathbb{C})$ transformation). To that end, we define,

$$\begin{aligned} \theta^1 &= \phi^1 + i\phi^2 \\ \bar{\theta}^1 &= \phi^1 - i\phi^2 \\ \theta^2 &= \phi^3 + i\phi^4 \\ \bar{\theta}^2 &= \phi^3 - i\phi^4. \end{aligned} \tag{5.14}$$

Notice that $(\theta^1)^* = \bar{\theta}^1$ and $(\theta^2)^* = \bar{\theta}^2$. This interdependence of θ and $\bar{\theta}$ is required for Weyl spinors over Minkowski space; we must have $(\theta^\alpha)^* = \bar{\theta}^{\dot{\alpha}}$. For future reference

we note that the inverse transformations are easily computed,

$$\begin{array}{ll} \phi^1 &= \frac{1}{2}(\theta^1 + \bar{\theta}^1) \\ \phi^2 &= \frac{1}{2i}(\theta^1 - \bar{\theta}^1) \\ \phi^3 &= \frac{1}{2}(\theta^2 + \bar{\theta}^2) \\ \phi^4 &= \frac{1}{2i}(\theta^2 - \bar{\theta}^2). \end{array} \tag{5.15}$$

5.5.1 Index Suppressing Conventions

In the physics literature there are certain canonical expressions of $\theta, \bar{\theta}$ and their products. It is customary to suppress the indices α and $\dot{\alpha}$ when possible, but that requires some care. To begin, we define how to lower indices,

$$\theta_\beta = \epsilon_{\beta\alpha}\theta^\alpha \qquad\qquad \bar{\theta}_{\dot{\beta}} = \epsilon^{\dot{\beta}\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} \tag{5.16}$$

where our convention is,

$$(\epsilon_{\alpha\beta}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (\epsilon_{\dot{\alpha}\dot{\beta}}) \qquad (\epsilon^{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (\epsilon^{\dot{\alpha}\dot{\beta}}). \tag{5.17}$$

We adopt the following convention for suppressed indices (up-down),

$$\theta\theta = \theta^\alpha\theta_\alpha = \epsilon_{\alpha\beta}\theta^\alpha\theta^\beta = -\theta^1\theta^2 + \theta^2\theta^1 = -2\theta^1\theta^2. \tag{5.18}$$

Sometimes the latter is written as $\theta\theta = \theta^2$, and at first glance it seems that such a quantity should be zero. After all it looks like the square of an odd variable. However, it is really just notation for a sort of fermionic dot product where instead of summing over a metric tensor we sum over the antisymmetric symbol.

Next we adopt a convention for suppressing dotted indices (down-up),

$$\bar{\theta}\bar{\theta} = \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^{\dot{\beta}}\bar{\theta}^{\dot{\alpha}} = \bar{\theta}^1\bar{\theta}^2 - \bar{\theta}^2\bar{\theta}^1 = 2\bar{\theta}^1\bar{\theta}^2. \tag{5.19}$$

Notice that it is important to distinguish between $\bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = \bar{\theta}\bar{\theta}$ and $\bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}} = -\bar{\theta}\bar{\theta}$, Grassmann spinor indices require some care.

Next, define $\sigma^m = (I, \sigma^i)$ and $\bar{\sigma}^m = (I, -\sigma^i)$. Here we use I to denote the 2x2 identity matrix and the σ^i are the Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.20}$$

The Pauli matrices possess dotted and undotted indices, $\sigma^m = (\sigma^m_{\alpha\dot{\alpha}})$. Consistent

with our previous conventions concerning the suppression of indices we introduce,

$$\theta\sigma^m\bar{\theta} = \theta^\alpha \sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}. \tag{5.21}$$

Finally we comment that the (up-down) convention applies to suppressing the sums of other undotted index carrying objects, and the (down-up) convention applies to suppressing the sums of other dotted index carrying objects. For example,

$$\theta\phi = \theta^\alpha\phi_\alpha \qquad\qquad \bar{\theta}\bar{\mathcal{X}} = \bar{\theta}_{\dot{\alpha}}\mathcal{X}^{\dot{\alpha}} \tag{5.22}$$

Our conventions for dotted and undotted Weyl spinors on $\mathbb{R}^{4|4}$ match those of [116].

5.5.2 More on the Conjugate Variable Reparametrization of $\mathbb{R}^{4|4}$

In this section we explain how products involving $(\theta^1, \theta^2, \bar{\theta}^1, \bar{\theta}^2)$ relate to the products of $(\phi^1, \phi^2, \phi^3, \phi^4)$. These are straightforward to compute; we leave these as an exercise for the reader,

$$\begin{array}{l}
\theta\theta = -2(\phi^1\phi^3 + i\phi^1\phi^4 + i\phi^2\phi^3 - \phi^2\phi^4) \\
\bar{\theta}\bar{\theta} = \ 2(\phi^1\phi^3 - i\phi^1\phi^4 - i\phi^2\phi^3 - \phi^2\phi^4) \\
\theta\sigma^0\bar{\theta} = -2i(\phi^1\phi^2 + \phi^3\phi^4) \\
\theta\sigma^1\bar{\theta} = -2i(\phi^1\phi^4 - \phi^2\phi^3) \\
\theta\sigma^2\bar{\theta} = -2i(\phi^1\phi^3 + \phi^2\phi^4) \\
\theta\sigma^3\bar{\theta} = -2i(\phi^1\phi^2 + \phi^3\phi^4).
\end{array} \tag{5.23}$$

In view of the above identities we easily calculate,

$$\begin{array}{l}
\theta\theta + \bar{\theta}\bar{\theta} = 4i(\phi^1\phi^4 + \phi^2\phi^3) \\
\theta\theta - \bar{\theta}\bar{\theta} = -4(\phi^1\phi^3 + \phi^2\phi^4).
\end{array} \tag{5.24}$$

It is then clear how to compute the inverse transformations,

$$\begin{array}{l}
\phi^1\phi^2 = \frac{1}{4i}(\theta\sigma^0\bar{\theta} + \theta\sigma^3\bar{\theta}) \\
\phi^3\phi^4 = \frac{-1}{4i}(\theta\sigma^0\bar{\theta} - \theta\sigma^3\bar{\theta}) \\
\phi^2\phi^3 = \frac{1}{8i}(\theta\theta + \bar{\theta}\bar{\theta} + 2\theta\sigma^1\bar{\theta}) \\
\phi^1\phi^4 = \frac{1}{8i}(\theta\theta + \bar{\theta}\bar{\theta} - 2\theta\sigma^1\bar{\theta}) \\
\phi^2\phi^4 = \frac{1}{8}(\theta\theta - \bar{\theta}\bar{\theta} + 2i\theta\sigma^2\bar{\theta}) \\
\phi^1\phi^3 = \frac{-1}{8}(\theta\theta - \bar{\theta}\bar{\theta} - 2i\theta\sigma^2\bar{\theta}).
\end{array} \tag{5.25}$$

The products of three Grassmanns are related as follows,

$$\begin{array}{l}
\theta\theta\bar{\theta}^1 = 4(\phi^1\phi^2\phi^4 - i\phi^1\phi^2\phi^3) \\
\theta\theta\bar{\theta}^2 = 4(i\phi^1\phi^3\phi^4 - \phi^2\phi^3\phi^4) \\
\bar{\theta}\bar{\theta}\theta^1 = -4(\phi^1\phi^2\phi^4 + i\phi^1\phi^2\phi^3) \\
\bar{\theta}\bar{\theta}\theta^2 = 4(\phi^2\phi^3\phi^4 + i\phi^1\phi^3\phi^4).
\end{array} \tag{5.26}$$

The inverse relations are,

$$\begin{array}{l} \phi^1\phi^2\phi^3 = \frac{-1}{8i}(\theta\theta\bar{\theta}^1 + \bar{\theta}\bar{\theta}\theta^1) \\ \phi^1\phi^2\phi^4 = \frac{1}{8}(\theta\theta\bar{\theta}^1 - \bar{\theta}\bar{\theta}\theta^1) \\ \phi^1\phi^3\phi^4 = \frac{1}{8i}(\theta\theta\bar{\theta}^2 + \bar{\theta}\bar{\theta}\theta^2) \\ \phi^2\phi^3\phi^4 = \frac{-1}{8}(\theta\theta\bar{\theta}^2 - \bar{\theta}\bar{\theta}\theta^2). \end{array} \tag{5.27}$$

Lastly, we relate the product of four Grassmans,

$$\phi^1\phi^2\phi^3\phi^4 = \frac{-1}{16}\theta\theta\bar{\theta}\bar{\theta}. \tag{5.28}$$

With all of these relations in hand it becomes a straight forward, but tedious, exercise to relate the component field expansion,

$$F = f + \theta\phi + \bar{\theta}\bar{\mathcal{X}} + \theta\theta m + \bar{\theta}\bar{\theta}n + \theta\sigma^m\bar{\theta}v^m + \theta\theta\bar{\theta}\bar{\lambda} + \bar{\theta}\bar{\theta}\theta\psi + \theta\theta\bar{\theta}\bar{\theta}d. \tag{5.29}$$

to the fermionic Taylor series expansion (relative to the ϕ^k, $k = 1, 2, 3, 4$ coordinates)

$$F = F_0 + F_i\phi^i + \frac{1}{2}F_{ij}\phi^i\phi^j + \frac{1}{6}F_{ijk}\phi^i\phi^j\phi^k + \frac{1}{24}F_{ijkl}\phi^i\phi^j\phi^k\phi^l. \tag{5.30}$$

For example, $\theta\phi + \bar{\theta}\bar{\mathcal{X}} = F_i\phi^i$ implies,

$$\begin{array}{l} \phi_1 = \frac{-1}{2}(F_1 - iF_2) \\ \phi_2 = \frac{-1}{2}(F_3 - iF_4) \\ \bar{\mathcal{X}}_1 = \frac{1}{2}(F_1 + iF_2) \\ \bar{\mathcal{X}}_2 = \frac{1}{2}(F_3 + iF_4). \end{array} \tag{5.31}$$

5.6 Chiral Coordinate Derivatives of $\mathbb{R}^{4|4}$

Throughout supersymmetric physics one finds chiral and antichiral coordinates are employed to facilitate an elegant solution to the chiral and antichiral field constraints. We introduce the reader to *chiral superfields* and show how chiral coordinates provide a natural solution to the chiral constraint equation.

To begin, we give an interpretation of derivatives with respect to "chiral coordinates" (they are not real so technically they do not take values in $\mathbb{R}^{4|4}$, although there is a bijective correspondence, see [29] for details on how to view $\mathbb{R}^{4|4}$ as a particular subset of $\mathbb{C}^{4|2}$). Let $(x, \theta, \bar{\theta})$ be coordinates on $\mathbb{R}^{4|4}$ where θ and $\bar{\theta}$ are conjugate as described in the preceding sections. Introduce "chiral coordinates" $(y, \beta, \bar{\beta})$ as follows,

$y^m = x^m + i\theta\sigma^m\theta$	$\beta^\alpha = \theta^\alpha$	$\beta^{\dot{\alpha}} = \theta^{\dot{\alpha}}$

We define that the derivatives with respect to $(y, \beta, \bar{\beta})$ as follows,

$\frac{\partial}{\partial y^m} = \frac{\partial}{\partial x^m}$	$\frac{\partial}{\partial \beta^\alpha} = \frac{\partial}{\partial \theta^\alpha} - i\sigma^n_{\alpha\dot\alpha}\theta^{\dot\alpha}\frac{\partial}{\partial x^n}$	$\frac{\partial}{\partial \bar\beta^{\dot\alpha}} = \frac{\partial}{\partial \bar\theta^{\dot\alpha}} + i\theta^\alpha\sigma^n_{\alpha\dot\alpha}\frac{\partial}{\partial x^n}$

The motivation for these definitions is revealed in the heuristic calculations below. Let f be a function on $\mathbb{R}^{4|4}$ and let g denote the same function in terms of chiral variables, $f(x,\theta,\bar\theta) = g(y,\beta,\bar\beta)$, then since $g(y,\beta,\bar\beta) = f(y - i\beta\sigma\bar\beta, \beta, \bar\beta)$ the calculations below follow from a formal chain rule.

$$\begin{aligned} \frac{\partial g}{\partial y^m} &= \frac{\partial}{\partial y^m}[f(y - i\beta\sigma\bar\beta, \beta, \bar\beta)] \\ &= \frac{\partial f}{\partial x^n}\frac{\partial}{\partial y^m}[y^n - i\beta\sigma^n\bar\beta] + \frac{\partial f}{\partial \theta^\alpha}\frac{\partial}{\partial y^m}[\beta^\alpha] + \frac{\partial f}{\partial \bar\theta^{\dot\alpha}}\frac{\partial}{\partial y^m}[\bar\beta^{\dot\alpha}] \\ &= \frac{\partial f}{\partial x^m}. \end{aligned} \tag{5.32}$$

We observe that $\frac{\partial}{\partial y^m} = \frac{\partial}{\partial x^m}$. Note that

$$\begin{aligned} \frac{\partial g}{\partial \beta^\alpha} &= \frac{\partial}{\partial \beta^\alpha}[f(y - i\beta\sigma\bar\beta, \beta, \bar\beta)] \\ &= \frac{\partial f}{\partial x^n}\frac{\partial}{\partial \beta^\alpha}[y^n - i\beta\sigma^n\bar\beta] + \frac{\partial f}{\partial \theta^\delta}\frac{\partial}{\partial \beta^\alpha}[\beta^\delta] + \frac{\partial f}{\partial \bar\theta^{\dot\alpha}}\frac{\partial}{\partial \beta^\alpha}[\bar\beta^{\dot\alpha}] \\ &= \frac{\partial f}{\partial x^n}(-i\sigma^n_{\alpha\dot\alpha}\bar\beta^{\dot\alpha}) + \frac{\partial f}{\partial \theta^\alpha} \\ &= [\frac{\partial}{\partial \theta^\alpha} - i\sigma^n_{\alpha\dot\alpha}\bar\beta^{\dot\alpha}\frac{\partial}{\partial x^n}](f). \end{aligned} \tag{5.33}$$

Thus $\frac{\partial}{\partial \beta^\alpha} = \frac{\partial}{\partial \theta^\alpha} - i\sigma^n_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\frac{\partial}{\partial x^n}$. Next consider,

$$\begin{aligned} \frac{\partial g}{\partial \bar\beta^{\dot\alpha}} &= \frac{\partial}{\partial \bar\beta^{\dot\alpha}}[f(y - i\beta\sigma\bar\beta, \beta, \bar\beta)] \\ &= \frac{\partial f}{\partial x^n}\frac{\partial}{\partial \bar\beta^{\dot\alpha}}[y^n - i\beta\sigma^n\bar\beta] + \frac{\partial f}{\partial \theta^\alpha}\frac{\partial}{\partial \bar\beta^{\dot\alpha}}[\beta^\alpha] + \frac{\partial f}{\partial \bar\theta^{\dot\gamma}}\frac{\partial}{\partial \bar\beta^{\dot\alpha}}[\bar\beta^{\dot\gamma}] \\ &= \frac{\partial f}{\partial x^n}(i\beta^\alpha\sigma^n_{\alpha\dot\alpha}) + \frac{\partial f}{\partial \bar\theta^{\dot\alpha}} \\ &= [\frac{\partial}{\partial \bar\theta^{\dot\alpha}} + i\beta^\alpha\sigma^n_{\alpha\dot\alpha}\frac{\partial}{\partial x^n}](f). \end{aligned} \tag{5.34}$$

Thus $\frac{\partial}{\partial \bar\beta^{\dot\alpha}} = \frac{\partial}{\partial \bar\theta^{\dot\alpha}} + i\theta^\alpha\sigma^n_{\alpha\dot\alpha}\frac{\partial}{\partial x^n}$.
The supersymmetric or "susy" covariant derivatives are defined in terms of the $(x,\theta,\bar\theta)$ coordinates on $\mathbb{R}^{4|4}$ as follows (see [116] for physical motivations)

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\frac{\partial}{\partial x^m} \qquad \bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^m_{\alpha\dot\alpha}\frac{\partial}{\partial x^m}.$$

Observe that we may rewrite the susy covariant derivatives in terms of the formal derivatives with respect to $(y,\beta,\bar\beta)$,

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\frac{\partial}{\partial x^m} = \frac{\partial}{\partial\beta^\alpha} + 2i\sigma^n_{\alpha\dot\alpha}\bar\beta^{\dot\alpha}\frac{\partial}{\partial y^n}$$

and,

$$\bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^m_{\alpha\dot\alpha}\frac{\partial}{\partial x^m} = -\frac{\partial}{\partial\bar\beta^{\dot\alpha}}.$$

A superfield Φ is a function on $\mathbb{R}^{4|4}$. A *chiral superfield* is a function on $\mathbb{R}^{4|4}$ that satisfies the constraint $\bar D_{\dot\alpha}\Phi = 0$ for $\dot\alpha = 1,2$. We see that in chiral coordinates this

condition is simply stated as,

$$-\frac{\partial \Phi}{\partial \bar{\beta}^{\dot{\alpha}}} = 0.$$

Evidentally the solution must be constant in $\bar{\beta}$ thus,

$$\Phi = A + \beta^{\alpha}\Psi_{\alpha} + \beta\beta F$$

where A, Ψ_{α}, F are functions of y alone. Wess and Bagger describe this calculation on page 30 in [116]. In their notation there is no β; they use θ for chiral coordinates and the real coordinates (in truth they are conjugate variables as we described in the previous sections so perhaps "real" is a misleading label for θ). One can also define antichiral coordinates and similar comments apply.

In chapter 10 we mention the concept of "partial derivative with respect to θ with x held fixed" and also "partial derivative with respect to θ with y held fixed". We denote them by $\frac{\partial}{\partial \theta^{\alpha}}|_x$ and $\frac{\partial}{\partial \theta^{\alpha}}|_y$ respectively. In the notation of this section we can interpret these statements as $\frac{\partial}{\partial \theta^{\alpha}}|_x = \frac{\partial}{\partial \theta^{\alpha}}$ and $\frac{\partial}{\partial \theta^{\alpha}}|_y = \frac{\partial}{\partial \beta^{\alpha}}$.

Chapter 6

Supermanifolds

A supermanifold is typically a curved space which is locally approximated by $\mathbb{K}^{p|q}$. Naturally, $\mathbb{K}^{p|q}$ is a special example of a supermanifold in that it possesses a global coordinate chart. Generally we have to insist that the transition functions between overlapping charts are supersmooth.

Mathematicians and physicists have been developing the theory of supermanifolds for over a quarter of a century. From almost the beginning, there have been at least two distinct approaches to the foundations of the superanalysis underlying the theory. Chronologically, the first of these is based on techniques reminiscent of ideas from algebraic geometry. We think of this approach as the sheaf theoretic development of supermathematics even when the theory of sheaves may not explicitly appear in some specific treatments of the subject. Certainly, Berezin, Leites, and Kostant [13], [76] were forerunners of this method and for that matter of the entire theory.

A second approach to the formulation of superanalysis and supermanifolds was initiated separately and differently by Rogers [98], Jadczyk and Pilch [68], and DeWitt [39]. Their work is more closely related to traditional ideas in manifold theory. Much work has been done describing both the sheaf theoretic and manifold theoretic descriptions of supermanifolds and how they are related, but we mention only a few whose work has directly impacted our work here, namely Rogers [98],[99],[100], Batchelor [11], and Bruzzo [23].

6.1 Definition of Supermanifold

This definition is due to Alice Rogers in [98].

Definition 6.1.1. *Let $\mathcal{M}$ be a Hausdorff topological space.*

1. *An $(p|q)$ open chart on $\mathcal{M}$ over Λ is a pair (U, ψ) with U open in $\mathcal{M}$ and ψ a homeomorphism of U onto an open subset of $\mathbb{K}^{p|q}$.*

2. *An $(p|q)$ G^s structure on $\mathcal{M}$ over Λ is a collection $\{(U_\alpha, \psi_\alpha) \mid \alpha \in \mathcal{I}\}$ of open charts on $\mathcal{M}$ such that (i) $\mathcal{M} = \cup_{\alpha\in\mathcal{I}} U_\alpha$, (ii) for $U_\alpha \cap U_\beta \neq \emptyset$ the mapping $\psi_\beta \circ \psi_\alpha^{-1}$ is a G^∞ mapping of $\psi_\alpha(U_\alpha \cap U_\beta)$ onto $\psi_\beta(U_\alpha \cap U_\beta)$, and (iii) the collection $\{(U_\alpha, \psi_\alpha) \mid \alpha \in \mathcal{I}\}$ is a maximal collection of open charts for which (i) and (ii) hold. A collection for which (i) and (ii) hold but is not necessarily maximal is called a $(p|q)$ G^s subatlas on $\mathcal{M}$ over Λ.*

3. *An $(p|q)$ dimensional G^s supermanifold over $\mathbb{K}^{p|q}$, is a Hausdorff topological space $\mathcal{M}$ with an $(p|q)$ G^s structure on $\mathcal{M}$ over Λ.*

4. *Each U_α is called a coordinate neighborhood, and each ψ_α is a coordinate map. For each $\alpha \in \mathcal{I}$, $p+q$ local coordinate functions are defined by,*

$$u^m = \Pi_m \circ \psi_\alpha \qquad v^\beta = \Pi_{p+\beta} \circ \psi_\alpha \quad \text{or by} \quad u^M = \Pi_M \circ \psi_\alpha$$

where $m = 1, 2, \ldots, p$, $\beta = 1, 2, \ldots, q$, and $M = 1, 2, \ldots, p+q$. We use lower case Latin indices for the commuting coordinates, Greek indices for the anti-commuting coordinates, and upper case Latin indices for both.

5. *Setting $r = \infty$ defines the structure of a G^∞ supermanifold.*

There are other definitions used in the literature for supermanifold. For example, graded manifolds of Kostant [76], or the DeWitt [39] or H^∞-manifold, and the definition due to Berezin and Leites [13]. All of these are included under the category of G^∞-manifold as is discussed in [98]. The G^∞ supermanifolds allow a richer class of topologies than the other definitions.

6.2 Supersmooth Functions on a Supermanifold

In traditional geometry the class of smooth C^∞-functions on a manifold are defined to be those whose local coordinate representatives are smooth. We define supersmooth G^∞-functions in a similar fashion.

Definition 6.2.1. *Let $\mathcal{M}$ be G^∞ supermanifold and $\{(U_\alpha, \psi_\alpha) \mid \alpha \in \mathcal{I}\}$ a subatlas of $\mathcal{M}$. If U is open in $\mathcal{M}$ we define G^∞ functions on U by*

$$G^\infty(U) = \{f \mid f : U \to \Lambda, \text{ with } f \circ \psi_\alpha^{-1} \in G^\infty[\psi_\alpha(U \cap U_\alpha)], \ \forall \alpha \in \mathcal{J}\}.$$

Then $G^\infty(p)$, the germ of G^∞ functions at a point $p \in \mathcal{M}$, is likewise defined by

$$G^\infty(p) = \{f \mid \exists \text{ an open neighborhood } N \text{ of } p \text{ such that } f \in G^\infty(N)\}$$

We say two functions in $G^\infty(p)$ are equivalent iff they agree on some open set about p. Consequently it would be more rigorous to say that $G^\infty(p)$ is the set of equivalence classes of functions defined near p.

Notice that we take the class of functions with values in Λ as the object of primary interest. This is a necessary step since we wish to deal with commuting and anticommuting fields to represent bosons and fermions in physics. Such fields parity is decided by their range so we must use Λ which includes both commuting and anticommuting superscalars.

Proposition 6.2.2. *Given U open in $\mathcal{M}$, then*

1. *$G^\infty(U)$ is a graded commutative algebra over $\mathbb{K}$ with,*

$$\begin{aligned} G^\infty(U)_0 &= \{f \in G^\infty(U) \mid f(U) \subset {}^0\Lambda\} \\ G^\infty(U)_1 &= \{f \in G^\infty(U) \mid f(U) \subset {}^1\Lambda\} \end{aligned} \tag{6.1}$$

2. *$G^\infty(U)$ is a graded left Λ module with parity defined as in (1.).*

The parity of functions is given by the parity of their range.

6.3 Derivations of Supersmooth Functions

Definition 6.3.1. *Let $End^+[G^\infty(U)]$ denote the set of all left vector space endomorphisms of $G^\infty(U)$, i.e. $L \in End^+[G^\infty(U)]$ iff it is an endomorphism over $\mathbb{K}$ in the traditional sense and*

$$L(fa) = L(f)a$$

for all $a \in \Lambda$ and all $f \in G^\infty(U)$.

We note that the super partial derivatives G_M are in $End^+[G^\infty(U)]$. Other authors prefer to use right endomorphisms, for example [55]. Our notation is a synthesis of [29] and [98].

Proposition 6.3.2. *Let U be open in $\mathcal{M}$ then*

1. *$End^+[G^\infty(U)]$ is a graded associative algebra over $\mathbb{K}$ with, composition as the multiplication and with,*

$$\begin{aligned} End^+[G^\infty(U)]_0 &= \{L \in End^+[G^\infty(U)] \mid \epsilon(Lf) = \epsilon(f)\ \} \\ End^+[G^\infty(U)]_1 &= \{L \in End^+[G^\infty(U)] \mid \epsilon(Lf) = \epsilon(f) + 1\ \}. \end{aligned}$$

   If $L \in End^+[G^\infty(U)]_0 \cup End^+[G^\infty(U)]_1$ and $f \in G^\infty(U)_0 \cup G^\infty(U)_1$ then

$$\epsilon(Lf) = \epsilon(L) + \epsilon(f)$$

2. *$End^+[G^\infty(U)]$ is a graded left Λ module with parity defined as in (1.).*

A similar proposition is true regarding $End^-[G^\infty(U)]$.

Definition 6.3.3. *Let U be open in $\mathcal{M}$. A G^∞ vector field on U is an element X of $End^+[G^\infty(U)]$ such that*

1. $X(fg) = (Xf)g + (-1)^{\epsilon(f)\epsilon(X)} fXg$ *for all* $f, g \in G^\infty(U)_0 \cup G^\infty(U)_1$
2. $X(af) = (-1)^{\epsilon(a)\epsilon(X)} aXf$ *for all* $f \in G^\infty(U)_0 \cup G^\infty(U)_1$ *and* $a \in {}^0\Lambda \cup {}^1\Lambda$

The set of all G^∞ vector fields is denoted $D^1(U)$.

Although our definition is given for pure elements it should be clear how to extend linearly to impure functions and supernumbers.

Remark 6.3.4. *We have affixed the qualifier G^∞ to distinguish these vector fields from the ordinary C^∞ vector fields which stem from the Banach space structure of $\mathcal{M}$. We will see in the next few sections that odd G^∞-vector fields cannot arise as the tangent to a curve whereas even G^∞-vector fields are in correspondence with tangents to curves.*

Proposition 6.3.5. *Let U be open in $\mathcal{M}$ then $D^1(U)$ is a graded Lie left Λ module with bracket*

$$[X, Y] = XY - (-1)^{\epsilon(X)\epsilon(Y)} YX$$

Since G^∞ vector fields are in $End^+[G^\infty(U)]$ we already know how to grade them. This is Proposition 5.5 of [98].

Definition 6.3.6. *Let (U, ψ) be a chart on a G^∞ supermanifold $\mathcal{M}$ where $\psi = (u^1, \ldots, u^p, v^1, \ldots, v^q)$. For $m = 1, 2, \ldots, p$, define*

$$\frac{\partial}{\partial u^m} : G^\infty(U) \to G^\infty(U), \quad \textit{where} \quad \frac{\partial f}{\partial u^m} \equiv [G_m(f \circ \psi^{-1})] \circ \psi$$

for all $f \in G^\infty(U)$. Also, for $\alpha = 1, 2, \ldots, q$ define

$$\frac{\partial}{\partial v^\alpha} : G^\infty(U) \to G^\infty(U), \quad \textit{where} \quad \frac{\partial f}{\partial v^\alpha} \equiv [G_{p+\alpha}(f \circ \psi^{-1})] \circ \psi$$

for all $f \in G^\infty(U)$. These are the coordinate derivatives.

Proposition 6.3.7. *Let (U, ψ) be a chart on a G^∞ supermanifold $\mathcal{M}$ of supermanifold dimension $(p|q)$. The coordinate derivatives are pure G^∞ vector fields on U. In particular, for $m = 1, 2, \ldots, p$ $\partial/\partial u^m \in D^1(U)_0$, and for $\alpha = 1, 2, \ldots, q$ $\partial/\partial v^\alpha \in D^1(U)_1$. In short, $\partial/\partial u^M \in D^1(U)_{\epsilon_M}$ for $M = 1, 2, \ldots, p + q$.*

Definition 6.3.8. *We say a supervector space W is graded left $G^\infty(U)$ module over and open set $U \subseteq \mathbb{K}^{p|q}$ iff $G^\infty(U)_r W_s \subseteq W_{r+s}$ for $r, s \in \mathbb{Z}_2$.*

Proposition 6.3.9. *Let (U, ψ) be a chart on a G^∞ supermanifold $\mathcal{M}$ where $\psi = (u^1, \ldots, u^p, v^1, \ldots, v^q)$,*

1. *$D^1(U)$ is a graded left $G^\infty(U)$ module.*
2. *$D^1(U)$ is a free left $G^\infty(U)$ module with basis $\{\partial/\partial u^M\}$ for $M = 1, 2, \ldots p + q$.*

6.4 Supermanifolds and the Banach Space Correspondence

Let $\mathcal{M}$ be a supermanifold. Then one has a maximal G^∞-atlas $\mathcal{A}_\mathcal{M}$ on $\mathcal{M}$ such that for $\phi, \psi \in \mathcal{A}_\mathcal{M}$, $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is a G^∞ mapping from an open subset $\psi(U \cap V)$ of $\mathbb{K}^{p|q}$ of $\mathbb{K}^{p|q}$ onto another open subset $\phi(U \cap V)$. By Proposition 2.8 of [98] $\phi \circ \psi^{-1}$ is also a C^∞ map.

Proposition 6.4.1. *If $\mathcal{M}$ is a supermanifold with G^∞-atlas $\mathcal{A}_\mathcal{M}$, then $\mathcal{M}$ is also a Banach manifold relative to the unique maximal C^∞-atlas, containing $\mathcal{A}_\mathcal{M}$. We denote this Banach manifold by $(\mathcal{BM}, \mathcal{A}_{\mathcal{BM}})$ where, as sets $\mathcal{BM} = \mathcal{M}$ and where $\mathcal{A}_{\mathcal{BM}}$ is the maximal C^∞-atlas containing $\mathcal{A}_\mathcal{M}$.*

We will use $\mathcal{BM}$ when we wish to emphasize the Banach manifold structure of $\mathcal{M}$. In practice we will work with the subatlas $\mathcal{A}_\mathcal{M}$ of the maximal atlas of the Banach manfold $\mathcal{BM}$ since it has the additional G^∞ structure.

6.5 Vector Fields and the Banach Space Correspondence

Recall that one definition of what it means to say v is tangent to a Banach manifold is the one which follows (see [80])

Definition 6.5.1. *Let M be a Banach manifold modeled on a Banach space B. We say that v is tangent to M at $x \in M$ and write $v \in T_x M$ iff v is a mapping from the set of all C^∞ charts of M at x into B such that if (U, ψ) and (V, ϕ) are C^∞ charts of M at x then*

$$v(\psi) = d_{\phi(x)}(\psi \circ \ \phi^{-1})(v(\phi)).$$

Remark 6.5.2. *A tangent vector v is uniquely determined by the latter transfomation law and its values on an atlas of M. So to define a tangent vector v to M at x it suffices to define v at all those charts of some atlas of M which contain x in their domain.*

We find the following slight modification of Rogers' definition in [98] to be useful in our context.

Definition 6.5.3. *Let $\mathcal{M}$ be a supermanifold and $x \in \mathcal{M}$. We say that v is a tangent to $\mathcal{M}$ at x and write $v \in T_x \mathcal{M}$ iff v is a mapping from $G^\infty(x)$ to Λ such that for some open set $U \subseteq \mathcal{M}$ such that $x \in U$ and for some G^∞ vector field $X \in D^1(U)$,*

$$v(f) = X(f)(x)$$

for all $f \in G^\infty(U)$. *We say that* v *is* **even** *and write* $v \in T^0_x\mathcal{M}$ *iff* $v(G^\infty(x)_\epsilon) \subseteq {}^\epsilon\Lambda$ *for* $\epsilon = 0, 1$. *Likewise,* v *is* **odd** *and write* $v \in T^1_x\mathcal{M}$ *iff* $v(G^\infty(x)_\epsilon) \subset {}^{\epsilon+1}\Lambda$ *for* $\epsilon = 0, 1$.

Note that $T_x\mathcal{M}$ is a graded vector space with $T_x\mathcal{M} = T^0_x\mathcal{M} \oplus T^1_x\mathcal{M}$. Moreover $T_x\mathcal{M}$ is a left Λ-module which is called the **tangent module** at $x \in \mathcal{M}$. If we suppose that $\mathcal{M}$ is a $(p|q)$-dimensional supermanifold then in the language of [68] we could give $T_x\mathcal{M}$ the structure of a (p, q)-dimensional supervector space. We also note that $T^0_x\mathcal{M}$ is a $(p|q)$-dimensional supervector space while $T^1_x\mathcal{M}$ is a $(\bar{p}|\bar{q})$-dimensional supervector space. It should be noted that $T^0_x\mathcal{M}$ is a trivial $(p|q)$-dimensional supermanifold. Likewise, $T^1_x\mathcal{M}$ is a trivial $(\bar{p}|\bar{q})$-dimensional supermanifold.

Definition 6.5.4. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be supermanifolds and* $g : \mathcal{M} \to \mathcal{N}$ *a class* G^1 *function we define* $d_xg : T_x\mathcal{M} \to T_{g(x)}\mathcal{N}$ *by,*

$$d_xg(X_x)(f) \equiv X_x(f \circ g) \tag{6.2}$$

for all $f \in G^\infty_{g(x)}$ *and* $X_x \in T_x\mathcal{M}$.

We pause to note that the differential on a supermanifold was just defined for the total tangent space. In contrast, in the preceding chapter we defined the total differential for $\mathbb{K}^{p|q}$ which generalizes to $T^0_x\mathcal{M}$ in our current context. There is no inconsistency since they match on the even sector and moreover due to Proposition 3.7.6 we know that this is the only possible consistent left-linear extension to the total space.

Proposition 6.5.5. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be supermanifolds and* $g : \mathcal{M} \to \mathcal{N}$ *then* $d_xg : T_x\mathcal{M} \to T_{g(x)}\mathcal{N}$ *is a parity preserving (even) right linear transformation, that is* $d_xg \in L^-(T_x\mathcal{M}, T_{g(x)}\mathcal{N})$.

This follows from the fact that the parity of a composite function is determined as follows,

$$f \circ g \in G^\infty(U)_\epsilon \iff f \in G^\infty_\epsilon. \tag{6.3}$$

Thus, the parity of g does not determine the parity of $f \circ g : \mathcal{M} \to \Lambda$. This means that d_xg is always parity preserving; for $y = g(x)$

$$d_xg(T^\epsilon_x\mathcal{M}) \subseteq T^\epsilon_y\mathcal{N}. \tag{6.4}$$

If (U, ψ) is a chart at x of $\mathcal{A}_\mathcal{M}$ with $\psi = (x^1, \ldots, x^p, \theta^1, \ldots, \theta^q)$ and $U \subset g^{-1}(V)$ for some chart $(V, \phi) \in A_\mathcal{N}$ with $\phi = (y^1, \ldots, y^r, \beta^1, \ldots, \beta^s)$ then the matrix of d_xg is,

$$[d_xg]_{\phi,\psi} = \begin{pmatrix} (d_x(y^j \circ g)(\frac{\partial}{\partial x^i}) & (d_x(y^j \circ g)(\frac{\partial}{\partial \theta^\alpha}) \\ (d_x(\beta^\gamma \circ g)(\frac{\partial}{\partial x^i}) & (d_x(\beta^\gamma \circ g)(\frac{\partial}{\partial \theta^\alpha}) \end{pmatrix}$$

where we note that the local coordinate representative of the Frechet derivative is a Grassmann valued matrix. Also notice that the matrix has the usual block decom-

position

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, D have entries from ${}^0\Lambda$ and B, C have entries from ${}^1\Lambda$.

Observation 6.5.6. *There are several supermanifold structures on Λ. Denote the projections onto ${}^0\Lambda$ and ${}^1\Lambda$ by Π_c and Π_a respective where if $(x+\theta) \in \Lambda$ with $x \in {}^0\Lambda$ and $\theta \in {}^1\Lambda$ then $\Pi_c(x+\theta) = x$ and $\Pi_a(x+\theta) = \theta$. Notice that $\psi = \Pi_c \times \Pi_a$ is a coordinate chart which makes Λ a $(1|1)$ dimensional supermanifold. On the other hand $\bar{\psi} = \Pi_a \times \Pi_c$ makes Λ a $(\bar{1}, \bar{1})$ dimensional supermanifold. Furthermore, both ${}^0\Lambda$ and ${}^1\Lambda$ can be given a variety of supermanifold structures: $(0|1)$, $(1|0)$, $(\bar{0}, \bar{1})$ or $(\bar{1}, \bar{0})$. This is largely a matter of book-keeping.*
For convenience we will assume that Λ is a $(1|1)$ dimensional supermanifold while ${}^0\Lambda$ is a $(1|0)$ dimensional supermanifold and ${}^1\Lambda$ is a $(0|1)$ dimensional supermanifold. Just to be clear let us write the standard coordinate charts for future reference: Let $z = x + \theta \in \Lambda$ where $x \in {}^0\Lambda$ and $\theta \in {}^1\Lambda$ then,

$$\psi(z) = (x, \theta)$$

so $\psi : \Lambda \to \mathbb{K}^{1|1}$. The identity map on ${}^0\Lambda$ makes ${}^0\Lambda$ a $(1|0)$ dimensional supermanifold. The identity map on ${}^1\Lambda$ makes ${}^1\Lambda$ a $(0|1)$ dimensional supermanifold.
Finally since the tangent module is twice as large as the parameter space we find that the following identifications are natural: $T_x({}^0\Lambda) = \Lambda$, $T_\theta({}^1\Lambda) = \Lambda$, $T_z\Lambda = \mathbb{K}^{2|2}$.

Obviously our definition of a tangent vector $v \in T_x\mathcal{M}$ depends on the vector field X used in the definition. We examine this dependence in more detail. Assume that U, V are open in $\mathcal{M}$, that $x \in U \cap V$, that X is a vector field on U, that Y is a vector field on V, and that

$$v(f) = X(f)(x), \qquad v(g) = Y(g)(x)$$

for all $f \in G^\infty(U)$, $g \in G^\infty(V)$. Then

$$X(f)(x) = Y(f)(x)$$

for all $f \in G^\infty(U \cap V)$. Moreover if $(\mathcal{O}, \psi)$ is a chart of $\mathcal{M}$ at x then, on $\mathcal{O} \cap U \cap V$,

$$X = \sum_{A=1}^{p+q} X_\psi^A \frac{\partial}{\partial z^A}, \qquad Y = \sum_{A=1}^{p+q} Y_\psi^A \frac{\partial}{\partial z^A}$$

where $\psi = (z^1, z^2, \ldots, z^{p+q})$ and where X_ψ^A, Y_ψ^A are G^∞ maps from $\mathcal{O} \cap U \cap V$ into Λ.

Moreover

$$\sum_{A=1}^{p+q} X_\psi^A(x)\frac{\partial f}{\partial z^A}(x) = X(f)(x) = Y(f)(x) = \sum_{A=1}^{p+q} Y_\psi^A(x)\frac{\partial f}{\partial z^A}(x)$$

for all $f \in G^\infty(\mathcal{O} \cap U \cap V)$. If we choose $f = z^B$, $1 \leq B \leq p+q$, we see that

$$X_\psi^B(x) = Y_\psi^B(x)$$

for all B.
Notice that if $\mathcal{M}$ is a supermanifold then $T\mathcal{M} = \cup_{p\in\mathcal{M}} T_p\mathcal{M}$ may be given a supermanifold structure just as in the case for ordinary manifolds. This follows using the G^∞ transformation laws relating two sets of components of tangent vectors to $\mathcal{M}$.
Observe that there exists a well-defined mapping $\beta_x : T_x^0\mathcal{M} \to T_x\mathcal{BM} \subseteq T_x\mathcal{M}$ defined by

$$\beta_x(v)(\psi) = (X_\psi^1(x), X_\psi^2(x), \cdots, X_\psi^{p+q}(x))$$

for $v \in T_x^0\mathcal{M}$ and ψ a chart of $\mathcal{M}$. Notice that we have defined $\beta_x(v)$ only on charts of $\mathcal{M}$ at x but if we show that the appropriate transformation law holds then $\beta_x(v)$ has a unique extension to all charts of $\mathcal{BM}$ at x (see Remark 6.5.2) and thus uniquely defines an element of $T_x\mathcal{BM}$. With this in mind let $(U,\psi),(V,\phi)$ be charts of $\mathcal{BM}$ at x, and observe that

$$\begin{aligned}\beta_x(v)(\psi) &= (X_\psi^1(x), X_\psi^2(x), \cdots, X_\psi^{p+q}(x)) \\ &= d_{\phi(x)}(\psi\circ\phi^{-1})(X_\phi^1(x), X_\phi^2(x), \cdots, X_\phi^{p+q}(x)) \\ &= d_{\phi(x)}(\psi\circ\phi^{-1})\beta_x(v)(\phi).\end{aligned}$$

Proposition 6.5.7. *If $\mathcal{M}$ is a supermanifold and $x \in \mathcal{M}$ then β_x is a ${}^0\Lambda$-linear vector space isomorphism from $T_x^0\mathcal{M}$ onto $T_x\mathcal{BM}$.*

Proof. It is clear that β_x is a ${}^0\Lambda$-linear vector space homomorphism. We show that β_x is injective. Assume that $v \in T_x^0\mathcal{M}$ such that $\beta_x(v) = 0$. Then there is an open set $U \subseteq \mathcal{M}$ and a vector field X on U such that $x \in U$, $v(f) = X(f)(x)$ for $f \in G^\infty(U)$ and $0 = \beta_x(v)(\psi) = (X_\psi^1(x), X_\psi^2(x), \cdots, X_\psi^{p+q}(x))$ for all charts ψ of $\mathcal{M}$ at x. Thus $X = 0$ and $v(f) = 0$ for all $f \in G^\infty(U)$. It follows that v is zero on the germ $G^\infty(x)$ and β_x is injective.

We now show that β_x is surjective. Let $X_x \in T_x\mathcal{BM}$ and recall that X_x is a mapping from the set of all charts of $\mathcal{BM}$ into $B = \mathbb{K}^{p|q}$. We want to find $v \in T_x^0\mathcal{M}$ such that $\beta_x(v) = X_x$. First we need to find a vector field defined on an open subset of $\mathcal{M}$ about x which agrees with X_x on charts of $\mathcal{M}$. Choose any chart (U,ψ) of $\mathcal{M}$

at x. Then $X_x(\psi) \in B = \mathbb{K}^{p|q}$ and we can define a constant vector field Y on U by

$$Y = \sum_{A=1}^{p+q} X_x^A(\psi) \frac{\partial}{\partial z^A}$$

where $\psi = (z^1, z^2, \ldots, z^{p+q})$. Thus the functions $Y_\psi^A : U \to \Lambda$ are the constant functions $Y_\psi^A(u) \equiv X_x^A(\psi)$ for all $u \in U$. Notice that $Y \in D^1(U)_0$. Define $v : G^\infty(x) \to \Lambda$ by

$$v(f) = Y(f)(x) = \sum_{A=1}^{p+q} X_x^A(\psi) \frac{\partial f}{\partial z^A}(x).$$

Then for any chart (V, ϕ) of $\mathcal{M}$ at x

$$\begin{aligned} \beta_x(v)(\phi) &= (Y_\phi^1(x), Y_\phi^2(x), \cdots, Y_\phi^{p+q}(x)) \\ &= d_{\psi(x)}(\phi \circ \psi^{-1})(Y_\psi^1(x), Y_\psi^2(x), \cdots, Y_\psi^{p+q}(x)) \\ &= d_{\psi(x)}(\phi \circ \psi^{-1})(X_x^1(\psi), X_x^2(\psi), \cdots, X_x^{p+q}(\psi)) \\ &= d_{\psi(x)}(\phi \circ \psi^{-1})(X_x(\psi)) \\ &= X_x(\phi). \end{aligned}$$

Thus $\beta_x(v)(\phi) = X_x(\phi)$ for all charts of $\mathcal{M}$, but since the charts of $\mathcal{M}$ form a subatlas of the manifold structure of $\mathcal{BM}$, $\beta_x(v)$ can be uniquely extended to agree with X_x at every chart of $\mathcal{BM}$. Thus β_x is surjective. The proposition follows. □

The mapping β_x induces a mapping of vector fields as follows. Recall that a vector field on a Banach manifold M is uniquely determined by defining a function Y from charts (U, ψ) of M into C^∞-maps from U into the Banach space B on which M is modeled. Of course if (U, ψ) and (V, ϕ) are charts of M such that $U \cap V \neq \emptyset$ the usual transformation holds,

$$Y(\psi)(x) = d_{\phi(x)}(\psi \circ \phi^{-1})(Y(\phi)(x))$$

for all $x \in U \cap V$.

Note that if $\mathcal{O} \subseteq \mathcal{M}$ is open and $X \in D^1(\mathcal{O})$, then for each $x \in \mathcal{O}$ we may define $X_x \in T_x\mathcal{M}$ by

$$X_x(f) = X(f)(x)$$

for all $f \in G^\infty(W)$ where W is open and $x \in W \subseteq \mathcal{O}$. Thus if (U, ψ) is a chart of $\mathcal{M}$ at x,

$$\beta_x(X_x)(\psi) = (X_\psi^1(x), X_\psi^2(x), \cdots, X_\psi^{p+q}(x))$$

and the mapping $\beta(X)(\psi)$ given by $x \mapsto \beta_x(X_x)(\psi)$ is a G^∞ function from V into $B = \mathbb{K}^{p|q}$. Since G^∞ maps are necessarily C^∞ maps we see that $\beta(X)$ is a **vector field** on $\mathcal{BM}$ since as a maps of charts of $\mathcal{M}$ it transforms correctly and thus can be

extended to all charts of $\mathcal{BM}$.

Thus we can write $v = \sum_{A=1}^{p+q} X_\psi^A(x)\partial/\partial z^A$ where $(X_x^1(x), X_x^2(x), \ldots, X_x^{p+q}(x)) \in \mathbb{K}^{p|q}$. If ϕ is another chart G^∞ related to ψ and $\phi = (w^1, w^2, \ldots, w^{p+q})$ then we can also write $v = \sum_{B=1}^{p+q} X_\phi^B(x)\partial/\partial w^A$. Moreover as in the classical case,

$$(X_\psi^1(x), X_\psi^2(x), \ldots, X_\psi^{p+q}(x)) = d_{\phi(x)}(\psi \circ \phi^{-1})(X_\phi^1(x), X_\phi^2(x), \ldots, X_\phi^{p+q}(x)).$$

Because the Banach space $B = \mathbb{K}^{p|q}$ is a Λ^0 module, vector fields on $\mathcal{BM}$ have a ${}^0\Lambda$-module structure.

Corollary 6.5.8. *If $\mathcal{O} \subseteq \mathcal{M}$ is an open subset of a supermanifold $\mathcal{M}$ then β is a ${}^0\Lambda$-linear vector space injection of the ${}^0\Lambda$-module of all* **EVEN** *vector fields $D^1(\mathcal{O})_0$ on $\mathcal{O}$ into the ${}^0\Lambda$-module of C^∞-vector fields of the Banach manifold $\mathcal{O} \subset \mathcal{BM}$.*

The mapping β is not surjective since for $X \in D^1(\mathcal{O})_0$ and for each chart (U, ψ) of $\mathcal{M}$, $\beta(X)(\psi) : U \to \mathbb{K}^{p|q}$ is a G^∞-mapping and not every C^∞-vector field on $\mathcal{O} \subset \mathcal{BM}$ has this property.

6.6 Higher Derivatives Banach Space Correspondence

We begin this section by working out things from basic principles. We find a useful technical characterization of a G^∞ function on supermanifolds in Theorem 6.6.3. Then we extend the differential to act on odd vectors (uniqueness follows from Proposition 3.7.6 or with proper interpretation in our context Proposition 4.2 of [68]). To conclude this section we give the most convenient characterizations that follows naturally from Theorem 4.3.3.

Recall that if $U \subseteq \mathbb{K}^{p|q} = \mathcal{B}$ is open and $\tilde{f} : U \to \Lambda$ is a class C^k mapping, then its p-fold Frechet derivative is a mapping from U into symmetric multi-linear maps from $\mathcal{B}^k = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B}$ into Λ. Thus for $x \in U$

$$d_x^p \tilde{f} : \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B} \to \Lambda$$

is symmetric. It is obtained by iterating the Frechet derivatives, for example,

$$d_x^2 \tilde{f}(v, w) = d_x(y \to (d_y \tilde{f})(w))(v).$$

Definition 6.6.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be supermanifolds and $f : \mathcal{M} \to \mathcal{N}$ a class C^p-*

mapping from $\mathcal{BM}$ into $\mathcal{BN}$. Define a mapping $d_x^p f$ by

$$d_x^p f : T_x\mathcal{BM} \times T_x\mathcal{BM} \times \cdots \times T_x\mathcal{BM} \to T_{f(x)}\mathcal{N}$$

where

$$d_x^p f(v_1, v_2, \ldots, v_p) = (d_{\phi(f(x))}\phi^{-1})\big(d^p_{\psi(p)}(\phi \circ f \circ \psi^{-1})(d_x\psi(v_1), d_x\psi(v_1), \ldots, d_x\psi(v_p))\big)$$

and where (U, ψ) is any chart of $\mathcal{M}$ and (V, ϕ) is any chart of $\mathcal{N}$ such that $f^{-1}(V) \subseteq U$. Note that it is sufficient to define $d_x^p f$ using charts of $\mathcal{M}$ since such charts are a subatlas of the atlas of $\mathcal{BM}$. Since $T_x^0\mathcal{M} \subseteq T_x\mathcal{BM}$ for each x, notice that there is an induced mapping

$$d_x^p f : T_x^0\mathcal{M} \times T_x^0\mathcal{M} \times \cdots \times T_x^0\mathcal{M} \to T_{f(x)}\mathcal{N}.$$

The definition of supersmoothness for Λ-valued functions was given in Section 6.5. We now give the definition of supersmoothness of functions whose domain and range reside in a supermanifold.

Definition 6.6.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be supermanifolds then $f : \mathcal{M} \to \mathcal{N}$ is a class G^l-mapping iff its local coordinate representatives are all G^l-mappings from $\mathbb{K}^{p|q}$ to $\mathbb{K}^{r|s}$. Likewise a function $f : \mathcal{M} \to \mathcal{N}$ a class G^∞-mapping iff its local coordinate representatives are all G^∞-mappings from $\mathbb{K}^{p|q}$ to $\mathbb{K}^{r|s}$.*

Recall that the definition of G^∞ for functions from $\mathbb{K}^{p|q}$ to $\mathbb{K}^{r|s}$ was given in Definition 4.7.1.

Theorem 6.6.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be supermanifolds of dimension $(p|q)$ and $(r|s)$ respectively and let $f : \mathcal{BM} \to \mathcal{BN}$ be a C^∞ function. The function $f : \mathcal{M} \to \mathcal{N}$ is a class G^l function iff for every chart (U, ψ) of $\mathcal{M}$ and (V, ϕ) of $\mathcal{N}$ such that $f^{-1}(V) \subset U$ there exist functions $b^{\psi J}_{A_1 \ldots A_k}$ with $1 \leq A_1 \ldots A_k \leq p + q$, $1 \leq J \leq r + s$ and $1 < k \leq l$, such that*

(1) *each function $b^{\psi J}_{A_1 \ldots A_k}$ is in $G^0(U)$, and*
(2) *for $x \in U$ and $X_1, X_2, \ldots, X_k \in T_x^0\mathcal{M}$,*
$d_x^k(\phi^J \circ f)(X_1, \ldots, X_k) = \sum_{A_1=1}^{p+q} \cdots \sum_{A_k=1}^{p+q} X_1^{A_1} \cdots X_k^{A_k} b^{\psi^J}_{A_1 \ldots A_k}(x)$.

Proof. If f is of class G^k for $k \leq l$ then for charts ψ, ϕ of $\mathcal{M}, \mathcal{N}$ respectively $\phi \circ f \circ \psi^{-1}$ is of class G^k. By Proposition 2.8 of [98], where the partials are of class G^0,

$$d^k_{\psi(x)}(\phi^J \circ f \circ \psi^{-1})(v_1, \ldots, v_k) = \sum_{A_1 \ldots A_k = 1}^{p+q} v^{A_1} \cdots v^{A_k} \frac{\partial^k(\phi^J \circ f \circ \psi^{-1})}{\partial u^{A_k} \cdots \partial u^{A_1}}(\psi(x))$$

for $1 \leq J \leq r + s$ and $v_1, v_2, \ldots, v_k \in \mathbb{K}^{p|q}$. We identify v_i with $d_x\psi(X_i)$ for arbitrary

given $X_1, X_2, \ldots, X_k \in T_x^0\mathcal{M}$ so that

$$d_x^k(\phi^J \circ f)(X_1, \ldots, X_k) = \sum_{A_1..A_k=1}^{p+q} X^{A_1} \cdots X^{A_k} \frac{\partial^k(\phi^J \circ f)}{\partial z^{A_k} \cdots \partial z^{A_1}}(x)$$

where $z^A \equiv \Pi^A \circ \psi$ (recall that Π^A is the projection of $\mathbb{K}^{p|q}$ onto its A-th factor). Thus we note that,

$$b^{\psi J}_{A_1 \ldots A_k}(x) = \frac{\partial^k(\phi^J \circ f)}{\partial z^{A_k} \cdots \partial z^{A_1}}(x) \qquad \text{for } x \in U.$$

and (1) and (2) hold.

Conversely, assume the existence of the functions $b^{\psi^J}_{A_1 \ldots A_k} : U \to \Lambda$ with $k \leq l$ that satisfy conditions (1) and (2) above. We show f is of class G^k for all $k \leq l$.

Begin with the case $k = 1$. Let ψ, ϕ of $\mathcal{M}, \mathcal{N}$ respectively and choose U open in $\mathcal{M}$ small enough so that $\phi \circ f \circ \psi^{-1}$ is defined on the open set $\psi(U)$. By hypothesis we have for $X \in T_x^0\mathcal{M}$ and $x \in U$,

$$d_x(\phi^J \circ f)(X) = \sum_{A=1}^{p+q} X^A b_A^{\psi J}(x) \tag{6.5}$$

where $b_A^{\psi J}(x) \in G^0(U)$. Thus there are supernumbers $b_A^{\psi J}(x)$ that encode the Frechet derivative of $(\phi^J \circ f)$ at x. Moreover, if we identify H with $d_x\psi(X)$ we find from eq.(6.5),

$$d_{\psi(x)}(\phi^J \circ f \circ \psi^{-1})(H) = \sum_{A=1}^{p+q} H^A b_A^{\psi J}(\psi(x)). \tag{6.6}$$

This identity implies that $\phi^J \circ f \circ \psi^{-1}$ is of class G^1 on $\psi(U)$ for each J and that $b_A^{\psi J}(x) = G_A(\phi^J \circ f \circ \psi^{-1})(\psi(x))$ in the notation of [98]. Hence $\phi \circ f \circ \psi^{-1}$ is of class G^1 on $\psi(U)$ and therefore, f is of class G^1 on U. The case $k = 1$ is proved.

Next we prove the case $k = 2$. Consider the mapping F from $\psi(U)$ to Λ defined by $F : y \mapsto d_y(\phi^J \circ f \circ \psi^{-1})(V_2)$ where $V_2 \in \mathbb{K}^{p|q}$ is given by $V_2 = d_x\psi(X_2)$ for an arbitrary, but fixed, $X_2 \in T_x^0\mathcal{M}$. Then by construction, V_2 is an arbitrary element of the Banach space $B = \mathbb{K}^{p|q}$ which does not change as y changes. Consider the Frechet

derivative of F at $u = \psi(x)$. We Have for $X_1 \in T_x^0\mathcal{M}$ and $V_1 = d_x\psi(X_1)$ and,

$$\begin{aligned} d_uF(V_1) &= d_u(d(\phi^J \circ f \circ \psi^{-1})(V_2))(V_1) \\ &= d_u^2(\phi^J \circ f \circ \psi^{-1})(V_1, V_2) \\ &= d_{\psi(x)}^2(\phi^J \circ f \circ \psi^{-1})(d_x\psi(X_1), d_x\psi(X_2)) \\ &= d_x^2(\phi^J \circ f)(X_1, X_2) \\ &= \sum_{A_1=1}^{p+q}\sum_{A_2=1}^{p+q} X_1^{A_1}X_2^{A_2}b_{A_1A_2}^{\psi J}(x) \\ &= \sum_{A_1=1}^{p+q}\sum_{A_2=1}^{p+q} V_1^{A_1}V_2^{A_2}b_{A_1A_2}^{\psi J}(x). \end{aligned} \tag{6.7}$$

Since we have already shown that f is of class G^1 on U we have that $d_y(\phi^J \circ f \circ \psi^{-1})(V_2) = \sum_{A_2=1}^{p+q} V_2^{A_2}\left(\frac{\partial(\phi^J \circ f \circ \psi^{-1})}{\partial u^{A_2}}\right)(y)$. From the definition of F we note

$$F(y) = \sum_{A_2=1}^{p+q} V_2^{A_2}\left(\frac{\partial(\phi^J \circ f \circ \psi^{-1})}{\partial u^{A_2}}\right)(y). \tag{6.8}$$

Thus for fixed V_2, we have

$$d_uF(V_1) = \sum_{A_2=1}^{p+q} (-1)^{\epsilon_{A_2}\epsilon(V_1)}V_2^{A_2}d_u\left(\frac{\partial(\phi^J \circ f \circ \psi^{-1})}{\partial u^{A_2}}\right)(V_1). \tag{6.9}$$

And so, comparing eq.(6.7) and eq.(6.9) we find,

$$\sum_{A_2=1}^{p+q} (-1)^{\epsilon_{A_2}\epsilon(V_1)}V_2^{A_2}d_u\left(\frac{\partial(\phi^J \circ f \circ \psi^{-1})}{\partial u^{A_2}}\right)(V_1) = \sum_{A_1=1}^{p+q}\sum_{A_2=1}^{p+q} V_1^{A_1}V_2^{A_2}b_{A_1A_2}^{\psi J}(x). \tag{6.10}$$

Thus,

$$\sum_{A_2=1}^{p+q} (-1)^{\epsilon_{A_2}\epsilon(V_1)}V_2^{A_2}d_u\left(\frac{\partial(\phi^J \circ f \circ \psi^{-1})}{\partial u^{A_2}}\right)(V_1) = \sum_{A_2=1}^{p+q} V_2^{A_2}\sum_{A_1=1}^{p+q} (-1)^{\epsilon_{A_1}\epsilon_{A_2}}V_1^{A_1}b_{A_1A_2}^{\psi J}(x). \tag{6.11}$$

This holds for all V_2 so,

$$d_u\left(\frac{\partial(\phi^J \circ f \circ \psi^{-1})}{\partial u^{A_2}}\right)(V_1) = \sum_{A_1=1}^{p+q} (-1)^{\epsilon_{A_1}\epsilon_{A_2}+\epsilon_{A_2}\epsilon(V_1)}V_1^{A_1}b_{A_1A_2}^{\psi J}(x). \tag{6.12}$$

It follows that $\left(\frac{\partial(\phi^J \circ f \circ \psi^{-1})}{\partial u^{A_2}}\right)$ is of class G^1 on $\psi(U) \subseteq \mathbb{K}^{p|q}$, and for $u \in \psi(U)$. Since $\frac{\partial(\phi^J \circ f \circ \psi^{-1})}{\partial u^{A_2}}$ is of class G^1 for each A, $\phi^J \circ f \circ \psi^{-1}$ is of class G^2 on $\psi(U)$ for each J. Thus f is of class G^2 on U. An inductive argument using similar computations will show that f is of class G^k for all $k \leq l$ □

Remark 6.6.4. *Given a C^∞-mapping $f : \mathcal{M} \to \mathcal{N}$ as in the theorem above we have conditions under which f is of class G^∞. One begins with maps*

$$d_x^k f : T_x\mathcal{BM} \times \cdots \times T_x\mathcal{BM} \to T_{f(x)}\mathcal{N}. \tag{6.13}$$

Then since $T_x^0\mathcal{M} \times \cdots \times T_x^0\mathcal{M} \subseteq T_x\mathcal{BM} \times \cdots \times T_x\mathcal{BM}$ one has a mapping on even vectors $X_1, X_2, \ldots X_k \in T_x^0\mathcal{M}$. Moreover, one obtains the formula for even vectors

$$d_x^k f(X_1, X_2, \ldots, X_k) = \sum_{A_1..A_k=1}^{p+q} (X_1^{A_1} X_2^{A_2} \cdots X_k^{A_k}) \left(\frac{\partial^k f}{\partial z^{A_k} \cdots \partial z^{A_2} \partial z^{A_1}} \right)(x). \tag{6.14}$$

It now follows that this mapping can be extended to a mapping from $T_x\mathcal{M} \times T_x\mathcal{M} \times \cdots \times T_x\mathcal{M}$ to $T_{f(x)}\mathcal{N}$ where the components of pure tangent vectors $X_1, X_2, \ldots, X_k$ may be in $\mathbb{K}^{p|q}$ or possibly in $({}^1\Lambda)^p \times ({}^0\Lambda)^q$. If the vectors $X_1, X_2, \ldots, X_k$ are not of definite parity then the components $(X_1^{A_1}, X_2^{A_2}, \ldots, X_k^{A_k})$ will reside in Λ^k in general. As an example, consider the case k=1. Observe that

$$d_x f(X) = X_x(f) = \sum_{A=1}^{p+q} X^A \frac{\partial f}{\partial z^A}(x) \tag{6.15}$$

makes sense for even and odd vectors $X_x \in T_x\mathcal{M}$. It is interesting that the operation of $d_x f$ on $T_x^0\mathcal{M}$ defines its operation on the other half of $T_x\mathcal{M}$ namely $T_x^1\mathcal{M}$.

In other words, a supermanifold $\mathcal{M}$ is modeled on $\mathbb{K}^{p|q}$ but the G^∞-tangent module "doubles the dimension". Even vectors are summed over **ALL** *of the even and odd coordinate vector fields (expanded against even and odd components in order that the vector field be even); so to have the derivative of some map preserve this property for even vector fields it is convenient to require that the derivative be defined on the coordinate vector field basis of the tangent module. This is why we extend the derivative to act on both even and odd vector fields. Even vector fields have the $(p|q)$ data hidden in them, the tangent module at a point is the direct sum of the Banach space $\mathbb{K}^{p|q}$ on which $\mathcal{M}$ is modeled and the Banach space $\mathbb{K}^{\bar{p}|\bar{q}}$.*

The next theorem is the natural generalization of Theorem 4.3.3.

Theorem 6.6.5. *Let $\mathcal{M}$ and $\mathcal{N}$ be supermanifolds of dimension $(p|q)$ and $(r|s)$ respectively and let $f : \mathcal{BM} \to \mathcal{BN}$ be a C^∞ function. The function $f : \mathcal{M} \to \mathcal{N}$ is a class G^∞ function iff for every chart (U, ψ) of $\mathcal{M}$ and (V, ϕ) of $\mathcal{N}$ such that $f^{-1}(V) \subset U$ there exist functions $b_{A_1}^{\psi J}$ with $1 \leq A_1 \leq p+q$, $1 \leq J \leq r+s$ such that*

1. *each function $b_{A_1}^{\psi J}$ is in $G^0(U)$, and*

2. *for $x \in U$ and $X_1 \in T_x^0\mathcal{M}$*

$$d_x(\phi^J \circ f)(X_1) = \sum_{A_1=1}^{p+q} X_1^{A_1} b_{A_1}^{\psi^J}(x).$$

To summarize, a function $f : \mathcal{M} \to \mathcal{N}$ is smooth and G^1 iff it is supersmooth.

Proof. To begin assume that there exist functions $b_{A_1}^{\psi J}$ with $1 \leq A_1 \leq p+q$, $1 \leq J \leq r+s$ satisfying (1.) and (2.) for a smooth function $f : \mathcal{M} \to \mathcal{N}$ then by Theorem 6.6.3 we have that f is G^1. Note then that each of the local coordinate representatives of f are G^1 and also by assumption they are smooth. We apply Theorem 4.3.3 to see that each coordinate representative of f is G^∞. Thus f is G^∞. The converse follows immediately from Theorem 6.6.3. □

This last theorem is the most efficient method of ascertaining if a function on supermanifolds is supersmooth. From the point of view of [68] this Theorem 6.6.5 might well become the definition for supersmoothness since it supercedes the definition in practice. We did not make use of this theorem in our paper [37] so we have chosen to treat Alice Rogers' definitions as primary and this theorem as a logical consequence. This dissertation differs from [37] in that the labor saving techniques of [68] will be applied to shorten certain proofs found in [37].

6.7 Differentiation on Banach Supervector Spaces

Definition 6.7.1. *Let $\mathcal{M}$ be a supermanifold and $\mathfrak{v}$ a Banach supervector space. Provide $\mathfrak{v}^0$ with the supermanifold structure obtained by defining the obvious single global chart obtained from a basis of $\mathfrak{v}$. Let f denote a smooth function from $\mathcal{M}$ into $\mathfrak{v}^0$ and let $\{f^B\}$ denote its components relative to a pure basis of $\mathfrak{v}$. We define the higher derivatives of f^B at $w \in \mathcal{M}$ inductively as follows. Define $d_w f^B : T_w\mathcal{M} \to \Lambda$ by $d_w f^B(X) = X(f^B)$ for $X \in T_w\mathcal{M}$. Define $d_w^{k+1} f^B : T_w\mathcal{M} \times T_w\mathcal{M} \times \cdots T_w\mathcal{M} \to \Lambda$ by $d_w^{k+1} f^B(X_1, X_2, \cdots, X_{k+1}) = d_w[d^k f^B(X_2, X_3, \cdots, X_{k+1})](X_1)$ for $X_1, X_2, \cdots, X_{k+1} \in T_w\mathcal{M}$. Here $d^k f^B(X_2, X_3, \cdots, X_{k+1})$ denotes the function from $\mathcal{M}$ into Λ defined by $x \to d_x^k f^B(X_2, X_3, \cdots, X_{k+1})$.*

We now consider an important special case of these ideas which we find useful in the last section of the chapter. Let $\mathfrak{g}$ and $\mathfrak{v}$ denote Banach super vector spaces. Consider $\mathfrak{g}^0$ as a supermanifold with a single global chart $\psi : \mathfrak{g}^0 \to \mathbb{K}^{p|q}$ whose components are defined by $\psi(x) = (u^1(x), u^2(x), \cdots, u^{p+q}(x))$, $x \in \mathfrak{g}^0$. For each $x \in \mathfrak{g}^0, T_x\mathfrak{g}^0$ may be identified with $\mathfrak{g}$ by identifying the basis $\{u^B\}$ of $T_x\mathfrak{g}^0$ with a given fixed basis $\{e_B\}$ of $\mathfrak{g}$. Similarly, choose a single coordinate chart on $\mathfrak{v}^0$. Moreover if f is a function from $\mathfrak{g}^0$ to $\mathfrak{v}^0$, then denote its components relative to the chart on $\mathfrak{v}^0$ by the functions $f^B : \mathfrak{g}^0 \to \Lambda$. Recall that if f is of class C^∞, then it is also of

class G^∞ iff each component function f^B is of class G^∞. Notice that the components $f^1, f^2, \cdots f^p$ are all even while $f^{p+1}, f^{p+2}, \cdots, f^{p+q}$ are all odd. Also notice that the derivatives $d_w^k f^B$ of each component function are maps from $\mathfrak{g}^k = \mathfrak{g} \times \cdots \times \mathfrak{g}$ to Λ at each $w \in \mathfrak{g}^0$ due to the identification of $\mathfrak{g}$ with $T_w \mathfrak{g}^0$.

Definition 6.7.2. *Let $\mathfrak{g}$ be a supervector space with basis $\{e_B\}$ and let $\beta : \mathfrak{g}^k \to \Lambda$. We say that β is multi-linear over $\mathfrak{g}^0$ iff for some pure basis $\{e_B\}$ of $\mathfrak{g}$,*

$$\beta(v_1, v_2, \cdots, v_k) = v_1^{A_1} v_2{}^{A_2} \cdots v_k^{A_k} \beta(e_{A_k}, \cdots, e_{A_2}, e_{A_1})$$

for $v_1, v_2, \cdots, v_k \in \mathfrak{g}^0$.

Notice that one must require that β be defined on all of $\mathfrak{g}^k$ rather than $(\mathfrak{g}^0)^k$, since it must be possible to evaluate β at arbitrary elements of a basis of $\mathfrak{g}$. This is also the case for higher derivatives such as $d^k f^B$ as defined above. This shows up explicitly in the proof of the following proposition.

Proposition 6.7.3. *Let $\mathfrak{g}$ and $\mathfrak{v}$ denote Banach super vector spaces and $f : \mathfrak{g}^0 \to \mathfrak{v}^0$ a C^∞ function. Then f is of class G^∞ iff for each $x \in \mathfrak{g}^0$ and each positive integer $k, d_x^k f^B : \mathfrak{g}^k \to \Lambda$ is multi-linear over $\mathfrak{g}^0$ for each component f^B of f.*

We give two proofs. The first proof shows how to do detailed calculations on a supervector space while second proof shortcuts much of this work via Theorem 6.6.5.

Proof. Assume first that $f : \mathfrak{g}^0 \to \mathfrak{v}^0$ is of class G^∞ and that f^B is a component of f. Choose a pure basis $\{e_B\}$ of $\mathfrak{g}$ and define u^B on $\mathfrak{g}^0$ by $u^B(\sum a^K e_K) = a^B$. Regard the (u^B) as coordinates on $\mathfrak{g}^0$. We first show for $x \in \mathfrak{g}^0$ and $v_1, v_2, \cdots v_k \in \mathfrak{g}^0$, that

$$d_x^k f^B(v_1, v_2, \cdots, v_k) = v_1^{A_1} v_2^{A_2} \cdots v_k^{A_k} \frac{\partial^k f}{\partial u^{A_k} \cdots \partial u^{A_2} \partial u^{A_1}}(x).$$

The proof proceeds by induction. First observe that $d_x f^B(v) = v(f^B) = v^A \frac{\partial f^B}{\partial u^A}$ so the result is true for $k = 1$ Now assume the result for arbitrary k and we show that

$$d_x^{k+1} f^B(v_1, v_2, \cdots, v_{k+1}) = v_1^{A_1} v_2^{A_2} \cdots v_{k+1}^{A_k+1} \frac{\partial^{k+1} f}{\partial u^{A_{k+1}} \cdots \partial u^{A_2} \partial u^{A_1}}(x).$$

By definition

$$\begin{aligned} d_x^{k+1} f^B(v_1, v_2, \cdots, v_{k+1}) &= d_x[d^k f^B(v_2, v_3, \cdots, v_{k+1})](v_1) \\ &= d_x[v_2^{A_2} v_3^{A_3} \cdots v_{k+1}^{A_{k+1}} \tfrac{\partial^k f}{\partial u^{A_{k+1}} \cdots \partial u^{A_3} \partial u^{A_2}}](v_1) \\ &= v_1^{A_1} \tfrac{\partial}{\partial u^{A_1}}[v_2^{A_2} v_3^{A_3} \cdots v_{k+1}^{A_{k+1}} \tfrac{\partial^k f}{\partial u^{A_{k+1}} \cdots \partial u^{A_3} \partial u^{A_2}}]. \end{aligned} \tag{6.16}$$

Now the partial derivative $\frac{\partial}{\partial u^{A_1}}$ can be pushed through the term $v_2^{A_2} v_3^{A_3} \cdots v_{k+1}^{A_{k+1}}$ but in doing so it produces a sign change $\varepsilon = (-1)^{\varepsilon(A_1)\varepsilon(A_2)}(-1)^{\varepsilon(A_1)\varepsilon(A_3)} \cdots (-1)^{\varepsilon(A_1)\varepsilon(A_{k+1})}$.

Thus one obtains

$$d_x^{k+1}f^B(v_1,v_2,\cdots,v_{k+1}) = \varepsilon[v^{A_1}v_2^{A_2}v_3^{A_3}\cdots v_{k+1}^{A_{k+1}}\frac{\partial^k f}{\partial u^{A_1}\partial u^{A_{k+1}}\cdots\partial u^{A_3}\partial u^{A_2}}].$$

Now one must permute the order of the partials but one finds that

$$\frac{\partial^k f}{\partial u^{A_1}\partial u^{A_{k+1}}\cdots\partial u^{A_3}\partial u^{A_2}} = \varepsilon[\frac{\partial^k f}{\partial u^{A_{k+1}}\cdots\partial u^{A_3}\partial u^{A_2}\partial u^{A_1}}].$$

The two signs cancel to give the desired result

$$d_x^{k+1}f^B(v_1,v_2,\cdots,v_{k+1}) = v_1^{A_1}v_2^{A_2}\cdots v_{k+1}^{A_k+1}\frac{\partial^{k+1}f}{\partial u^{A_{k+1}}\cdots\partial u^{A_2}\partial u^{A_1}}(x).$$

This finishes the first part of the proof.

To complete the proof we must show that for each positive integer k,

$$d_x^k f^B(\frac{\partial}{\partial u^{A_1}},\frac{\partial}{\partial u^{A_2}},\cdots\frac{\partial}{\partial u^{A_k}}) = \frac{\partial^k f}{\partial u^{A_1}\partial u^{A_2}\cdots\partial u^{A_k}}(x).$$

This proof also proceeds by induction. The result is obvious when $k=1$, since $d_x f^B(\frac{\partial}{\partial u^A}) = \frac{\partial f^B}{\partial u^A}$. Assume, inductively, that for some positive k,

$$d_x^k f^B(\frac{\partial}{\partial u^{A_2}},\frac{\partial}{\partial u^{A_3}},\cdots\frac{\partial}{\partial u^{A_{k+1}}}) = \frac{\partial^k f}{\partial u^{A_2}\partial u^{A_3}\cdots\partial u^{A_{k+1}}}(x).$$

By the definition of $d_x^{k+1}f^B$ we have

$$d_x^{k+1}f^B(\frac{\partial}{\partial u^{A_1}},\frac{\partial}{\partial u^{A_2}},\cdots\frac{\partial}{\partial u^{A_{k+1}}}) = d_x[d^k f^B(\frac{\partial}{\partial u^{A_2}},\frac{\partial}{\partial u^{A_3}},\cdots,\frac{\partial}{\partial u^{A_{k+1}}}))(\frac{\partial}{\partial u^{A_1}})]$$

$$= d_x(\frac{\partial^k f}{\partial u^{A_2}\partial u^{A_3}\cdots\partial u^{A_{k+1}}})(\frac{\partial}{\partial u^{A_1}}) = \frac{\partial^{k+1}f}{\partial u^{A_1}\partial u^{A_2}\cdots\partial u^{A_{k+1}}}(x)$$

and the result follows. From these two results, we have that for for all k and for $v_1,v_2,\cdots v_k\in\mathfrak{g}^0$

$$d_x^k f^B(v_1,v_2,\cdots,v_k) = v_1^{A_1}v_2^{A_2}\cdots v_k^{A_k}\frac{\partial^k f}{\partial u^{A_k}\cdots\partial u^{A_2}\partial u^{A_1}}(x)$$

$$= (v_1^{A_1}v_2^{A_2}\cdots v_k^{A_{k+1}})d_x^k f^B(\frac{\partial}{\partial u^{A_k}},\cdots,\frac{\partial}{\partial u^{A_2}},\frac{\partial}{\partial u^{A_1}}).$$

Thus $d_x^k f^B$ is k-multi-linear and consequently if f is of class G^∞, then all the derivatives of the components of f are multi-linear over $\mathfrak{g}^0$.

Conversely, assume that all the derivatives of the components of f are multi-linear over $\mathfrak{g}^0$. We show that f is of class G^∞. In fact the result is an immediate consequence

of Theorem 6.6.3 since we have that

$$d_x^k f^B(v_1, v_2, \cdots, v_k) = (v_1^{A_1} v_2^{A_2} \cdots v_k^{A_k}) d_x^k f^B(\frac{\partial}{\partial u^{A_k}}, \cdots, \frac{\partial}{\partial u^{A_2}}, \frac{\partial}{\partial u^{A_1}})$$

$$= v_1^{A_1} v_2^{A_2} \cdots v_k^{A_k} \frac{\partial^k f}{\partial u^{A_k} \cdots \partial u^{A_2} \partial u^{A_1}}(x)$$

and the hypothesis of Theorem 6.6.3 holds with $b^J_{A_1 A_2 \cdots A_k} = \frac{\partial^k f}{\partial u^{A_k} \cdots \partial u^{A_2} \partial u^{A_1}}$. The proposition follows. □

There is an easier alternate proof of the converse.

Proof. Let $f : \mathfrak{g}^0 \to \mathfrak{v}^0$ be a C^∞ function where $\mathfrak{g}^0$ and $\mathfrak{v}^0$ are supervector spaces which clearly have natural supermanifold structures. Suppose that for each $x \in \mathfrak{g}^0$ and each positive integer k, $d_x^k f^B : \mathfrak{g}^k \to \Lambda$ is multi-linear over $\mathfrak{g}^0$ for each component function f^B of f. Then in particular the assumption holds for $k = 1$; thus $d_x f^B : \mathfrak{g} \to \Lambda$ is multi-linear over $\mathfrak{g}^0$ for each component f^B of f.. But, this means that (recall Observation 3.8.4)

$$d_x f^B(v^A e_A) = v^A d_x f^B(e_A)$$

for each B with respect to a pure basis $\{e_A\}$ of $\mathfrak{g}$. Note then $df^B(cV) = cdf^B(V)$ thus df^B is Λ-linear and thus f^B is G^1. We also know that f^B is smooth hence f^B is supersmooth. Since this holds for each B and there is only one coordinate representative we find that f is supersmooth by Theorem 6.6.5. □

The second converse proof suggests we can refine the proposition as follows:

Proposition 6.7.4. *Let $\mathfrak{g}$ and $\mathfrak{v}$ denote Banach super vector spaces and $f : \mathfrak{g}^0 \to \mathfrak{v}^0$ a C^∞ function. Then f is of class G^∞ iff for each $x \in \mathfrak{g}^0$, $d_x f^B : \mathfrak{g} \to \Lambda$ is ${}^0\Lambda$-linear for each component f^B of f.*

6.8 Submanifolds of Supermanifolds

We find in this section that supermanifolds share many of the same submanifold constructions as in traditional finite dimensional manifold theory. The essential technical difficulty is to verify supersmoothness of the newly constructed sub supermanifolds or immersed sub supermanifolds, but this does not present too much difficulty thanks to the fact that supermanifolds are also Banach manifolds.

Definition 6.8.1. *Let $\mathcal{M}$ be a $(p|q)$ supermanifold and $\mathcal{S} \subseteq \mathcal{M}$. A chart of (U, ψ) of $\mathcal{M}$ is called an $(r|s)$-submanifold chart of $\mathcal{M}$ relative to $\mathcal{S}$ iff*

$$\psi(U \cap \mathcal{S}) = \psi(U) \cap (\mathbb{K}^{r|s} \times \{(0,0)\})$$

where $(0,0) \in \mathbb{K}^{(p-r|q-s)}$. *We say that* $\mathcal{S}$ *is a* $(r|s)$ *submanifold of* $\mathcal{M}$ *iff for each* $x \in \mathcal{S}$ *there exists a* $(r|s)$*-submanifold chart* (U,ψ) *of* $\mathcal{M}$ *relative to* $\mathcal{S}$ *such that* $x \in U$. *There is a subtle point to be made here and that is that the definition depends on a specific splitting* $\mathbb{K}^{p|q} = \mathbb{K}^{r|s} \times \mathbb{K}^{(p-r|q-s)}$. *In general many such splittings are possible. In our definition we choose one specific splitting and all submanifold charts are required to respect this particular splitting.*

Remark 6.8.2. *If* $\mathcal{S}$ *is a* $(r|s)$*-submanifold of* $\mathcal{M}$ *let* $\mathcal{A}_{\mathcal{S}}$ *denote the set of all pairs* $(U\cap\mathcal{S}, \psi_{\mathcal{S}})$ *such that there exists an* $(r|s)$*-submanifold chart* (ψ, U) *of* $\mathcal{M}$ *relative to* $\mathcal{S}$ *such that* $\mathcal{S}\cap U \neq \emptyset$ *and* $\psi_{\mathcal{S}} : U\cap\mathcal{S} \to \mathbb{K}^{r|s}$ *is defined in terms of* ψ *by requiring that* $\psi_{\mathcal{S}}$ *be the restriction of* ψ *to* $U\cap\mathcal{S}$ *composed with the obvious projection of* $\mathbb{K}^{r|s}\times\{(0,0)\}$ *to* $\mathbb{K}^{r|s}$ *which discards the* $\{(0,0)\} \in \mathbb{K}^{(p-r|q-s)}$. *It is obvious and well-known that if* (U,ψ) *and* (V,ϕ) *are such charts with* $U\cap V\cap\mathcal{S} \neq \emptyset$ *then*

$$\phi_{\mathcal{S}} \circ \psi_{\mathcal{S}}^{-1} : \psi_{\mathcal{S}}(U\cap V\cap\mathcal{S}) \to \psi_{\mathcal{S}}(U\cap V\cap\mathcal{S})$$

is a C^∞ *mapping. Thus* $\mathcal{S}$ *inherits a* C^∞*-manifold structure from* $\mathcal{BM}$ *which we denote by* $\mathcal{BS}$ *when we wish to emphasize that it is a Banach manifold. Moreover* $\phi_{\mathcal{S}} \circ \psi_{\mathcal{S}}^{-1}$ *is essentially the restriction of* $\phi\circ\psi^{-1} : \psi(U\cap V) \to \phi(U\cap V)$ *to* $\psi(U\cap V)\cap(\mathbb{K}^{r|s}\times\{(0,0)\})$ *which maps this set to* $\phi(U\cap V)\cap(\mathbb{K}^{r|s}\times\{(0,0)\})$ *and consequently it is easy to see that* $\phi_{\mathcal{S}} \circ \psi_{\mathcal{S}}^{-1}$ *is a* G^∞*- mapping. Indeed the inclusion mapping*

$$i : \mathbb{K}^{r|s} \hookrightarrow \mathbb{K}^{r|s}\times\{(0,0)\} \hookrightarrow \mathbb{K}^{p|q}$$

is a G^∞*-mapping as is also its restriction* i_Q *to the open set* $Q = \psi_{\mathcal{S}}(U\cap V\cap\mathcal{S}) \subseteq \mathbb{K}^{r|s}$. *For* $1 \leq i \leq r$ *and* $1 \leq \alpha \leq s$

$$\phi_{\mathcal{S}}{}^i \circ \psi_{\mathcal{S}}^{-1} = \phi^i \circ \psi^{-1} \circ i_Q \qquad \text{and} \qquad \phi_{\mathcal{S}}{}^{r+\alpha} \circ \psi_{\mathcal{S}}^{-1} = \phi^{r+\alpha} \circ \psi^{-1} \circ i_Q.$$

Consequently the components of $\phi_{\mathcal{S}} \circ \psi_{\mathcal{S}}^{-1}$ *are* G^∞ *maps and thus so is* $\phi_{\mathcal{S}} \circ \psi_{\mathcal{S}}^{-1}$. *This proves the next proposition.*

Proposition 6.8.3. *If* $\mathcal{S}$ *is a* $(r|s)$*-submanifold of a* $(p|q)$*-supermanifold* $\mathcal{M}$ *then* $\mathcal{S}$ *is a* $(r|s)$*-supermanifold.*

Corollary 6.8.4. *If* $\mathcal{S}$ *is a* $(r|s)$*-submanifold of a* $(p|q)$*-supermanifold* $\mathcal{M}$ *then the inclusion* $i : \mathcal{S} \hookrightarrow \mathcal{M}$ *is a* G^∞*-mapping.*

Proof. Let (U,ψ) be a $(r|s)$ submanifold chart of $\mathcal{M}$ relative to $\mathcal{S}$. We must show that $\psi \circ i \circ \psi_{\mathcal{S}}^{-1}$ is a G^∞-mapping. But $\psi_{\mathcal{S}}^{-1} = \psi^{-1} \circ i_Q$ where $Q = \psi_{\mathcal{S}}(U\cap\mathcal{S}) \subseteq \mathbb{K}^{r|s}$ and i_Q is the inclusion $Q \hookrightarrow Q\times\{(0,0)\} \hookrightarrow \mathbb{K}^{p|q}$. Thus $\psi \circ i \circ \psi_{\mathcal{S}}^{-1} = \psi\circ\psi^{-1}\circ i_Q = i_Q$ which is a G^∞-mapping. □

Definition 6.8.5. *Let $\mathcal{M}$ be a supermanifold of dimension $(p|q)$ with $\mathcal{S} \subseteq \mathcal{M}$. A chart $(U, \psi) \in \mathcal{A}_{\mathcal{M}}$ is called an initial submanifold chart relative to $\mathcal{S}$ centered at $x \in U$ iff*

$$\psi(C_x(U \cap \mathcal{S})) = \psi(U) \cap (\mathbb{K}^{r|s} \times \{(0,0)\}) \tag{6.17}$$

relative to a specific splitting

$$\mathbb{K}^{p|q} = \mathbb{K}^{r|s} \times \mathbb{K}^{(p-r|q-s)} \tag{6.18}$$

and $C_x(U \cap \mathcal{S})$ denotes the set of all $y \in U \cap \mathcal{S}$ such that there is a smooth curve in $\mathcal{M}$ from x to y lying in $U \cap \mathcal{S}$. We say $\mathcal{S}$ is an initial super submanifold of $\mathcal{M}$ of dimension $(r|s)$ iff for each $x \in \mathcal{S}$ there exists an initial submanifold chart relative to $\mathcal{S}$ centered at x whose image is contained in $\mathbb{K}^{r|s} \subseteq \mathbb{K}^{p|q}$. See [73] for details regarding initial submanifolds of an ordinary manifold.

The author is grateful to Ratiu for the last reference and for clarifying the status of these concepts for Banach manifolds.

Theorem 6.8.6. *Let $\mathcal{M}$ be a supermanifold and $\mathcal{S} \subseteq \mathcal{M}$ an initial super submanifold of $\mathcal{M}$ of dimension $(r|s)$. Then there exists a unique C^∞-manifold structure on $\mathcal{S}$ such that the injection $i : \mathcal{BS} \hookrightarrow \mathcal{BM}$ is an injective immersion. Moreover, $\mathcal{S}$ is in fact a supermanifold and i is a G^∞-mapping.*

Proof. Given that $\mathcal{S}$ is an initial super submanifold of $\mathcal{M}$ it is clear that as a subset of $\mathcal{BM}$, $\mathcal{BS}$ is an initial submanifold of $\mathcal{BM}$. It is known that an initial submanifold of a Banach manifold, such as $\mathcal{BM}$, possesses a unique C^∞-structure relative to which $i : \mathcal{BS} \hookrightarrow \mathcal{BM}$ is smooth. Thus given an atlas $\mathcal{A}_{\mathcal{M}}$ of $\mathcal{M}$ and $\mathcal{A}_{\mathcal{BM}} = \mathcal{A}_{\mathcal{M}}$ we have that the set of pairs

$$(C_x(U \cap \mathcal{S}), \psi | C_x(U \cap \mathcal{S}))$$

such that $x \in U, (U, \psi) \in \mathcal{A}_{\mathcal{M}}$, and $U \cap \mathcal{S}$ is nonempty is an atlas of $\mathcal{S}$. Moreover $\mathcal{S}$ is a Banach manifold relative to this atlas and $i : \mathcal{S} \hookrightarrow \mathcal{BM}$ is smooth. To see that it is a supermanifold we must show that for two overlapping charts (U, ψ), (V, ϕ) in $\mathcal{A}_{\mathcal{M}}$ which are used to define charts on $\mathcal{S}$ we have that

$$\bar{\phi} \circ \bar{\psi}^{-1} : \bar{\psi}(\bar{U}_x \cap \bar{V}_x) \to \bar{\phi}(\bar{U}_x \cap \bar{V}_x) \tag{6.19}$$

is of class G^∞ where $\bar{U}_x = C_x(U \cap \mathcal{S})$, $\bar{V}_x = C_x(V \cap \mathcal{S})$ and $\bar{\psi} = \psi | \bar{U}_x$, $\bar{\phi} = \phi | \bar{V}_x$. Let ϕ^J denote the J-th component of ϕ and observe that for $u \in \psi(U \cap V)$

$$d^k_u(\phi^J \circ \psi^{-1})(V_1, V_2, \ldots, V_k) = \sum_{A_1 .. A_k = 1}^{p+q} V_1^{A_1} V_2^{A_2} \cdots V_k^{A_k} \left(\frac{\partial^k (\phi^J \circ \psi^{-1})}{\partial z^{A_k} \cdots \partial z^{A_2} \partial z^{A_1}} \right)(u). \tag{6.20}$$

for $V_1, V_2, \ldots, V_k \in \mathbb{K}^{p|q}$. Eq.(6.20) holds by the definition of supermanifold which implies that the transition maps $\phi^J \circ \psi^{-1}$ are G^∞. If we restrict to $u \in \psi(\bar{U}_x \cap \bar{V}_x)$ and $V_1, V_2, \ldots, V_k \in \mathbb{K}^{r|s}$ where we identify $\mathbb{K}^{r|s}$ with $\mathbb{K}^{r|s} \times \{(0,0)\} \subseteq \mathbb{K}^{p|q}$, then

$$d^k_u(\bar{\phi}^J \circ \bar{\psi}^{-1})(V_1, V_2, \ldots, V_k) = d^k_u(\phi^J \circ \psi^{-1})(V_1, V_2, \ldots, V_k) \tag{6.21}$$

Thus,

$$d^k_u(\bar{\phi}^J \circ \bar{\psi}^{-1})(V_1, V_2, \ldots, V_k) = \sum_{A_1..A_k=1}^{p+q} V_1^{A_1} V_2^{A_2} \cdots V_k^{A_k} \left(\frac{\partial^k (\phi^J \circ \psi^{-1})}{\partial z^{A_k} \cdots \partial z^{A_2} \partial z^{A_1}} \right)(u). \tag{6.22}$$

Therefore, $\bar{\phi}^J \circ \bar{\psi}^{-1}$ is G^∞ on $\psi(\bar{U}_x \cap \bar{V}_x)$ by Theorem 6.6.3. We simply take $f = \bar{\psi}^{-1}$ and $\mathcal{N} = \mathbb{K}^{p|q}$ which is of course a trivial supermanifold.

To see that $i : \mathcal{S} \hookrightarrow \mathcal{M}$ is G^∞ note that, using the same notation as above, $\psi \circ i \circ \bar{\psi}^{-1}$ is the inclusion of $\bar{\psi}(\bar{U}_x)$ into $\psi(U)$. To be more explicit, it is the inclusion

$$\psi(U) \cap (\mathbb{K}^{r|s} \times \{(0,0)\}) \hookrightarrow \psi(U) \subseteq \mathbb{K}^{p|q}$$

which is clearly class G^∞ because the inclusion

$$\mathbb{K}^{r|s} \overset{h}{\hookrightarrow} \mathbb{K}^{p|q}$$

is G^∞ since its components h^I are. □

Corollary 6.8.7. *Assume $\mathcal{M}$ is a supermanifold of dimension $(p|q)$ and that $\mathcal{S}$ is a leaf of a foliation of the Banach manifold $\mathcal{BM}$ such that, for each $x \in T_x\mathcal{S}$ is a subspace of $T_x\mathcal{M}$ of dimension (r, s). Then $\mathcal{S}$ is an initial super submanifold of $\mathcal{M}$ of dimension $(r|s)$ and consequently $\mathcal{S}$ is a supermanifold whose inclusion of $\mathcal{S}$ into $\mathcal{M}$ is a G^∞ mapping.*

Proof. It is known that each leaf of a foliation of a Banach manifold $\mathcal{BM}$ is an initial submanifold of $\mathcal{BM}$ and consequently if $\mathcal{S}$ is such a leaf then it follows from the theorem that $\mathcal{S}$ is an initial super submanifold of $\mathcal{M}$. The corollary follows. □

Proposition 6.8.8. *Assume that $\mathcal{M}, \mathcal{N}$ are supermanifolds, that $\mathcal{P}$ is a supermanifold of dimension $(r|s)$, that $\psi : \mathcal{M} \to \mathcal{N}$ is a G^∞ mapping, and that $i : \mathcal{P} \to \mathcal{N}$ is a class G^∞ injective immersion onto an initial submanifold $i(\mathcal{P})$ of $\mathcal{N}$ of dimension $(r|s)$. If $\psi_{x_o} : \mathcal{M} \to \mathcal{P}$ is the unique mapping such that $i \circ \psi_{x_o} = \psi$, then it is of class G^∞.*

Proof. First assume that $i(\mathcal{P})$ is an initial submanifold of $\mathcal{N}$ of dimension $(r|s)$ and that the inclusion $i : \mathcal{P} \to \mathcal{N}$ is a class G^∞ injective immersion. Notice that $B\mathcal{P}$ is an initial submanifold of the Banach manifold $B\mathcal{N}$ and that $\psi : \mathcal{BM} \to B\mathcal{N}$

is a C^∞ mapping. It is known [73] that for Banach manifolds the unique mapping $\psi_o : \mathcal{BM} \to B\mathcal{P}$ such that $i \circ \psi_o = \psi$ is necessarily continuous and is in fact of class C^∞.

To finish the proof, it suffices to show that each point $p \in \mathcal{P}$ is in the domain U of a chart (U, y) of $\mathcal{P}$ such that $y \circ \psi_o|_{\psi_o^{-1}(U)}$ is of class G^∞ (observe that $\psi_o^{-1}(U)$ is open in $\mathcal{M}$). Let $p \in \mathcal{P}$ and let (V, x) be a chart of $\mathcal{N}$ at $i(p)$. There exists $j_1 < j_2 < \cdots < j_t$ such that $x^{j_1} \circ i, x^{j_2} \circ i, \ldots, x^{j_t} \circ i$ are components of a chart on a neighborhood U_p of $U = i^{-1}(V) \subseteq \mathcal{P}$. If $y = (x^{j_1} \circ i, x^{j_2} \circ i, \ldots, x^{j_t} \circ i)$ then for $q \in \psi_o^{-1}(U_p)$, $1 \leq k \leq t$,

$$(y^k \circ \psi_o)(q) = (x^{j_k} \circ i \circ \psi_o)(q) = (x^{j_k} \circ \psi)(q)$$

and $y^k \circ \psi_o = x^{j_k} \circ \psi$ which is a class G^∞ mapping. Since $y \circ \psi_o$ is of class G^∞, it follows that ψ_o is a class G^∞ mapping. □

Chapter 7

Super Lie Groups

7.1 Introduction

Although mathematicians and physicists have been developing the theory of super Lie groups for over a quarter of a century, there remains a gap in one of the formulations of this theory. The gap which we perceive to exist has to do with the treatment of super Lie groups due to Rogers [100]. She, in fact, has laid out the basic theory of supermanifolds based on a space Λ of supernumbers which is in fact a Banach algebra generated by either a finite or a countably infinite number of Grassmann generators. Her supermanifolds are locally modeled on Banach spaces $\mathbb{K}^{p|q} = (\Lambda^0)^p \times (\Lambda^1)^q$ where either $\Lambda = \Lambda_N$ has N generators or $\Lambda = \Lambda_\infty$ has an infinite number of such generators. In her paper on super Lie groups [100] she derives basic theorems about super Lie groups, but the deeper results are obtained only when $\Lambda = \Lambda_N$. In this case it turns out that, with considerable effort, one can reduce the deeper theorems to corresponding theorems for ordinary finite dimensional Lie groups. It is asserted that it would be interesting to develop these ideas in case $\Lambda = \Lambda_\infty$ and that there are explicit areas of quantum field theory where such results would be useful. This same conclusion is asserted in the book by Freund [43].

It is our purpose to fill this gap in the Rogers approach to super Lie groups. Infinitely generated Grassmann algebras are both more and less complicated than in the finitely generated case. Since there is no generator of maximal order, there are no ambiguities in the top dimension. In the finitely generated case, the highest order derivatives of a function are not unique; this ambiguity sporadically surfaces and can be a source of difficulty which continually requires consideration. On the other hand, in the infinitely generated case, we are not able to appeal to corresponding theory of finite dimensional Lie groups. We *are* able to utilize the theory of Banach Lie groups at various points of our development, but even when we are able to do so, we often must develop the machinery needed to assure that we remain in the "supersmooth category". It came to our attention after the completion of this work that many

of our results have been obtained in the superanalytic category [25], [94], but these results have little impact on our work here. Our notation throughout the thesis is an amalgam of that of Rogers [98] and Buchbinder and Kuzenko [29].

We determine when a sub-super Lie algebra $\mathfrak{h}$ of the super Lie algebra $\mathcal{L}(\mathcal{G})$ of a super Lie group $\mathcal{G}$ is in fact the super Lie algebra of a sub-super Lie group of $\mathcal{G}$. We also find conditions under which the even part of an abstract Banach super Lie algebra is the even part of the super Lie algebra of some super Lie group $\mathcal{G}$. Given a super Lie algebra $\mathfrak{g}$ we show that there exists a super Lie group whose G^∞ structure is determined by the even part of $\mathfrak{g}$. Moreover, the super Lie structure on $\mathfrak{g}$ is recovered from the super Lie group $\mathcal{G}$. Along the way we also show that if $\mathcal{H}$ is a closed sub-super Lie group of a super Lie group $\mathcal{G}$, then $\mathcal{G} \to \mathcal{G}/\mathcal{H}$ is a principal fiber bundle. We emphasize that all of this work assumes an infinite number of Grassmann generators of our space of supernumbers.

Finally, in the last section of the chapter, we show how to apply our results to those types of super Lie groups prevalent in the physics literature. In that context super Lie groups often arise by beginning with a super Lie algebra which is used to construct a super Lie group using the exponential mapping and the Baker-Campbell-Hausdorff formula. This is an effective procedure but does not address the issue of finding a super smooth atlas for the group. In particular, one also has no way of determining the topology of the super Lie group. Our theory settles these issues when the underlying module structures utilize infinitely generated supernumbers as scalars; we emphasize that the finitely generated case was dealt with by Rogers [100]. In this last section we show how our results relate to procedures utilized in the physics literature especially for super Lie groups and super Lie algebras of matrices with supernumbers as entries. Additionally, we show that for every graded Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, there exists a super Lie group $\mathcal{G}$ whose super Lie algebra is the Grassmann shell $\widehat{\mathfrak{g}_{Lie}}$ of the Lie algebra $\mathfrak{g}$.

The author is grateful to T. Ratiu who provided him with information and references regarding the theory of Banach Lie groups. He is, of course, in no way responsible for any misunderstanding or misuse of these ideas in this dissertation.

7.2 Left Invariant Vector Fields as a Banach Lie Algebra

Definition 7.2.1. *A supermanifold $\mathcal{G}$ which is also an abstract group is called a super Lie group if the group operations are G^∞ with respect to the supermanifold structure on $\mathcal{G}$.*

When the supermanifold $\mathcal{G}$ is given the Banach manifold structure implicit in its definition the resulting Banach manifold is denoted by $\mathcal{BG}$.

Definition 7.2.2. *A Banach manifold $\mathcal{B}$ which is also an abstract group is called a*

Banach Lie group if the group operations are C^∞ with respect to the manifold structure on $\mathcal{B}$.

Remark 7.2.3. *Since G^∞ functions are always class C^∞ functions, it follows that the Banach manifold $\mathcal{BG}$ corresponding to a super Lie group $\mathcal{G}$ is necessarily a Banach Lie group.*

Left invariant vector fields are defined just as in the classical case,

Definition 7.2.4. *Let $\mathcal{G}$ be a super Lie group with left translation map $l_x(g) = xg$. Then a vector field X on $\mathcal{G}$ is said to be left invariant if for $g, x \in \mathcal{G}$*

$$X(gx) = d_x l_g(X(x)).$$

For each $v \in T_e\mathcal{G}$ the vector field X^v defined by

$$X^v(x) = d_e l_x(v)$$

for all $x \in \mathcal{G}$ is left invariant and for every left invariant vector field X there exists a $v \in T_e\mathcal{G}$ such that $X = X^v$. We denote the set of all left invariant vector fields on $\mathcal{G}$ by $\mathcal{L}(\mathcal{G})$. Moreover $\mathcal{L}(\mathcal{G})^0$ denotes the set of even left invariant vector fields while $\mathcal{L}(\mathcal{G})^1$ denotes those which are odd.

The first assertion of the following theorem is Theorem 3.4 in [100].

Theorem 7.2.5. *Let $\mathcal{G}$ be an $(p|q)$-dimensional super Lie group, then $\mathcal{L}(\mathcal{G})$ is an $(p|q)$-dimensional graded Lie left Λ module subject to the bracket operation $[\ ,\ ] : \mathcal{L}(\mathcal{G}) \times \mathcal{L}(\mathcal{G}) \to \mathcal{L}(\mathcal{G})$ defined by*

$$[X, Y] = XY - (-1)^{\epsilon(X)\epsilon(Y)} YX$$

for all $X, Y \in \mathcal{L}(\mathcal{G})$. Moreover, there is a norm $||\cdot||$ on $\mathcal{L}(\mathcal{G})$ such that it is a Banach space and

(1) $\mathcal{L}(\mathcal{G})^0$ and $\mathcal{L}(\mathcal{G})^1$ are closed subspaces of $\mathcal{L}(\mathcal{G})$,

(2) $\mathcal{L}(\mathcal{G})$ is a Banach super Lie algebra in the sense that there exists $M > 0$ such that $||[X, Y]|| \leq M||X||\ ||Y||$ for all $X, Y \in \mathcal{L}(\mathcal{G})$,

(3) the Banach Lie algebra of the Banach Lie group $\mathcal{BG}$ is $\mathcal{L}(\mathcal{G})^0$.

Proof. The first assertion is proved in [100]. To obtain a norm on $\mathcal{L}(\mathcal{G})$ we first define a norm on $\mathfrak{g} = T_e\mathcal{G}$. Choose a chart $\psi = (u^1, u^2, \cdots, u^{p+q})$ at the identity e of $\mathcal{G}$. For $X \in T_e\mathcal{G}$, let

$$X_\psi = (X^1_\psi, X^2_\psi, \cdots, X^{p+q}_\psi) \in \Lambda^{p+q}$$

where $X = \sum_A X^A_\psi e_A$ and the basis $\{e_A\}$ of $\mathfrak{g} = T_e\mathcal{G}$ is that defined by $e_A = \frac{\partial}{\partial u^A}$. Now define $||X|| = ||(X^1_\psi, X^2_\psi, \cdots, X^{p+q}_\psi)|| = \sum_A ||X^A_\psi||$ which is the norm of

$(X^1_\psi, X^2_\psi, \cdots, X^{p+q}_\psi)$ in Λ^{p+q}. Clearly, $\mathfrak{g}$ is a Banach space with respect to this norm. It is equally clear that $\mathfrak{g}^0 = \mathcal{L}(\mathcal{G})^0$ and $\mathfrak{g}^1 = \mathcal{L}(\mathcal{G})^1$ are closed subspaces of $\mathfrak{g}$.

We show that the norm satisfies condition (2) of the Theorem. In this part of the proof we abandon the notation used in the first paragraph choosing to represent elements of $\mathfrak{g}$ as the value X_e of some left invariant vector field $X \in \mathcal{L}(\mathcal{G})$. Using this notation we define a norm on $\mathcal{L}(\mathcal{G})$ by $||X|| = ||X_e||$ where $||X_e||$ is the norm of X_e as defined in the first paragraph. Let $(\tilde{e}_A)_x = d_e l_x(e_A), x \in \mathcal{G}$, denote the left invariant vector field defined by an element e_A of the basis of $\mathfrak{g}$. For $Z \in \mathcal{L}(\mathcal{G})$ note that, because $d_e l_x$ is even for $x \in \mathcal{G}$, $Z_x = d_e l_x(Z_e) = \sum_A d_e l_x(Z^A e_A) = \sum_A Z_A(\tilde{e}_A)_x$, for $Z_A \in \Lambda$. Define structure constants $f^C_{AB} \in \Lambda$ by $[\tilde{e}_A, \tilde{e}_B] = \sum_C f^C_{AB}\tilde{e}_C$ and let $M > 0$ be a number such that $||f^C_{AB}|| \leq M$ for all A, B, C. We have for appropriate $\epsilon(A, B) \in \mathbb{Z}_2$,

$$||[X, Y]|| = ||\sum_A \sum_B (-1)^{\epsilon(A,B)} X^A Y^B [\tilde{e}_A, \tilde{e}_B]|| \leq \sum_{A,B,C} ||X^A||||Y^B||||f^C_{AB}\tilde{e}_C||$$

$$\leq M(p+q) \sum_A ||X^A|| \sum_B ||Y^B|| = M(p+q)||X||||Y||$$

and (2) follows. Part (3) follows from the fact that as Banach spaces $\mathcal{L}(\mathcal{G})^0$ is isometric and isomorphic to $\mathfrak{g}^0 = T^0_e\mathcal{G}$ which can be identified with the tangent space to $\mathcal{BG}$. □

Remark 7.2.6. *Notice that the norm defined on $\mathcal{L}(\mathcal{G})$ above depends on the chart chosen at the identity e and that, relative to this norm, $\mathcal{L}(\mathcal{G})$ is isometric to the Banach space $\mathbb{K}^{p|q} \oplus \mathbb{K}^{\bar{p}|\bar{q}}$. Another chart produces a different norm on $\mathcal{L}(\mathcal{G})$ but also provides an isometry from $\mathcal{L}(\mathcal{G})$ onto $\mathbb{K}^{p|q} \oplus \mathbb{K}^{\bar{p}|\bar{q}}$. It follows that $\mathcal{L}(\mathcal{G})$ relative to the first norm is isometric to $\mathcal{L}(\mathcal{G})$ with the second norm, but the two spaces are not identical. Thus the topology on $\mathcal{L}(\mathcal{G})$ is chart independent and so a subspace of $\mathcal{L}(\mathcal{G})$ is closed relative to one norm iff it is relative to the other. This becomes important in our next theorem. We refer to a norm which is defined by some chart at the identity as an* **admissible norm.**

Definition 7.2.7. *Assume that $\mathfrak{g}$ is a super Lie algebra of graded dimension (p, q). We say that it is a Banach super Lie algebra if there is a norm on $\mathfrak{g}$ such that*

1. *$\mathfrak{g}$ is a Banach space relative to the norm such that both $\mathfrak{g}^0$ and $\mathfrak{g}^1$ are closed subspaces of $\mathfrak{g}$, and*
2. *there exists a number $M > 0$ such that $||[X, Y]|| \leq M||X||||Y||$ for all $X, Y \in \mathfrak{g}$.*

7.3 Inducing Sub Super Lie Groups from Sub Super Lie Algebras

We prepare to determine when a sub-super Lie algebra of $\mathcal{L}(\mathcal{G})$ is $\mathcal{L}(\mathcal{H})$ for some super Lie group $\mathcal{H}$. If $\mathfrak{h}$ is a sub-super Lie algebra of $\mathcal{L}(\mathcal{G})$, then we say that it is closed and

split in $\mathcal{L}(\mathcal{G})$ iff it is closed with respect to some admissible norm on $\mathcal{L}(\mathcal{G})$ and there is a closed complementary subspace $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathcal{L}(\mathcal{G})$. More precisely, we require that $\mathcal{L}(\mathcal{G}) = \mathfrak{h} \oplus \mathfrak{m}$ as graded normed linear spaces. Notice that if $\mathfrak{h}$ is closed and split in $\mathcal{L}(\mathcal{G})$ such that $\mathcal{L}(\mathcal{G}) = \mathfrak{h} \oplus \mathfrak{m}$ then since $\mathcal{L}(\mathcal{G})^0 = \mathfrak{h}^0 \oplus \mathfrak{m}^0$ we see that $\mathfrak{h}^0$ is closed and split in $\mathcal{L}(\mathcal{G})^0$.

Definition 7.3.1. *Suppose $\mathcal{M}, \mathcal{N}$ are supermanifolds and that ϕ is a G^∞ mapping from $\mathcal{M}$ into $\mathcal{N}$. If X is a vector field on $\mathcal{M}$ and Y is a vector field on $\mathcal{N}$, then we say X is ϕ-related to Y if and only if $d_x\phi(X_x) = Y_{\phi(x)}$ for each $x \in \mathcal{M}$.*

Remark 7.3.2. *For ordinary manifolds, M, N it is well-known that if X_1, X_2 are vector fields on M and Y_1, Y_2 are vector fields on N such that X_i is ϕ-related to Y_i for $i = 1, 2$, then $[X_1, X_2]$ is ϕ-related to $[Y_1, Y_2]$. This also holds in the present case for supermanifolds $\mathcal{M}, \mathcal{N}$ when ϕ is a G^∞ mapping. The proof is identical to the classical proof and is left to the reader. This fact is needed in the proof of the next theorem.*

Theorem 7.3.3. *Let $\mathcal{G}$ denote a type $(p|q)$ dimensional super Lie group and $\mathfrak{g} = \mathcal{L}(\mathcal{G})$ its super Lie algebra of left invariant vector fields. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a (r, s) dimensional sub-super Lie algebra of $\mathfrak{g}$ which is closed and split in $\mathcal{L}(\mathcal{G})$. Then there is a type $(r|s)$ super Lie group $\mathcal{H}$ which is a subgroup of $\mathcal{G}$ such that $\mathcal{L}(\mathcal{H}) = \mathfrak{h}$ and the inclusion $i : \mathcal{H} \to \mathcal{G}$ is a G^∞ injective immersion.*

Proof. Let $\mathcal{G}$ be a super Lie group of type $(p|q)$ and $\mathfrak{g}$ its Banach super Lie algebra of left invariant vector fields. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a sub-super Lie algebra of type (r, s) which is closed and split. Then $\mathfrak{h}^0 \subseteq \mathfrak{g}^0$ is a closed and split sub-Lie algebra of the Banach Lie algebra $\mathfrak{g}^0$. Moreover $\mathfrak{g}^0$ is the Lie algebra of the Banach Lie group $\mathcal{BG}$. Since $\mathfrak{h}^0$ is closed and split in $\mathfrak{g}^0$ it is known (see [80]) that there is a Banach Lie subgroup H of $\mathcal{BG}$ with Lie algebra $\mathfrak{h}^0$.

Moreover H can be obtained as the maximal integral submanifold through the identity of $\mathcal{BG}$ of the subbundle $E \to \mathcal{BG}$ of the tangent bundle $T\mathcal{BG} \to \mathcal{BG}$ defined by $E_x = d_e l_x(\mathfrak{h}_e^0)$ for each $x \in \mathcal{BG}$ where $\mathfrak{h}_e = \{X_e | X \in \mathfrak{h}\}$ and $\mathfrak{g}_e = \{X_e | X \in \mathfrak{g}\}$. Here $\mathfrak{h}_e^0$ is identified as a closed split subspace of $\mathfrak{g}_e^0$ which is identified with $T_e\mathcal{BG}$. It is known that a leaf of a foliation is an initial submanifold (see the book by Kolar, Michor, and Slovak [73]). Moreover it is known that the inclusion $i : H \hookrightarrow \mathcal{BG}$ is a smooth injective immersion. It follows from Corollary 6.8.7 that H can be given a supermanifold structure and if we call H with this structure $\mathcal{H}$, then the corollary also assures that the inclusion $i : \mathcal{H} \hookrightarrow \mathcal{G}$ is a G^∞ mapping. Note that E_x has dimension $(r|s)$ for each $x \in \mathcal{G}$, and $E_e = \mathfrak{h}_e^0$. So $T_eH = \mathfrak{h}_e^0$ and charts take their values in the appropriate subspace $\mathbb{K}^{r|s}$ of $\mathbb{K}^{p|q}$. Since the charts of $\mathcal{H}$ take their values in $\mathbb{K}^{r|s}$, $\mathcal{H}$ has dimension $(r|s)$.

Let $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ denote the group multiplication on $\mathcal{G}$. It follows that $\mu \circ (i \times i) : \mathcal{H} \times \mathcal{H} \to \mathcal{G}$ is a G^∞ mapping. Since $\mathcal{H}$ is an initial submanifold of $\mathcal{G}$ and

$\mu(\mathcal{H}\times\mathcal{H})\subseteq\mathcal{H}$ it follows from Proposition 6.8.4 that the mapping $\mu_{\mathcal{H}}:\mathcal{H}\times\mathcal{H}\to\mathcal{H}$ such that $i\circ\mu_{\mathcal{H}}=\mu\circ(i\times i)$ is a class G^∞ mapping. A similar application of Proposition 6.8.4 shows that $inv_{\mathcal{H}}(x)=x^{-1}$ is also a class G^∞ mapping. Thus $\mathcal{H}$ is a super Lie group and $i:\mathcal{H}\to\mathcal{G}$ is a G^∞-immersion.

Finally, since $\mathcal{H}$ is a super Lie group, $l_x:\mathcal{H}\to\mathcal{H}$ is a G^∞-mapping for each $x\in\mathcal{H}$ and d_yl_x maps $T_y\mathcal{H}$ into $T_{xy}\mathcal{H}$ for all $x,y\in\mathcal{H}$. In particular d_yl_x also maps $T^0_y\mathcal{H}$ into $T^0_{xy}\mathcal{H}$ so that for each $x\in\mathcal{H}, d_el_x(T^0_e\mathcal{H})=T^0_x\mathcal{H}$ and $\mathfrak{h}^0_e=T_e\mathcal{B}\mathcal{H}=T^0_e\mathcal{H}$. Thus $T^0_e\mathcal{H}$ may be identified with $\mathfrak{h}^0$. It is perhaps, not as obvious that $T^1_e\mathcal{H}$ can be identified with $\mathfrak{h}^1$.

We show that $\mathfrak{h}$ is isomorphic to $\mathcal{L}(\mathcal{H})$ as super Lie algebras in a succession of steps. To do this first observe that $\mathcal{L}(\mathcal{G})$ can be identified with $T_e\mathcal{G}$ by identifying $v\in T_e\mathcal{G}$ with $X^v_{\mathcal{G}}\in\mathcal{L}(\mathcal{G})$ where $X^v_{\mathcal{G}}(x)=d_el_x(v)$ for all $x\in\mathcal{G}$. Notice that since $\mathfrak{h}$ is a sub-super Lie algebra of $\mathcal{L}(\mathcal{G}),\mathfrak{h}$ is identified with $\mathfrak{h}_e\equiv\{X_e|X\in\mathfrak{h}\subseteq\mathcal{L}(\mathcal{G})\}$ (notice the change in notation, $\mathfrak{h}_e$ here and below is a subset of $\mathcal{L}(\mathcal{G})$ not $\mathcal{L}(\mathcal{H})$). Both $\mathfrak{h}_e$ and $T_e\mathcal{G}$ are given a super Lie algebra structure by defining $[v,w]_{\mathcal{G}}$ for $v,w\in T_e\mathcal{G}$ via

$$X^{[v,w]_{\mathcal{G}}}_{\mathcal{G}}=[X^v_{\mathcal{G}},X^w_{\mathcal{G}}]. \tag{7.1}$$

Thus $\mathfrak{h}\cong\mathfrak{h}_e$ which is a sub-super Lie algebra of $T_e\mathcal{G}$.

We now show that $T_e\mathcal{H}$ can also be identified as a sub-super Lie algebra of $T_e\mathcal{G}$ to be followed later by a proof that $\mathfrak{h}_e=T_e\mathcal{H}$. To do this recall that $\iota:\mathcal{H}\to\mathcal{G}$ is an immersed initial submanifold of $\mathcal{G}$ and consequently that $d_e\iota:T_e\mathcal{H}\to T_e\mathcal{G}$ is a right Λ-linear injection of $T_e\mathcal{H}$ into $T_e\mathcal{G}$. For $v\in T_e\mathcal{H}$ let $X^v_{\mathcal{H}}$ denote the left invariant vector field on $\mathcal{H}$ defined by $X^v_{\mathcal{H}}(y)=d_el_y(v), y\in T_e\mathcal{H}$. For $v,w\in T_e\mathcal{H}$ define $[v,w]_{\mathcal{H}}$ by

$$X^{[v,w]_{\mathcal{H}}}_{\mathcal{H}}=[X^v_{\mathcal{H}},X^w_{\mathcal{H}}].$$

Notice that for every $v\in T_e\mathcal{H}$, the vector field $X^v_{\mathcal{H}}$ is ι-related to $X^{d_e\iota(v)}_{\mathcal{G}}$. It follows from Remark 7.3.2 that for $v,w\in T_e\mathcal{H}$,

$$X^{[v,w]_{\mathcal{H}}}_{\mathcal{H}}=[X^v_{\mathcal{H}},X^w_{\mathcal{H}}]\qquad and\qquad X^{[d_e\iota(v),d_e\iota(w)]_{\mathcal{G}}}_{\mathcal{G}}=[X^{d_e\iota(v)}_{\mathcal{G}},X^{d_e\iota(w)}_{\mathcal{G}}]$$

are ι-related. Consequently $d_e\iota[v,w]_{\mathcal{H}}=[d_e\iota(v),d_e\iota(w)]_{\mathcal{G}}$, and $(T_e\mathcal{H},[,]_{\mathcal{H}})$ may be identified as a sub-super Lie algebra of $(T_e\mathcal{G},[,]_{\mathcal{G}})$.

It remains only to show that $\mathfrak{h}_e$ and $T_e\mathcal{H}$ are equal as subsets of $T_e\mathcal{G}$. To see this notice that a pure basis of $\mathfrak{h}_e$ can be extended to a pure basis of $T_e\mathcal{G}$. It follows that there exists a pure basis $\{e_A|1\leq A\leq p+q\}$ of $T_e\mathcal{G}$ such that $\{e_A|A\in\mathcal{A}\},\mathcal{A}=\{1,2,\cdots,r,p+1,p+2,\cdots,p+s\}$ is a pure basis of $\mathfrak{h}_e$. Choose a chart $\psi:U\to T^0_e\mathcal{G}$ of $\mathcal{G}$ at $e\in U$. Then $\psi\circ\iota:\iota^{-1}(U)\to\mathfrak{h}^0_e$ is a chart of $\mathcal{H}$ at $e\in\iota^{-1}(U)$. If we define coordinate functions (u^A) of ψ by $\psi(x)=\sum_{A=1}^{p+q}u^A(x)e_A, x\in U$, then we have coordinate functions defined on $\mathcal{H}$ by $(\psi\circ\iota)(y)=\sum_{A\in\mathcal{A}}u^A(\iota(y))e_A, y\in\iota^{-1}(U)$. Thus

$\{\frac{\partial}{\partial u^A}|A \in \mathcal{A}\}$ in $T_e\mathcal{H}$ is identified with $\{e_A|A \in \mathcal{A}\}$ in $\mathfrak{h}_e$ and

$$T_e\mathcal{H} = \{\sum_{A\in\mathcal{A}} \lambda^A e_A \quad |\lambda^A \in \Lambda\} = \mathfrak{h}_e. \tag{7.2}$$

Consequently, we have that as super Lie algebras

$$\mathfrak{h} = \mathfrak{h}_e = T_e\mathcal{H} = \mathcal{L}(\mathcal{H}), \tag{7.3}$$

from which the theorem follows. □

Definition 7.3.4. *Let $\mathcal{G}$ be a super Lie group and $\mathfrak{g}$ its tangent module $T_e\mathcal{G}$ at the identity e of $\mathcal{G}$. For each $v \in \mathfrak{g}^0$ we define a left invariant vector field X^v on $\mathcal{BG}$ by*

$$X^v(x) = d_e l_x(v) \in T^0_x\mathcal{G} = T_x\mathcal{BG}$$

for $x \in \mathcal{BG}$. Let $\phi_v : \mathbb{R} \times \mathcal{BG} \to \mathcal{BG}$ denote the flow of the vector field X^v on $\mathcal{BG}$. Thus

$$\frac{d}{dt}\phi_v(t,x) = X^v(\phi_v(t,x)) \qquad \text{where} \qquad \phi_v(0,x) = x. \tag{7.4}$$

Definition 7.3.5. *exp is the mapping from $\mathfrak{g}^0$ into $\mathcal{BG}$ defined by $exp(v) \equiv \phi_v(1,e)$.*

Note that *exp* is C^∞ mapping which is also a local diffeomorphism. Also, we can regard *exp* as a mapping from $\mathfrak{g}^0$ into $\mathcal{G}$ since as sets $\mathcal{BG} = \mathcal{G}$. In fact it can be shown that $exp : \mathfrak{g}^0 \to \mathcal{G}$ is a G^∞-mapping. We now establish several lemmas towards that goal.

We fix the notation from this point up through the proof of Theorem 4.15. Let $\mathcal{G}$ denote an arbitrary super Lie group and $\mathfrak{g}$ its tangent module $T_e\mathcal{G}$ at the identity. Even vectors are denoted $\mathfrak{g}^0$. We have a fixed pure basis $\{e_a\}$ of $\mathfrak{g}$ which can be taken to be the partials relative to a chart at e.

Definition 7.3.6. *The adjoint mapping defined on $\mathfrak{g}$ is the mapping $ad : \mathfrak{g} \to End(\mathfrak{g})$ where, for $x, y \in \mathfrak{g}$,*

$$ad(x)(y) = ad_x(y) = [x,y]$$

Observe that $ad_{\alpha x} = \alpha ad_x$ for all $\alpha \in \Lambda$; the adjoint ad on $\mathfrak{g}$ is right-Λ-linear, thus $ad \in L^-(\mathfrak{g}, End(\mathfrak{g}))$. However, for a particular $x \in \mathfrak{g}$, we note that $ad_x(y\alpha) = ad_x(y)\alpha$ for all $\alpha \in \Lambda$, thus $ad_x \in End^+(\mathfrak{g})$.

Lemma 7.3.7. *Let $(End^+\mathfrak{g})^0$ denote the linear space of all even left endomorphisms of $\mathfrak{g}$. Once for all, identity these linear mappings with their matrices relative to our fixed basis of $\mathfrak{g}$. For each matrix M (representing such a linear mapping), define,*

$$|| \ M \ || = \sum_{i=1}^{p}\sum_{j=1}^{q} || \ M_{ij} \ ||.$$

Let $R > 0$ and assume that $\{a_k\}_{k=0}^{\infty}$ are numbers in $\mathbb{K}$ such that $\sum_{k=0}^{\infty} |a_k| \ ||M||^k$ converges for all $M \in (End^+\mathfrak{g})^0$ such that $||M|| \leq R$. Let $B_R(0)$ be the **open** *ball at zero in $(End^+\mathfrak{g})^0$, then $f : B_R(0) \to (End^+\mathfrak{g})^0$ defined by $f(M) = \sum_{k=0}^{\infty} a_k M^k$ is of class G^∞.*

Remark 7.3.8. *Having chosen a basis $\{e_i, \tilde{e}_\alpha\}$ of $\mathfrak{g}$ the* **even** *endomorphisms of $\mathfrak{g}$ are identified with matrices with a (p,q) block-form,*

$$M \in (End^+\mathfrak{g})^0 \implies M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{7.5}$$

where $A_{mn}, D_{\alpha,\beta} \in {}^0\Lambda$ and $B_{m\beta}, C_{\alpha,n} \in {}^1\Lambda$. Notice that as a module over ${}^0\Lambda$, $(End^+\mathfrak{g})^0$ may be identified with $\mathbb{K}^{(p^2+q^2|2pq)}$.

We now prove the lemma.

Proof. Note that for $||M|| < R$,

$$f(M+H) = f(M) + \sum_{k=1}^{\infty} a_k \sum_{i=0}^{k} (M^{k-i-1} H M^i) + O(H^2).$$

Thus,

$$d_M f(H) = \sum_{k=1}^{\infty} a_k \sum_{i=0}^{k} (M^{k-i-1} H M^i).$$

The components of this matrix are

$$d_M f_{bc}(H) = \sum_{k=1}^{\infty} a_k \sum_{i=0}^{k} \sum_{m,n} (M^{k-i-1})_{bm} (H)_{mn} (M^i)_{nc} = \sum_{m,n} H_{mn} \Lambda_{bc}^{mn}(M) \tag{7.6}$$

where, for some $\epsilon_{bmn} \in \mathbb{Z}_2$

$$\Lambda_{bc}^{mn}(M) = \sum_{k=1}^{\infty} a_k \sum_{i=0}^{k} (-1)^{\epsilon_{bmn}} (M^{k-i-1})_{bm} (M^i)_{nc}.$$

Thus $\frac{\partial f_{bc}}{\partial z_{mn}}$ exists and is equal to Λ_{bc}^{mn}; moreover the components f_{bc} of f are of class G^1 on $B_R(0)$. Thus f is of class G^1 on $B_R(0)$. By construction f is analytic on $B_R(0)$ hence it is C^∞ on $B_R(0)$. So f is C^∞ and G^1 on $B_R(0)$ thus by Theorem 6.6.5 we conclude that f is G^∞ on $B_R(0)$. □

Corollary 7.3.9. *Let $\mathfrak{g}$ be any super Lie algebra such as the one defined above and let $\mathfrak{g}^0$ be its even elements. Define a mapping f from $\mathfrak{g}^0$ into $(End^+\mathfrak{g})^0$ by*

$$X \overset{f}{\longmapsto} \int_0^1 e^{-s \ ad_X} ds.$$

Then f is of class G^∞.

Proof. First note that if $X \in \mathfrak{g}^0$ and $ad_X(Y) = [X, Y]$ then since X is even and $\epsilon([X, Y]) = \epsilon(X) + \epsilon(Y) = \epsilon(Y)$ we find that ad_X is an even left endomorphism of $\mathfrak{g}$. The composite of even endomorphisms is even, thus the series

$$e^{-s\ ad_X} = \sum_{k=0}^{\infty} \frac{1}{k!}(-s)^k\ (ad_X)^k$$

is an even left endomorphism of $\mathfrak{g}$. This series is absolutely and uniformly convergent on every ball about zero relative to the matrix norm defined in the lemma. It follows from the lemma that the mapping from $(End^+\mathfrak{g})^0$ to itself defined by

$$M \longmapsto e^{-s\ M}$$

is a G^∞ mapping.

To finish the proof we must show that $ad : X \to ad_X$ is a class G^∞ mapping. The mapping $ad : \mathfrak{g}^0 \to (End^+\mathfrak{g})^0$ is linear over $\mathbb{K}$ as is clear from $ad_X(Y) = [X, Y]$ and the definition of the Lie bracket. Hence the best linear approximation to the adjoint mapping is itself; $d_X(ad) = ad$. Thus the mapping $X \mapsto d_X(ad)$ is constant, its higher derivatives are zero. To see that ad is class G^∞ we have only to show that it is of class G^1, so we must show that for $X \in \mathfrak{g}^0, H \in \mathfrak{g}^0, d_X(ad^c_a)(H)$ is linear in the components of H where the $ad^c_a : \mathfrak{g}^0 \to \Lambda$ are the component mappings of ad defined by representing ad_X as a matrix .

Since ad_X is right linear its matrix is defined by $ad_X(e_a) = [X, e_a] = \sum X^b[e_b, e_a] = \sum X^b f^c_{ba} e_c$ so that $ad^c_a(X) = \sum X^b f^c_{ba}$. Now observe that $d_X ad^c_a(H) = ad^c_a(H) = \sum H^b f^c_{ba}$ which is linear in the components of H. It follows from Proposition 6.7.3 that ad is a class G^∞ mapping, hence $X \mapsto e^{-s\ ad_X}$ is the composite of G^∞ maps and is consequently G^∞ for each $s \in \mathbb{R}$. Finally integrate to obtain the desired result. □

Notice that the proof that $ad : \mathfrak{g} \to End(\mathfrak{g})$ is a class G^∞ mapping is completely analogous to this proof since it is also linear over $\mathbb{K}$ and possesses the required properties with respect to the module operations over Λ. Moreover the mapping ad regarded as a mapping from $\mathfrak{g}^0$ to $(End^+\mathfrak{g}^0)$ is also a class G^∞ mapping. Its components ad^c_a are obtained as before even though the basis is not a basis of $\mathfrak{g}^0$.

Theorem 7.3.10. *$exp : \mathfrak{g}^0 \to \mathcal{G}$ is a class G^∞ mapping.*

Proof. We need to compute the Frechet derivative of exp at $X \in \mathfrak{g}^0$. Since $\mathcal{BG}$ is a Banach Lie group we have the following formula for the Frechet derivative (see [40]),

$$d_X(exp)(H) = d_e l_{exp(X)} \left(\int_0^1 e^{-s\ ad_X}(H) ds \right). \tag{7.7}$$

Define a function $F : \mathfrak{g}^0 \to (End^+\mathfrak{g})^0$ by

$$F(X)(H) = \int_0^1 e^{-s\ ad_X}(H)ds. \tag{7.8}$$

It follows from Corollary 7.3.9 that F is a class G^∞ mapping. Notice that $d_X(exp)(H) = d_e l_{exp(X)}(F(X)(H))$ even though, in this formula, not only is H restricted to $\mathfrak{g}^0$, but it is also the case that $F(X)(H) \in \mathfrak{g}^0$. We have insisted, however, that $F(X)$ be defined on all of $\mathfrak{g}$ since we need the identity $F(X)(H) = H^i F(X)(e_i) + H^{\tilde{\alpha}} F(X)(e_{\tilde{\alpha}})$ which requires that $F(X)$ be defined on odd elements of $\mathfrak{g}$. On the other hand this very formula shows that the mapping from $\mathfrak{g}^0$ to $(End^+\mathfrak{g}^0)^0$ defined by $X \to F(X)|_{\mathfrak{g}^0}$ is also a class G^∞ mapping. We will occasionally abuse notation by failing to distinguish between the two mappings. Let $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ be the class G^∞ group multiplication of the supergroup $\mathcal{G}$. We have that,

$$\begin{aligned} d_X(exp)(H) &= d_e l_{exp(X)}\big(F(X)(H)\big) \\ &= d_e\big[\mu(exp(X), \cdot)\big](F(X)(H)) \\ &= (d_2\mu)(exp(X), e)(F(X)(H)). \end{aligned} \tag{7.9}$$

Where $d_2\mu$ denotes the Frechet derivative with respect to the second slot of μ. If $H = \sum_{i=1}^p H^i e_i + \sum_{\alpha=1}^q \tilde{H}^\alpha \tilde{e}_\alpha$ with respect to the pure basis $\{e_i, \tilde{e}_\alpha\}$ of $\mathfrak{g}$ then

$$\begin{aligned} d_X(exp)(H) &= (d_2\mu)(exp(X), e)(F(X)(\textstyle\sum_{i=1}^p H^i e_i + \sum_{\alpha=1}^q \tilde{H}^\alpha \tilde{e}_\alpha)) \\ &= \textstyle\sum_{i=1}^p H^i (d_2\mu)(exp(X), e)(F(X)(e_i)) \\ &\quad + \textstyle\sum_{\alpha=1}^q \tilde{H}^\alpha (d_2\mu)(exp(X), e)(F(X)(\tilde{e}_\alpha)) \\ &= \textstyle\sum_{i=1}^p H^i d_X(exp)(e_i) + \sum_{\alpha=1}^q \tilde{H}^\alpha d_X(exp)(\tilde{e}_\alpha). \end{aligned}$$

To pull the "scalars" out of $d_2\mu$ in the above we used the following observation. Since $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is G^∞ so is the mapping with one argument fixed, that is $(\mu_a)(x) \equiv \mu(a, x)$ is G^∞. Therefore $d_e\mu_a$ is a mapping from the full tangent module $T_e\mathcal{G} = \mathfrak{g}$ into $T_a\mathcal{G}$ such that

$$(d_e\mu_a)(H) = \sum_{i=1}^p H^i (d_e\mu_a)(e_i) + \sum_{\alpha=1}^q \tilde{H}^\alpha (d_e\mu_a)(\tilde{e}_\alpha).$$

We have shown that $d_X(exp)$ is linear over the components $H^i, \tilde{H}^\alpha$ and hence that exp is superdifferentiable at X for each $X \in \mathfrak{g}^0$. It follows that exp is of class G^1. It is known from the theory of Banach Lie groups that exp is C^∞ on $\mathcal{BG} = \mathfrak{g}^0$ hence by Theorem 6.6.5 we find that exp is G^∞ on $\mathfrak{g}^0$. □

Remark 7.3.11. *The proof given here replaces a lengthy proof in our paper [37]. It is easy to see the wisdom of [68] in elevating Theorem 6.6.5 to be the definition of G^∞.*

Theorem 7.3.12. *Let $\mathcal{G}$ be a $(p|q)$-super Lie group and $\mathcal{S}$ a subgroup which is also an initial $(r|s)$-submanifold of $\mathcal{G}$. Then $\mathcal{S}$ is a $(r|s)$-super Lie group.*

Proof. Let $i : \mathcal{S} \hookrightarrow \mathcal{G}$ denote the inclusion mapping and $\mu_{\mathcal{S}}, \mu_{\mathcal{G}}$ the group "multiplications" on $\mathcal{S}$ and $\mathcal{G}$ respectively, then by Theorem 4.5 $\mu_{\mathcal{G}} \circ (i \times i)$ is the composite of G^∞ mappings and so is of class G^∞. Since $\mathcal{S}$ is an initial submanifold, it follows from Proposition 4.7 6.8.4 that the unique mapping $\mu_{\mathcal{S}} : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ such that $i \circ \mu_{\mathcal{S}} = \mu_{\mathcal{G}} \circ (i \times i)$ is of class G^∞. A similar argument shows that $inv_{\mathcal{S}}$ is a class G^∞ mapping. The theorem follows. □

Definition 7.3.13. *If $\mathcal{G}$ is a $(p|q)$ super Lie group and $\mathcal{S} \subseteq \mathcal{G}$ is a subgroup which is also a $(r|s)$-submanifold of $\mathcal{G}$ then we say $\mathcal{S}$ is a sub-super Lie group of $\mathcal{G}$.*

Remark 7.3.14. *If $\mathcal{S}$ is a closed sub-super Lie group of a super Lie group of $\mathcal{G}$ then $\mathcal{BS}$ is a closed sub-Lie group of $\mathcal{BG}$ as Banach Lie groups. Moreover the coset space $\mathcal{BG}/\mathcal{BS}$ is known to be a Banach manifold and $\mathcal{BG} \to \mathcal{BG}/\mathcal{BS}$ is a principal fiber bundle with structure group $\mathcal{BS}$.*

Theorem 7.3.15. *If $\mathcal{G}$ is a $(p|q)$ super Lie group and $\mathcal{S}$ is a closed $(r|s)$ sub-super Lie group of $\mathcal{G}$ then $\mathcal{G}/\mathcal{S}$ is a $(p-r|q-s)$ supermanifold. Moreover $\mathcal{G} \to \mathcal{G}/\mathcal{S}$ is a G^∞-mapping and is a principal fiber bundle with structure group the super Lie group $\mathcal{S}$. All local trivializing maps are G^∞-maps.*

Proof. One only needs to check that the mappings which define the bundle structure of $\mathcal{BG} \to \mathcal{BG}/\mathcal{BS}$ are in fact G^∞-maps so there is little to prove. We sketch the main features of the proof for the convenience of the reader but in fact the argument is borrowed from Bröcker and Dieck [22]
First notice that since $i : \mathcal{S} \hookrightarrow \mathcal{G}$ is G^∞ the mapping $d_e i : T_e^0\mathcal{S} \hookrightarrow T_e^0\mathcal{G}$ is injective. Choose a pure basis $\{e_i, \tilde{e}_\alpha\}$, $1 \leq i \leq r$ and $1 \leq \alpha \leq s$ of $T_e\mathcal{S}$, and extend it to a pure basis $\{e_i, \tilde{e}_\alpha\}$, $1 \leq i \leq p$ and $1 \leq \alpha \leq q$ of $T_e\mathcal{G}$. Thus,

$$T_e^0\mathcal{S} \cong \mathbb{K}^{r|s} \hookrightarrow \mathbb{K}^{r|s} \times \mathbb{K}^{(p-r|q-s)} \cong T_e^0\mathcal{G}$$

and one may factor $T_e^0\mathcal{G} = T_e^0\mathcal{S} \times \mathcal{M}_e$ as Banach spaces where

$$v \in \mathcal{M}_e \iff v = \sum_{j=r+1}^{p} v^j e_j + \sum_{\alpha=s+1}^{q} \tilde{v}^\alpha \tilde{e}_\alpha$$

where $v^j, \tilde{v}^\alpha \in {}^0\Lambda$. The Banach structure is given by the norm on $T_e^0\mathcal{G}$ which is defined by,

$$|| \sum_{i=1}^{p} H^i e_i + \sum_{\alpha=1}^{q} \tilde{H}^\alpha \tilde{e}_\alpha || = \sum_{i=1}^{p} |H^i|_\Lambda + \sum_{\alpha=1}^{q} |\tilde{H}^\alpha|_\Lambda$$

for $H^i \in {}^0\Lambda$ and $\tilde{H}^\alpha \in {}^1\Lambda$ and $|\cdot|_\Lambda$ is the norm on the Banach algebra of supernumbers Λ. The definition for the norm on subspaces of $T_e^0\mathcal{G}$ is obvious.

Now define $\mathcal{M}_e^\epsilon = \{X \in \mathcal{M}_e \mid ||X|| < \epsilon\}$ for $\epsilon > 0$ and let $\mathcal{D}_\epsilon = exp(\mathcal{M}_e)$. Recall that $exp : T_e^0\mathcal{G} \to \mathcal{G}$ is both a local C^∞ diffeomorphism and a G^∞-mapping. Consider $\mu : \mathcal{D}_\epsilon \times \mathcal{S} \to \mathcal{G}$ defined by $\mu(g, s) = gs$ in $\mathcal{G}$. We claim that for ϵ small enough μ is an embedding. To see this first note that $(d\mu)_{(e,e)}|(T_e^0\mathcal{D}_\epsilon \times \{0\})$ and $(d\mu)_{(e,e)}|(\{0\} \times T_e^0\mathcal{S})$ are identity maps on $T_e^0\mathcal{D}_\epsilon$ and $T_e^0\mathcal{S}$ respectively. So $(d\mu)_{(e,e)}(v, w) = v + w$ and if $(d\mu)_{(e,e)}(v, w) = 0$ then $v = -w \in T_e^0\mathcal{D}_\epsilon \cap T_e^0\mathcal{S} = \{0\}$ and $ker(d\mu)_{(e,e)} = \{(0, 0)\}$. By the inverse function theorem for Banach manifolds there exists an open set U about e in $\mathcal{S}$ and $\epsilon > 0$ small enough
so that $\mu : \mathcal{D}_\epsilon \times U \to \mathcal{D}_\epsilon U$ is a C^∞ diffeomorphism. It is also a G^∞-mapping since the group operation on $\mathcal{G}$ is a G^∞-mapping and since the inclusions $\mathcal{D}_\epsilon \times U \hookrightarrow \mathcal{D}_\epsilon \times \mathcal{S} \hookrightarrow \mathcal{G} \times \mathcal{S} \hookrightarrow \mathcal{G} \times \mathcal{G}$ are all G^∞-mappings. Note that for $s \in \mathcal{S}$ the right multiplication map $R_s : \mathcal{G} \to \mathcal{G}$ defined by $R_s(x) = xs$ is a G^∞-mapping and so is

$$\mu|(\mathcal{D}_\epsilon \times (Us)) = R_s \circ [\mu|(\mathcal{D}_\epsilon \times \mathcal{S})] \circ [id_{\mathcal{D}_\epsilon} \times R_{s^{-1}}].$$

Moreover $\mu|(\mathcal{D}_\epsilon \times (Us))$ is a C^∞ diffeomorphism from $\mathcal{D}_\epsilon \times (Us)$ onto $\mathcal{D}_\epsilon Us$ for each $s \in \mathcal{S}$ and $\mu|(\mathcal{D}_\epsilon \times \mathcal{S})$ is a local C^∞-diffeomorphism and a G^∞-mapping. We claim that for small enough ϵ, $\mu|(\mathcal{D}_\epsilon \times \mathcal{S})$ is injective. Indeed if one chooses $V \subseteq \mathcal{G}$ open about e such that $(V^{-1}V) \cap \mathcal{S} \subseteq U$ then for each $\epsilon' < \epsilon$, $\epsilon' > 0$ such that $\mathcal{D}_{\epsilon'} \subseteq V$ one can show that $\mu|(\mathcal{D}_{\epsilon'} \times \mathcal{S})$ is injective. Thus we have the existence of $\epsilon > 0$ such that $\mu : \mathcal{D}_\epsilon \times \mathcal{S} \to \mathcal{D}_\epsilon\mathcal{S}$ is an embedding.

We now show how to obtain a G^∞ structure on the coset space $\mathcal{G}/\mathcal{S}$. Let $\eta : \mathcal{G} \to \mathcal{G}/\mathcal{S}$ denote the mapping which sends $x \in \mathcal{G}$ to the coset $\eta(x) \in \mathcal{G}/\mathcal{S}$. For $g \in \mathcal{G}$ let $U_g = g\mathcal{D}_\epsilon\mathcal{S}$ and notice that $U_g = \mu(\mathcal{D}_\epsilon \times \mathcal{S})$ is open in $\mathcal{G}$. Since U_g is the union of cosets $\eta(U_g)$, it is open in the quotient topology on $\mathcal{G}/\mathcal{S}$. Let ψ_g^{-1} denote the inverse of a chart where $\psi_g^{-1} : \mathcal{D}_\epsilon \to \eta(U_g)$ is defined by

$$\mathcal{D}_\epsilon \longrightarrow \mathcal{D}_\epsilon \times \{e\} \to \mathcal{D}_\epsilon \times \mathcal{S} \xrightarrow{\mu} \mathcal{D}_\epsilon\mathcal{S} \xrightarrow{l_g} g\mathcal{D}_\epsilon\mathcal{S} = U_g \xrightarrow{\eta} \eta(U_g).$$

For $g, h \in \mathcal{G}$ such that the relevant maps are well defined,

$$\begin{aligned}
(\psi_h \circ \psi_g^{-1})(x) &= \psi_h(\psi_g^{-1}(x)) \\
&= \psi_h(\eta(l_g(\mu(x, e)))) \\
&= \psi_h(\eta(l_h(l_{h^{-1}}l_g)(\mu(x, e)))) \\
&= \psi_h(\eta(l_h(\mu(h^{-1}gx, e)))) \\
&= \psi_h(\psi_h^{-1}(h^{-1}gx, e)) \\
&= l_{h^{-1}g}(x).
\end{aligned}$$

Thus $\psi_h \circ \psi_g^{-1}$ is a G^∞-mapping and consequently $\{(\eta(U_g), \psi_g) \mid g \in \mathcal{G}\}$ is a G^∞ structure on $\mathcal{G}/\mathcal{S}$.

We now produce a G^∞ local trivialization of $\mathcal{G}$ as a bundle over $\mathcal{G}/\mathcal{S}$. For $g \in \mathcal{G}$ let $\phi_g^{-1} : \eta(U_g) \times \mathcal{S} \to U_g \subseteq \mathcal{G}$ be the inverse of our proposed trivialization mapping where ϕ_g^{-1} is defined by

$$\eta(U_g) \times \mathcal{S} \overset{\psi_g \times id}{\longrightarrow} \mathcal{D}_\epsilon \times \mathcal{S} \overset{\mu}{\longrightarrow} \mathcal{D}_\epsilon \mathcal{S} \overset{l_g}{\longrightarrow} g\mathcal{D}_\epsilon \mathcal{S} = U_g$$

meaning,

$$(x, s) \mapsto (\psi_g(x), s) \mapsto \mu(\psi_g(x), s) \mapsto l_g(\mu(\psi_g(x), s)).$$

For appropriate $g, h \in \mathcal{G}$

$$\begin{aligned}
(\phi_h \circ \phi_g^{-1})(x, s) &= \phi_h(l_g(\mu(\psi_g(x), s))) \\
&= \phi_h(l_h((h^{-1}g)\mu(\psi_g(x), s))) \\
&= \phi_h(l_h((h^{-1}g)\psi_g(x)s)) \\
&= \phi_h(l_h((\psi_h \circ \psi_g^{-1})(\psi_g(x)s))) \\
&= \phi_h(l_h(\psi_h(x)s)) \\
&= \phi_h(l_h(\mu(\psi_h(x), s))) \\
&= \phi_h(\phi_h^{-1}(x, s)) \\
&= (x, s).
\end{aligned}$$

Thus two "adjacent" local trivializing maps agree and one has a principal bundle structure on $\mathcal{G} \to \mathcal{G}/\mathcal{S}$. □

7.4 A Super Version of Lie's Third Theorem

We have seen that a super Lie group induces a super Lie algebra of left invariant vector fields. A natural question to ask is if we are given an abstract super Lie algebra can we find a super Lie group which induces an isomorphic copy of the give super Lie algebra in its left invariant vector fields ? The answer to this question in the non-super case is affirmative and the result is known as Lie's Third Theorem. In the super case other authors have addressed this question, but their proofs and assumptions differ from those given in this section which is based on [37].

We begin with a few technical preliminaries concerning the supersmoothness of flows of vector fields then in the second part of the section we state and prove Lie's Third Theorem for the G^∞ category.

7.4.1 Technical Preliminaries

Our next result requires us to show that if one has a supersmooth (G^∞) vector field on the even part of a super Lie algebra and if this vector field depends supersmoothly on a parameter then the solution depends supersmoothly on both the parameter and the initial condition.

Consider then a Banach super Lie algebra $\mathfrak{g}$ and a function $F : \mathfrak{g}^0 \times \mathfrak{g}^0 \to \mathfrak{g}^0$ which we interpret as a parametrized vector field on $\mathfrak{g}^0$. What does it mean to say F is a G^∞ function? We choose a basis of $\mathfrak{g}$ and identify $\mathfrak{g}^0$ with $\mathbb{K}^{p|q}$ via the obvious globally defined chart. We actually choose two copies of the same chart but denote the components of the first by $(u_1, u_2, \cdots, u^{p+q})$ and its copy by $(v^1, v^2, \cdots, v^{p+q})$. So coordinates on $\mathfrak{g}^0 \times \mathfrak{g}^0$ will be denoted by $(u^1, u^2, \cdots, u^{p+q}, v^1, v^2, \cdots, v^{p+q})$ although strictly speaking these should be reordered so that all even coordinates come first in the $2(p+q)$-tuple and the odd coordinates last so that the chart has its values in $\mathbb{K}^{2p|2q}$. Throughout this section, E will denote the Banach space $\mathfrak{g}^0 \times \mathfrak{g}^0$ with the norm defined below. Thus E is a G^∞ manifold with a single global chart. Now F is a G^∞ function iff all its component functions are. In that which follows we will assume F is of class G^∞ in which case it is necessarily of class C^∞ on the Banach space E.

If $\mathfrak{g}$ denotes such a Banach super Lie algebra let $E = \mathfrak{g}^0 \times \mathfrak{g}^0$ denote the Banach space with norm defined by $||(X,Y)||_E = max\{||X||_{\mathfrak{g}^0}, ||Y||_{\mathfrak{g}^0}\}$ for $(X,Y) \in E$. We write $B_r(0)$ to denote $\{X \in \mathfrak{g}^0 | \quad ||X||_{\mathfrak{g}^0} < r\}$ and $B_r^E(0)$ for $\{(X,Y) \in E | \quad ||(X,Y)|| < r\}$. We also drop the subscripts on both $||\cdot||_E$ and $||\cdot||_{\mathfrak{g}^0}$ below since it should be obvious from the context which norm is intended.

Lemma 7.4.1. *Let $F : \mathfrak{g}^0 \times \mathfrak{g}^0 \to \mathfrak{g}^0$ be a class G^∞ mapping such that $F(0,0) = 0$, and such that for some positive number M and each positive number r, $||(d_2F)_u|| \leq rMe^{rM}$ for all $u \in B_r^E(0)$. Then, for some $r > 0$, there exists a unique mapping $f : [0,1] \times B_r^E(0) \to \mathfrak{g}^0$ such that*

1. *for each $t \in [0,1]$ the mapping from E into $\mathfrak{g}^0$ defined by $u \to f(t,u)$ is a class G^∞ mapping and*
2. *$\frac{df}{dt}(t,X,Y) = F(X, f(t,X,Y))$ and $f(0,X,Y) = Y$ for all $(X,Y) \in B_r^E(0)$.*

Proof. First we show that there exists a C^∞ mapping f which satisfies condition (2) of the lemma. For this purpose consider the mapping $\tilde{F} : E \to E$ defined by $\tilde{F}(X,Y) = (0, F(X,Y))$. Then $\tilde{F}$ is a smooth vector field on E such that $\tilde{F}(0,0) = 0$ and by Corollary 4.1.25 of [2] there exists $r > 0$ such that whenever $u \in E$ and $||u|| < r$ there exists an integral curve of $\tilde{F}$ through u which is defined on $[-1,1]$. Since there exists a flow box of $\tilde{F}$ at $(0,0)$ on E it follows that, for some $r > 0$, there exists a smooth function $\tilde{f} : [-1,1] \times B_r^E(0) \to E$ such that for $(t,u) \in [-1,1] \times B_r^E(0)$

$$\frac{d\tilde{f}}{dt}(t,u) = \tilde{F}(\tilde{f}(t,u)) \qquad \tilde{f}(0,u) = u.$$

Since, for $u = (X, Y) \in B_r^E(0)$, $\tilde{F}(\tilde{f}(t,u)) = (0, F(\tilde{f}(t,u)))$ and $\frac{d\tilde{f}}{dt} = (\frac{d\tilde{f}_1}{dt}, \frac{d\tilde{f}_2}{dt})$ it follows that $\frac{d\tilde{f}_1}{dt} = 0$ and $\frac{d\tilde{f}_2}{dt} = F(\tilde{f}(t,u)) = F(X, \frac{d\tilde{f}_2}{dt}(t,u))$. Consequently, $f \equiv \tilde{f}_2$ is a smooth mapping from $[0,1] \times B_r^E(0)$ into $\mathfrak{g}^0$ such that

$$\frac{df}{dt}(t, X, Y) = F(X, f(t, X, Y)) \quad \textit{and} \quad f(0, X, Y) = Y$$

for all $(X, Y) \in B_r^E(0)$.

We must now show that (1) of the lemma holds. To do this we require an explicit formula which shows how the derivatives of the function $(X, Y) \to f(t, X, Y)$ depend on X and Y for each fixed $t \in [0, 1]$.

Let $\mathcal{F}$ denote the Banach space of all continuous maps g from $I = [0, 1]$ into $\mathfrak{g}^0$ equipped with the sup-norm:

$$|| \ g \ || = lub_{t \in I} |g(t)|.$$

It is our intent to show that the mapping $h : B_r^E(0) \to \mathcal{F}$ defined by $h(u)(t) = f(t, u)$ for $u \in B_r^E(0), t \in [0, 1]$ is of class G^∞, that is, we will show that the mapping from $B_r^E(0)$ to $\mathfrak{g}^0$ defined by $u \to h(u)(t)$ is of class G^∞ for each $t \in [0, 1]$. It will then follow that the solution of our differential equation is a G^∞ function of (X, Y) where $X \in \mathfrak{g}^0$ is a parameter and Y is an initial condition of the differential equation. To avoid excessive language we simply say h is of class G^∞ in this situation.

Notice that $\mathfrak{g} \times \mathfrak{g}$ is a Banach super vector space such that $(\mathfrak{g} \times \mathfrak{g})^0 = \mathfrak{g}^0 \times \mathfrak{g}^0 = E$ and $(\mathfrak{g} \times \mathfrak{g})^1 = \mathfrak{g}^1 \times \mathfrak{g}^1$. A function such as $h : B_r^E(0) \to \mathcal{F}$ is of class G^∞ iff the function $h_t : E \to \mathfrak{g}^0$ defined by $h_t(w) = h(w)(t)$ is of class G^∞ for each t and by Proposition 6.7.4 this is true iff it is of class C^∞ and the derivatives $d_w h_t^B$ of the components h_t^B are Λ-linear. Here $d_w h_t^B$ is a mapping from $T_w E = E$ into Λ. Since this is a condition on the components h_t^B of h_t we may write $d_w h_t = d_w h_t^B e_B$ and think of it as a $\mathfrak{g}^0$-valued function. Indeed, $h_t = h_t^B e_B$ where $\{e_B\}$ is a basis of $\mathfrak{g}$ (not $\mathfrak{g}^0$) and so $h_t^1, h_t^2, \cdots h_t^p$ are even functions while $h_t^{p+1}, h_t^{p+2}, \cdots, h_t^{p+q}$ are odd. Thus $d_w h_t^B$ maps into Λ^0 for $1 \leq B \leq p$ and maps into Λ^1 for $p+1 \leq B \leq p+q$ from which it follows that $d_w h_t = d_w h_t^B e_B$ is $\mathfrak{g}^0$-valued.
Define $K : E \times \mathcal{F} \to \mathcal{F}$ by

$$K((X, Y), g)(t) = Y + \int_0^t F(X, g(s)) ds$$

for $(X, Y) \in E$. Notice that $K((X, Y), g) = g$ iff

$$\frac{dg}{dt} = F(X, g(t)) \qquad \text{and} \qquad g(0) = K((X, Y), g)(0) = Y.$$

If f is the smooth solution of the vector field F obtained above and $h : B_r^E(0) \to \mathcal{F}$ is defined by $h(u)(t) = f(t,u)$ for $t \in [0,1], u \in B_r^E(0)$, then h is smooth (since solutions depend smoothly on parameters and initial conditions) and

$$K(u, h(u)) = h(u) \qquad \forall\ u \in B_r^E(0) \subset E$$

Thus if $H(u,f) \equiv f - K(u,f)$, then

$$H(u, h(u)) \equiv 0$$

and for $\lambda \to u_\lambda$ a curve through u in $B_r^E(0)$ and $\delta = \frac{d}{d\lambda}(u_\lambda)|_{\lambda=0}$ such that

$$H(u_\lambda, h(u_\lambda)) = 0$$

we have

$$(d_1H)_{(u,h(u))}(\delta) + (d_2H)_{(u,H(u))}((dh)_u(\delta)) = 0.$$

If we can show that $r > 0$ can be chosen small enough so that $(d_2H)_{(u,h(u))} : \mathcal{F} \to \mathcal{F}$ has an inverse for all $(u, h(u))$, then it will follow that

$$(d_2H)((dh)_u(\delta)) = -(d_1H)(\delta)$$

and that

$$\boxed{(dh)_u(\delta) = -(d_2H)^{-1}_{(u,h(u))}((d_1H)_{(u,h(u))}(\delta).} \tag{7.10}$$

This explicit formula for dh will enable us to show that h is of class G^1 and hence by Theorem 6.6.5 that it is of class G^∞. In order to obtain the required $r > 0$ first notice that d_2H can be written in terms of d_2K which, in turn, can be written in terms of d_2F. Indeed, if $\lambda \to f_\lambda$ is a curve in $\mathcal{F}$ through $f \in \mathcal{F}$, then

$$\begin{aligned}(d_2H)_{(u,f)}\big(\tfrac{d}{d\lambda}(f_\lambda)|_{\lambda=0}\big) &= \tfrac{d}{d\lambda}(H(u,f_\lambda))|_{\lambda=0} \\ &= \tfrac{d}{d\lambda}(f_\lambda - K(u,f_\lambda))|_{\lambda=0} \\ &= \tfrac{d}{d\lambda}(f_\lambda)|_{\lambda=0} - (d_2K)_{(u,f)}(\tfrac{d}{d\lambda}(f_\lambda)|_{\lambda=0})\end{aligned}$$

and denoting $\delta_f \equiv \frac{d}{d\lambda}(f_\lambda)|_{\lambda=0}$, we have

$$\boxed{(d_2H)_{(u,f)}(\delta_f) = \delta_f - (d_2K)_{(u,f)}(\delta_f)} \tag{7.11}$$

It follows that $(d_2H)_{(u,f)} = I_{\mathcal{F}} - (d_2K)_{(u,f)}$ as operators and so if the operator norm of $(d_2K)_{(u,f)}$ is smaller than 1, then $(d_2H)_{(u,f)}$ will be invertible. But we also know

that $K((X,Y),f_\lambda)(t) = Y + \int_0^t F(X, f_\lambda(s))ds$ so that

$$\begin{aligned}(d_2K)_{((X,Y),f)}(\delta_f)(t) &= \tfrac{d}{d\lambda}(K((X,Y),f_\lambda)(t)) \\ &= \textstyle\int_0^t \frac{d}{d\lambda}(F(X,f_\lambda(s))ds \\ &= \textstyle\int_0^t (d_2F)_{(X,f(s))}(\delta_f(s))ds.\end{aligned}$$

Note that,

$$||(d_2K)_{(u,f)}(\delta_f)|| \leq \int_0^1 ||(d_2F)_{(X,f(s))}|| \quad ||\delta_f(s)||ds \leq rMe^{rM}||\delta_f||$$

and $||(d_2K)_{(u,f)}|| \leq rMe^{rM} < 1$ for appropriately chosen $r > 0$. Now let $w_\lambda = (X_\lambda, Y_\lambda) \in B_r^E(0)$ be a curve through $w = (X,Y)$ and $\delta = \frac{d}{d\lambda}(w_\lambda)|_{\lambda=0} = (\delta_1, \delta_2) \in E = \mathfrak{g}^0 \times \mathfrak{g}^0$, then

$$H(w_\lambda, f)(t) = f(t) - K(w_\lambda, f)(t) = f(t) - Y_\lambda - \int_0^t F(X_\lambda, f(s))ds$$

and $(d_1H)_{(w,f)}(\delta)(t) = -\delta_2 - \int_0^t (d_1F)_{(X,f(s))}(\delta_1)ds$. Let $\hat{\delta}(t) = (d_1H)_{(w,h(w))}(\delta)(t)$, then

$$\hat{\delta}(t) = -\delta_2^B \frac{\partial}{\partial v^B} - \delta_1^A \int_0^t (d_1F)_{(X,h(w)(s))}(\frac{\partial}{\partial u^B})ds.$$

It follows that eqn.[7.10] becomes $(dh)_w(\delta) =$

$$\begin{aligned}&= -(d_2H)^{-1}((d_1H)_{(w,h(w))}(\delta)) \\ &= (d_2H)^{-1}(\hat{\delta}) \\ &= [I_{\mathcal{F}} - (d_2K)_{(w,h(w))}]^{-1}(\hat{\delta}) \qquad \text{by eqn.[7.11]} \\ &= [I_{\mathcal{F}} + (d_2K)_{(w,h(w))} + (d_2K)_{(w,h(w))} \circ (d_2K)_{(w,h(w))} + \cdots](\hat{\delta}) \\ &= \hat{\delta} + (d_2K)_{(w,h(w))}(\hat{\delta}) + (d_2K)_{(w,h(w))}(d_2K_{(w,h(w))}(\hat{\delta})) + \cdots + (d_2K)^l_{(w,h(w))}(\hat{\delta}) + \cdots\end{aligned} \tag{7.12}$$

We now show that h is of class G^1. For $\hat{\delta}$ as defined above,

$$\begin{aligned}(d_2K)_{(w,h(w))}(\hat{\delta})(t) &= \textstyle\int_0^t (d_2F)_{(w,h(w)(s))}(-\delta_2^B \frac{\partial}{\partial v^B} - \delta_1^A \int_0^s (d_1F)_{(X,h(w)(r))}(\frac{\partial}{\partial u^B})dr)ds \\ &= \delta_2^B \gamma_B^v(w)(t) + \delta_1^A \gamma_A^u(w)(t)\end{aligned} \tag{7.13}$$

where

$$\gamma_A^u(w)(t) = -\int_0^t (d_2F)_{(w,h(w)(s))}(\int_0^s (d_1F)_{(X,h(r))}(\frac{\partial}{\partial u^B})dr)ds \tag{7.14}$$

and

$$\gamma_B^v(w)(t) = -\int_0^t (d_2F)_{(w,h(w)(s))}(\frac{\partial}{\partial v^B})ds. \tag{7.15}$$

Write $(d_2K)_{(w,h(w))}(\hat{\delta})(t) = \sum_C \delta^C \gamma_C(w)(t)$ where the components δ^C include all the components of both δ_1 and δ_2 and the γ_C include both types of indexed functions γ_A^u and γ_B^v. We have

$$\begin{aligned}[(d_2K)_{(w,h(w))}]^l(\hat{\delta}) &= [(d_2K)_{(w,h(w))}]^{l-1}(\textstyle\sum_B \delta^C \gamma_C(w)) \\ &= \textstyle\sum_C \delta^C [(d_2K)_{(w,h(w))}]^{l-1}(\gamma_C(w)) \\ &= \textstyle\sum_C \delta^C \hat{b}_C^l(w)\end{aligned} \tag{7.16}$$

for some set of continuous functions $\hat{b}_C^l$ from $B_r^E(0)$ to $\mathcal{F}$. It follows that if $w \in B_r^E(0)$ and $\delta = (\delta_1, \delta_2) \in E = \mathfrak{g}^0 \times \mathfrak{g}^0$ then,

$$d_w h_t(\delta) = (dh)_w(\delta)(t) = \sum_C \delta^C \sum_{l=0}^{\infty} \hat{b}_B^{lv}(w)(t) \tag{7.17}$$

and by Theorem 6.6.3, h is of class G^1. Moreover since h is C^∞ by construction, by Theorem 6.6.5 it is G^∞.

□

Corollary 7.4.2. *Let $F : \mathfrak{g}^0 \times \mathfrak{g}^0 \to \mathfrak{g}^0$ be defined by*

$$F(X, Z) = X + \sum_{k=1}^{\infty} \frac{B_k}{k!} ad_Z^k(X)$$

where $B_0, B_1, B_2, \cdots$ are the Bernoulli numbers. Then there exists a positive number r and a function $W : [0,1] \times B_r^E(0) \to \mathfrak{g}^0$ such that

1. *for each $t \in [0,1]$ the mapping from $B_r^E(0)$ into $\mathfrak{g}^0$ defined by $u \to W(t,u)$ is a class G^∞ mapping and*
2. *$\frac{dW}{dt}(t, X, Y) = F(X, W(t, X, Y))$ and $W(0, X, Y) = Y$ for all $(X, Y) \in B_r^E(0)$.*

Proof. Given F as defined above, observe that it follows from Lemma 7.3.7 and the second half of the proof of Corollary 7.3.9 that F is of class G^∞ since

$$F(X, Z) = \sum_{k=1}^{\infty} F_k(X, Z) \tag{7.18}$$

where

$$F_k(X, Z) = ad_Z^k(X) \tag{7.19}$$

is Λ-linear in X and is the diagonal of a Λ-multi-linear mapping in Z up to signs. (Here $(z_1, z_2, \ldots, z_k) \mapsto [z_1, [z_2, [\cdots, [z_k, X], \cdots]$ is multi-linear up to signs in the z_i over Λ for each $i = 1, 2, \ldots k$ and $ad_Z^k(X)$ is the diagonal of this map in the z

variables.) Also, notice that for $X, Z \in B_r(0)$,

$$\begin{aligned} ||(d_2F)_{(X,Z)}(H)|| &\leq \sum_{k=1}^{\infty} k\frac{|B_k|}{k!}M^k||Z||^{k-1}||X||||H|| \\ &\leq rM\sum_{k=1}^{\infty}\frac{1}{(k-1)!}(rM)^{k-1}||H|| \\ &= rMe^{rM}||H||. \end{aligned} \tag{7.20}$$

Since $F(0,0) = 0$ it follows from Lemma 7.4.1that there exists a function $W : [0,1] \times B_r(0) \times B_r(0) \to \mathfrak{g}^0$ which satisfies (1) and (2) of Lemma 7.4.1. The corollary follows. □

7.4.2 Statement and Proof of Lie's Third Theorem

We mention that the question of enlargeability has been studied by Teofilatto for the superanalytic category in [114].

Theorem 7.4.3. *Assume that $\mathfrak{g}$ is a Banach super Lie algebra of finite graded dimension such that*

1. *$\mathfrak{g}^0$ is enlargible with Lie group the Banach Lie group G, and*
2. *for all $g \in G$, $Ad_g : \mathfrak{g}^0 \to \mathfrak{g}^0$ is ${}^0\Lambda$-linear.*

Then there exists a G^∞-atlas on G such that the corresponding supermanifold $\mathcal{G}$ is a super Lie group with respect to the group operation on G. Moreover the even factor $\mathcal{L}(\mathcal{G})^0$ of the super Lie algebra of left-invariant vector fields on $\mathcal{G}$ is Lie algebra isomorphic to $\mathfrak{g}^0$ and consequently the super Lie algebra $\mathcal{L}(\mathcal{G})$ is isomorphic to $\mathfrak{g}$.

Proof. Using our lemma and corollary, the proof follows that of Duistermaat and Kolk [40]. Let

$$\mathfrak{g}_e^0 = \{X \in \mathfrak{g}^0 \mid \frac{e^{ad_X} - I}{ad_X} = \sum_{k=0}^{\infty}\frac{1}{(k+1)!}ad_X^k \text{ has an inverse } \}$$

If $\eta : \mathfrak{g}_e^0 \to \mathfrak{g}^0$ is defined by $\eta(X) = \frac{e^{ad_X}-I}{ad_X}$ then the inverse of $\eta(X)$ is given by $\zeta(X)$ where ζ is the mapping from $\mathfrak{g}_e^0$ to $\mathfrak{g}^0$ defined by

$$\zeta(X) = \frac{ad_X}{e^{ad_X} - I} = \sum_{k=0}^{\infty}\frac{B_k}{k!}ad_X^k$$

where $B_0, B_1, B_2, \ldots$ are the Bernoulli numbers (see [40]). We know that $X \mapsto ad_X$ is a G^∞ mapping from $\mathfrak{g}^0$ to $End(\mathfrak{g}^0)$, moreover we also know that the mappings from $End(\mathfrak{g}^0)$ to $End(\mathfrak{g}^0)$ defined by

$$A \mapsto \sum_{k=0}^{\infty}\frac{1}{(k+1)!}A^k \qquad \text{and} \qquad A \mapsto \sum_{k=0}^{\infty}\frac{B_k}{k!}A^k$$

are G^∞-mappings (by Lemma 7.3.7 and the proof of the second half of Corollary 7.3.9 Define a mapping $F : \mathfrak{g}_e^0 \times \mathfrak{g}_e^0 \to \mathfrak{g}^0$ by

$$F(X,Z) = \zeta(Z)(X) = \sum_{k=0}^{\infty} \frac{B_k}{k!} ad_Z^k(X).$$

It follows from the last corollary that there exists a function $W : [0,1] \times B_r(0) \times B_r(0) \to \mathfrak{g}^0$ such that

1. for each $t \in [0,1]$ the mapping from $B_r(0) \times B_r(0)$ into $\mathfrak{g}^0$ defined by $(X,Y) \to W(t,X,Y)$ is a class G^∞ mapping and
2. $\frac{dW}{dt}(t,X,Y) = F(X, W(t,X,Y))$ and $W(0,X,Y) = Y$ for all $(X,Y) \in B_r^E(0)$.

Thus if we define $\mu : \mathfrak{g}^0 \times B_r(0) \to \mathfrak{g}^0$ by

$$\mu(X,Y) = W(1,X,Y),$$

then μ is a class G^∞-mapping. It follows from the argument of the proof of Theorem 1.6.1 of Duistermaat and Kolk [40] that

$$exp(\mu(X,Y)) = exp(X)exp(Y)$$

for all $X, Y \in B_r(0) \subset \mathfrak{g}_e^0$. Now we know $exp : \mathfrak{g}^0 \to G$ is a C^∞-diffeomorphism on a small ball about $0 \in \mathfrak{g}^0$ (here G is the Banach Lie group having $\mathfrak{g}^0$ as its Lie algebra). For each $x \in G$ define $\kappa^x(y) = log(l_{x^{-1}}(y))$ where $log = exp^{-1}$. Then κ^x is a local C^∞ diffeomorphism. Duistermaat and Kolk show that for x, y such that $\kappa^y \circ (\kappa^x)^{-1}$ is defined it follows that

$$(\kappa^y \circ (\kappa^x)^{-1})(X) = Y \qquad \Longleftrightarrow \qquad Y = \mu(\mu(Y_o, -X_o), X)$$

for a choice of X_o, Y_o in $dom(\kappa^x) \cap dom(\kappa^y)$. Thus

$$(\kappa^y \circ (\kappa^x)^{-1})(X) = \mu(\mu(Y_o, -X_o), X)$$

and consequently, $\kappa^y \circ (\kappa^x)^{-1}$ is a G^∞ mapping. It follows that the family of maps $\{\kappa^x\}$ is a G^∞-atlas on G and we denote the resulting supermanifold by $\mathcal{G}$. Following Duistermaat and Kolk once more, let $m : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ be defined by $m(x,y) = xy^{-1}$ where m is just the group operations on G. We show m is of class G^∞. We have

$$\begin{aligned} \big(\kappa^{xy^{-1}} \circ m \circ [(\kappa^x)^{-1} \times (\kappa^y)^{-1}]\big)(X,Y) &= \kappa^{xy^{-1}}\big(m((\kappa^x)^{-1}(X), (\kappa^y)^{-1}(Y))\big) \\ &= \kappa^{xy^{-1}}\big((\kappa^x)^{-1}(X)(\kappa^y)^{-1}(Y))^{-1}\big) \\ &= ad_y(\mu(X,-Y)) \end{aligned}$$

since

$$\begin{aligned} xexp(X)(yexp(Y))^{-1} &= xexp(X)exp(-Y)y^{-1} \\ &= xy^{-1}yexp(\mu(X,-Y))y^{-1} \\ &= xy^{-1}exp(ad_y(\mu(X,-Y))). \end{aligned}$$

Since $ad_y : \mathfrak{g}^0 \to \mathfrak{g}^0$ is a G^∞-mapping for all $y \in G$ we see that

$$\kappa^{xy^{-1}} \circ m \circ [(\kappa^x)^{-1} \times (\kappa^y)^{-1}]$$

is a G^∞ mapping since it is just the map

$$(X, Y) \mapsto ad_y(\mu(X, -Y)).$$

Thus $\mathcal{G}$ is a super Lie group. Finally notice that $\mathcal{L}(\mathcal{G})$ can be identified as a super Lie algebra with $T_e\mathcal{G}$ and consequently $\mathcal{L}(\mathcal{G})^0 = T_e^0\mathcal{G} = T_e\mathcal{B}\mathcal{G} = \mathfrak{g}^0$ as Lie algebras.

Notice, however, that if $\mathfrak{h}$ is a supervector space of dimension (m, n) which supports two brackets $[\cdot,\cdot]$ and $[\cdot,\cdot]^*$ such that $\mathfrak{h}$ is a super Lie algebra with respect to both brackets and if the brackets agree on even vectors then they necessarily agree on odd vectors as well. This observation can be proven by choosing a pure basis $\{e_i, \tilde{e}_\alpha\}$ of $\mathfrak{h}$ and observing that since the brackets agree on even vectors one has that for fixed indices i, α, $[e_i, \lambda\tilde{e}_\alpha] - [e_i, \lambda\tilde{e}_\alpha]^* = 0$ for all odd supernumbers λ. It is easy to show, in the case one has infinitely many generators of Λ (as we do), that $[e_i, \tilde{e}_\alpha] - [e_i, \tilde{e}_\alpha]^* = 0$. Similarly, for fixed indices α, β, $[\lambda e_\beta, \mu\tilde{e}_\alpha] - [\lambda e_\beta, \mu\tilde{e}_\alpha]^* = 0$ for all odd supernumbers λ and μ from which it follows that $[e_\beta, \tilde{e}_\alpha] - [e_\beta, \tilde{e}_\alpha]^* = 0$. Consequently, $[v, w] = [v, w]^*$ for all $v, w \in \mathfrak{h}$. It follows from this observation that $\mathcal{L}(\mathcal{G}) = T_e\mathcal{G}$ as super Lie algebras and the theorem follows. □

Remark 7.4.4. *If $\mathcal{G}$ and $\mathcal{H}$ are super Lie groups and $\phi : \mathcal{G} \to \mathcal{H}$ is a class G^∞ homomorphism then the mapping $d_e\phi$ is a homomorphism from the super Lie algebra $T_e\mathcal{G}$ to the super Lie algebra $T_e\mathcal{H}$ (using their obvious identifications with the super Lie algebras of left invariant vector fields). Moreover the diagram*
is commutative. The proof of this result is almost identical to the proof in the usual Lie group case and is left to the reader. The point is that since $d_e\phi$ is Λ left-linear it is also a class G^∞ mapping so the entire diagram is in the G^∞ category. In particular notice that if ϕ is an injective immersion then this shows that the exponential mapping on $\mathcal{H}$ is simply the restriction of the exponential mapping on $\mathcal{G}$ to $\phi(\mathcal{H})$. This fact makes it possible to make contact with the physicist usual technique for identifying the super Lie groups of matrices of given super Lie algebras of matrices.

7.5 Formal Supergroups

The physics literature sometimes defines supergroups in terms of a formal multiplication. The rules for multiplying group elements are obtained from the Baker-Campbell-

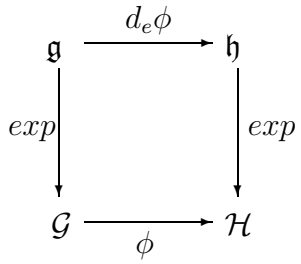

Figure 7.1: Subgroups Induce Subalgebras

Hausdorff formula (see [29]). We examine the formal group calculations in Chapter 9. Apparently this is related to the approach taken by Berezin and Leites [13], Kac [69], and Kostant [76] which are analogous to the formal groups in ordinary Lie theory (see [113] for example). The formal approach assumes a certain algebraic structure as the starting point. We, in contrast, have shown that the exponential function is a G^∞ mapping and can use our geometric results to prove that the relevant algebraic structure is correct.

Let $\mathcal{W} = \mathcal{W}^0 \oplus \mathcal{W}^1$ denote a graded left Λ-module which is finitely and freely generated over Λ. Once for all, select a fixed pure basis of $\mathcal{W}$ of type (p, q) and recall that pure left Λ endomorphisms of $\mathcal{W}$ may be represented by matrices.

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{7.21}$$

where A, B, C, D are respectively, $p \times p, p \times q, q \times p, q \times q$ matrices over Λ which respect the grading. Also recall that M is even iff both A and D have only even entries while B and C have only odd entries. Similarly M is odd iff both A and D have only odd entries while B and C have only even entries.

We denote the set of all matrices M defined above by $gl(\mathcal{W}) = gl(p, q, \Lambda)$ and observe that it is a super Lie algebra with respect to the bracket

$$[M, N] = MN - (-1)^{\epsilon(M)\epsilon(N)} NM. \tag{7.22}$$

Moreover it is a Banach super Lie algebra relative to the norm $||M|| = \sum_{i,j=1}^{p+q} ||M_{ij}||_\Lambda$. Clearly the subspace of even elements $gl^0(\mathcal{W})$ is a Banach Lie algebra whose Lie bracket is induced by the multiplication of the associative Banach algebra structure on $gl^0(\mathcal{W})$. It is well-known that the group of units $Gl(\mathcal{W})$ of this associative Banach algebra is open in the associative algebra. Moreover it is a Banach Lie group whose Lie algebra is precisely the Lie algebra structure on $gl^0(\mathcal{W})$ induced by the associative structure (see [88]).

Clearly $gl^0(\mathcal{W})$ is an enlargible super Banach Lie algebra with Banach Lie group $Gl(\mathcal{W})$ and the adjoint mapping is ${}^0\Lambda$ linear. Therefore, Lie's Third Theorem of G^∞ supergroups tells us that there exists a G^∞ atlas of charts on $Gl(\mathcal{W})$ having values in the even part of the super Lie algebra $gl(\mathcal{W})$. We denote the resulting supermanifold by $Gl^s(\mathcal{W})$. The proof that follows the proposition is more direct.

Proposition 7.5.1. *The super Lie algebra of left invariant vector fields $\mathcal{L}(Gl^s(\mathcal{W}))$ of $Gl^s(\mathcal{W})$ is isomorphic to the super Lie algebra $gl(\mathcal{W})$.*

Proof. We omit most of the details as they closely follow the usual proof that the Lie algebra of $Gl(n,\mathbb{R})$ is $gl(n,\mathbb{R})$. One shows that if B in $gl^0(\mathcal{W})$ is identified with the vector tangent to $Gl^s(\mathcal{W})$ at the identity e, then the left invariant vector field on $Gl^s(\mathcal{W})$ is given by

$$X^B(A) = d_e l_A(B) = \sum_{i,j} (AB)_{ij} \frac{\partial}{\partial x_{ij}}|_A \tag{7.23}$$

for $A \in Gl^s(\mathcal{W})$. Here x_{ij} is a chart on $Gl^s(\mathcal{W})$ where $x_{ij}(M)$ denotes the ij-component of the matrix M; thus x_{ij} is a Λ-valued function on $Gl^s(\mathcal{W})$. Notice that x_{ij} is even for $1 \leq i,j \leq p$ and for $p+1 \leq i,j \leq p+q$ but otherwise is odd. Using the fact that $X^B = \sum_{i,j,k} x_{ik} B_{kj} \frac{\partial}{\partial x_{ij}}$ one can now show that $X^{[M,N]} = [X^M, X^N]$ as in the usual case. The tedious details require careful, but straightforward, considerations of parities; they are left to the reader. □

It should now be clear that the various supergroups of matrices which occur in the physics literature are indeed super Lie groups relative to Rogers' definition of a supermanifold when the supernumbers are infinitely generated. The usual physics treatments of the subject begin with a super Lie algebra of matrices and then define the corresponding super Lie group by a formula which is sometimes tacitly assumed to describe a supermanifold. The Baker-Campbell-Hausdorff formula is then used to link the super Lie algebra to its super Lie group. We have developed the machinery necessary to understand the supermanifold structure of the underlying super Lie group and have shown that the exponential mapping is indeed a G^∞ mapping. These results then justify the physicist's intuition and also show how the super Lie group structure in the matrix case arises from more geometric principles.

Definition 7.5.2. *If $\mathfrak{g}_{Lie}$ is a graded Lie algebra over $\mathbb{C}$, then its Grassmann shell is the super Lie algebra $\widehat{\mathfrak{g}_{Lie}} = \Lambda \otimes \mathfrak{g}_{Lie}$ defined by $[\lambda X, \mu Y] = \lambda\mu(-1)^{\epsilon(\mu)\epsilon(X)}[X,Y]_{\mathfrak{g}_{Lie}}$ for $\lambda, \mu \in \Lambda$ and $X, Y \in \mathfrak{g}_{Lie}$. More generally, one says that a super Lie algebra $\mathfrak{g}$ is a conventional Berezin superalgebra of dimension (p,q) if and only if it possesses a pure basis for which the structure constants have no soul (see Definition 2.4.2).*

Notice that if we choose a basis $\{E_m, \tilde{E}_\alpha\}$, $m = 1,2,\ldots p$ and $\alpha = 1,2,\ldots q$ of a graded Lie algebra $\mathfrak{g}_{Lie}$ over $\mathbb{C}$ where we define E_m to be even and $\tilde{E}_\alpha$ to be odd, then

$$\begin{array}{ccc} \mathfrak{g}_{Lie} & \longrightarrow & gl(r,s,\mathbb{C}) \\ \downarrow & & \downarrow \\ \widehat{\mathfrak{g}_{Lie}} & \longrightarrow & gl(r,s,\Lambda) \end{array}$$

Figure 7.2: Grassmann Shell

it is also a pure basis of the Grassmann shell of $\widehat{\mathfrak{g}_{Lie}}$ and the corresponding structure constants relative to this basis are complex numbers. Thus the Grassmann shell of a graded Lie algebra over $\mathbb{C}$ is a special type of conventional Berezin superalgebra.

Theorem 7.5.3. *Let $\mathfrak{g}_{Lie} = \mathfrak{g}^0_{Lie} \oplus \mathfrak{g}^1_{Lie}$ denote a (p,q) graded Lie algebra over $\mathbb{C}$. Then there exists a super Lie group $\mathcal{H}$ whose super Lie algebra of left invariant vector fields is isomorphic to the Grassmann shell $\widehat{\mathfrak{g}_{Lie}}$ of $\mathfrak{g}_{Lie}$.*

Proof. First apply Ado's Theorem for the case of graded Lie algebras over $\mathbb{C}$ (see [69] page 79). This theorem assures us that there exists an even injective homomorphism $\phi : \mathfrak{g}_{Lie} \hookrightarrow gl(r,s,\mathbb{C})$. We choose $gl(r,s,\Lambda)$ to be the set of all left endomorphisms on a (r,s) supervector space $\mathcal{V}$ and identify these endomorphisms with their corresponding $(r+s)\times(r+s)$ Λ-valued matrices. Now identify $\mathfrak{g}_{Lie}$ with its image in $gl(r,s,\Lambda)$ and choose a basis $\{E_m, \tilde{E}_\alpha\}, 1 \le m \le p, 1 \le \alpha \le q$ of $\mathfrak{g}_{Lie}$. Finally, extend this (p,q) basis to a basis $\{E_m, \tilde{E}_\alpha\}, \quad m = 1,2,\ldots(r+s)^2, \quad \alpha = 1,2,\ldots(r+s)^2$ of $gl(r,s,\mathbb{C})$. The Grassmann shell of $gl(r,s,\mathbb{C})$ is

$$gl(r,s,\Lambda) \equiv \{\sum_{m=1}^{r} \xi^m E_m + \sum_{\alpha=1}^{s} \xi^\alpha \tilde{E}_\alpha \mid \xi^m, \xi^\alpha \in \Lambda\}.$$

Likewise the Grassmann shell of $\mathfrak{g}_{Lie}$, denoted $\widehat{\mathfrak{g}_{Lie}}$, is constructed by replacing complex scalars by Grassmann scalars. Notice that the injective homomorphism naturally extends to the Grassmann shell, thus we injectively embed the Grassmann shell of the graded Lie algebra into matrices having Grassmann supernumbers as entries:

$$\phi : \widehat{\mathfrak{g}_{Lie}} \hookrightarrow gl(r,s,\Lambda).$$

Now we know that $gl(r,s,\Lambda) = gl(\mathcal{V})$ is the Lie super algebra of the super Lie group $G^s(\mathcal{V})$ and that $\widehat{\mathfrak{g}_{Lie}}$ is a sub-super Lie algebra of $gl(\mathcal{V})$. It follows from Theorem 7.3.3 that there is a super Lie group $\mathcal{H}$ which is a sub-super Lie group of $Gl^s(\mathcal{V})$ having $\widehat{\mathfrak{g}_{Lie}}$ as its super Lie algebra of left invariant vector fields. Thus we have the commutative diagram. □

Remark 7.5.4. *While [37] was under review we discovered a number of papers related to our work. For the most part, these are peripheral to our present endeavor but two of these, [25] and [94], are concerned with a "super" version of Lie's third*

theorem. It turns out that both of these papers are formulated in the category of superanalytic manifolds and superanalytic morphisms while our results are formulated in the category of G^∞ supermanifolds with G^∞ morphisms. A cursory examination of this chapter or [37] will reveal that practically all our proofs are concerned with assuring that the various mappings we consider are of class G^∞ and that these proofs are not minor modifications of their superanalytic counterparts.

In addition in [94] it is shown that there is a gap in the proof of [25] regarding the normed structure defined on a certain enveloping algebra. A counterexample is provided which to show that such a norm is not always possible.

Both [25] and [94] are excellent papers and in [94] Pestov shows how to complete the proof in [25]. On the other hand we believe that the impact on our work here is minimal since both [25] and [94] require superanalyticity and utilize sheaf theoretic techniques while it is our intent to provide a framework more closed related to differential-geometric influences as opposed to algebraic-geometric ones. Additionally, Pestov uses nonstandard analysis techniques to complete the proof of the third theorem which we find mildly distracting.

The author is grateful to the referee of [37] for directing our attention to the paper by Jadczyk and Pilch [68]. Their methods have helped streamline certain proofs in this chapter.

Chapter 8

Supergeometry of Super Yang-Mills Theory

Abelian gauge theory has played an important role in physics since the mid nineteenth century due to the fact that electromagnetism can be viewed as a gauge theory with gauge group $U(1)$. This viewpoint which was powerfully argued by Weyl in his fascinating 1929 paper which laid the groundwork for the $SU(2)$ gauge theory of Yang and Mills and also the general nonabelian gauge group theory of Utiyama in the late 1950's. All of this eventually culminated in the electroweak theory of Weinberg and Salam; it gained acceptance after some difficult calculations by t'Hooft which showed their theory had a consistent quantum formulation. During the 1970's it was understood that gauge theory in physics used the mathematics of principle fiber bundles. The inhomogeneous transformation law of the gauge potential matches precisely the transformation law relating two pullbacks of a connection. This understanding has lead to many interesting results in both mathematics and physics, certainly too many to list here. See *The Dawning of Gauge Theory* [92] for a more detailed history, including some of the original papers alluded to above.

Super Yang-Mills theory seeks to implement the usual Yang-Mills theory for a multiplet of fields which simultaneously form a representation of supersymmetry. In short, this means that there must be both bosons and fermions in the model. In contrast, the usual Yang-Mills theory makes no particular restriction on the overall field content of the model except perhaps the existence of the gauge boson. The Wess-Zumino model is an example of a super Yang-Mills theory. The superfield technique of Salam and Strathdee was employed to formulate the theory elegantly and compactly. One finds a good summary of these matters in the introductory text by Wess and Bagger [116].

The mathematics used in Wess and Bagger is internally consistent. However, the arguments are typically local and the existence and/or construction of supercalculus

is not dealt with. The transformation laws are made at the level of superfields and do not obviously follow from a principle fiber bundle construction. Francois Gieres transformed and expanded the arguments of Wess and Bagger and obtained a reformulation of the theory which was much closer to that required for a principle fiber bundle description of the super gauge theory. However, Gieres' arguments were also local, and he admits to ignoring technical issues relating to supermanifold structures. We believe we have taken Gieres work to its logical conclusion. We have constructed a global formulation of super Yang-Mills theory over a supermanifold with relatively weak assumptions. Our formulation reproduces the superfield transformation laws found in Wess and Bagger from a geometric construction very much reminiscent of the principle fiber bundle formulation of Yang-Mills theory. The base manifold becomes a supermanifold, and the gauge group becomes a supergroup, etc... We derive general results that apply to a variety of gauge groups, but super Yang-Mills theory itself has a rather simple gauge group, it is simply the "superization" of an ordinary gauge group.

The geometry of super Yang-Mills theory has been studied by many other authors. We mention only those who influenced us in preparing for this work. Schwarz has a series of papers (for example [105] and [74]) which use a more sheaf-theoretic definition of supermanifolds. Rosly's paper [104], which also was connected to the work of Schwarz, certainly inspired us in no small way. Also the paper by E.A. Ivanov [65] has played an important motivating role in our work. In particular, the presentation of Ivanov's work in [44] stimulated our intuition. Although we will not make an effort to relate our work to twistor methods, we should mention that there are many papers that followed from Witten's the 1978 paper on the subject [117] (see [52] for a mathematical survey). Additionally, other authors have focused on the relation of superspace to real physical spacetime. We on the other hand have basically not faced the difficulty of finding a well-defined body of the supermanifold. Probably a more physical treatment would form some sort of merger of the ideas introduced in this chapter and those given in [51].

8.1 Special Sections of a Super Principle Fiber Bundle

In this chapter we are forced to change the topology on our basic space of supernumbers. Throughout this chapter Λ will denote the algebra of supernumbers $z = \sum_J z_J \zeta^J$ (the sum extends over all multi-indices) such that $\sum_J |z_J|^2$ converges. The norm of z throughout this chapter will be defined by $||z|| = \sqrt{\sum_J |z_J|^2}$. Additionally, all supermanifolds will be defined by atlases whose charts take their values in Banach spaces which are coordinated supervector spaces modeled on $\mathbb{R}^{p|q}$, for some

p, q with norm $||z|| = \sqrt{\sum_J |z_J|^2}$. This change turns out to be necessary in order that certain sections of a principal bundle, called special sections by Lichernowicz in [81], be smooth maps. This issue is dealt with in the first proposition in this chapter below.

Let $\mathcal{M}$ be a $(p|q)$ dimensional supermanifold and let $\tau : \mathcal{P} \to \mathcal{M}$ be a super principle fiber bundle with group a super Lie group $\mathcal{G}$. Moreover, suppose ω is a connection on $\mathcal{P}$ and Ω its curvature. We assume that ω is even, i.e., we assume that $\omega(T^i\mathcal{P}) \subseteq \mathfrak{g}^i$, where $\mathfrak{g}$ is the Lie superalgebra of $\mathcal{G}$ and $i = 0, 1$. For each point $u \in \mathcal{P}$ we denote the space of horizontal vectors by H_u and $T_u\mathcal{P} = H_u \oplus V_u$.

Recall that if $\gamma : I \to \mathcal{M}$ is a path in $\mathcal{M}$ and $z_o \in \tau^{-1}(\gamma(0))$, then there exists a unique path $\tilde{\gamma} : I \to \mathcal{P}$ such that $\tilde{\gamma}(0) = z_o$, $\tau \circ \tilde{\gamma} = \gamma$ and $\omega(\dot{\tilde{\gamma}}(t)) = 0$ for all $t \in I$. The path $\tilde{\gamma}$ is called the horizontal lift of γ to z_o since $\dot{\tilde{\gamma}}(t)$ is a horizontal vector for each $t \in I$.

Definition 8.1.1. *Assume that $\mathfrak{s} : U \to \mathcal{P}$ is a local section of τ and that $\gamma : I \to U$ is a path in U. The standard argument that one finds in the literature (see, for example [86]) shows that even in the supercategory there exists a unique mapping $g : I \to U$ such that $\tilde{\gamma}(t) = \mathfrak{s}(\gamma(t))g(t)$ for each $t \in I$. The curve g satisfies the equation*

$$\dot{g}(t)g(t)^{-1} = -(\mathfrak{s}^*\omega)(\dot{\gamma}(t))$$

for all $t \in I$ and is called the development of γ relative to the local section $\mathfrak{s}$. Notice that $\dot{g}(t)$ is necessarily an even tangent vector for each $t \in I$.

Observation 8.1.2. *If we consider a principal fiber bundle with a structure group which acts on the left (we assume a right action in the first part of this chapter), then the development g would instead satisfy $g(t)^{-1}\dot{g}(t) = -(\mathfrak{s}^*\omega)(\dot{\gamma}(t))$.*

Given $x_o \in \mathcal{M}$ and $z_o \in \mathcal{P}$ such that $\tau(z_o) = x_o$ we follow Lichernowicz [81] who constructs a local section $\mathfrak{s}$ of τ as follows. First choose a chart $\mathcal{X}$ at x_o defined on an open set U containing x_o such that $\mathcal{X}(x_o) = 0$ and $\mathcal{X}(U)$ is an open ball centered at 0 in $\mathbb{R}^{p|q}$ of radius 1. We refer to the curves $\{\ \mathcal{X}^{-1}(tu) \mid 0 \leq t \leq 1\ \}$ as "rays" in U. The section $\mathfrak{s}$ is obtained by horizontally lifting each of these rays to z_o. Thus $\mathfrak{s}(U)$ is the union of the horizontal lifts of the paths $t \mapsto \mathcal{X}^{-1}(tu)$ to z_o. A section arising in this manner is called a special section at z_o by Lichernowicz, and we will use this terminology as well.

Now one might rightly question whether such special sections exist which are G^∞ mappings, and in fact Lichernowicz is not clear in [81] as to why such sections are even differentiable. We sketch an argument in the following remark which shows that G^∞ special local sections exist .

Proposition 8.1.3. *Each special local section $\mathfrak{s}$ defined in a deleted neighborhood of an arbitrary point m_0 of $\mathcal{M}$ is of class G^∞*

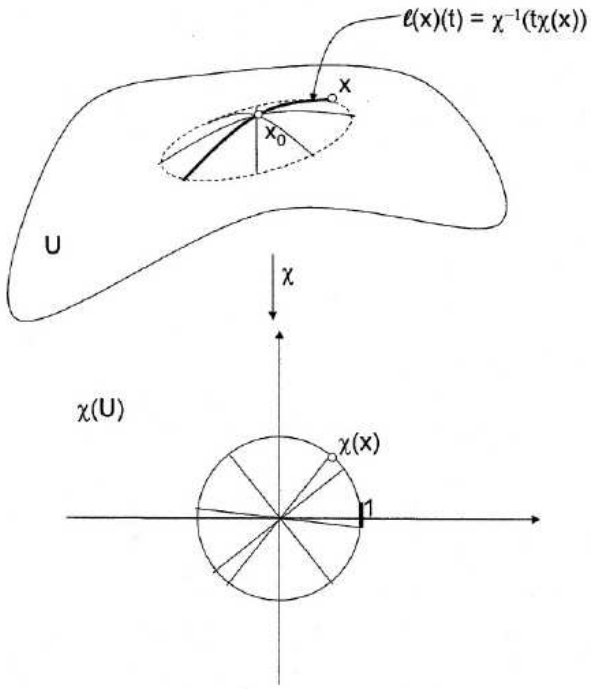

Figure 8.1: Special Chart Domain

Proof. Choose a local section $\hat{s} : U \to \mathcal{P}$ of τ such that $\hat{s}$ is a G^∞ mapping and $m_0 \in U$. We can choose such a local section since $\tau : \mathcal{P} \to \mathcal{M}$ is locally trivial. We may choose U small enough so that it is the domain of a chart $\mathcal{X} : U \to \mathcal{X}(U) \subseteq \mathbb{R}^{p|q}$. Moreover the chart may be chosen such that $\{z \in \mathbb{R}^{p|q} | \ ||z|| \leq 1\} \subseteq \mathcal{X}(U)$ and $\mathcal{X}(m_0) = 0$. Here $||z|| = \sqrt{\sum_{j,J} |z^j_J|^2}$ where $z^j = \sum_J z^j_J \zeta^J$ and for each j and J the Grassmann coefficient z^j_J is either real or pure imaginary. See Section 2.10 for details on why the Grassmann coefficients must be either real or pure imaginary in this case. Let $\gamma : I \times S^1 \to B_1$ be defined by $\gamma(t, z) = tz$, where $I = (0, 1), S^1 = \{z \in \mathbb{R}^{p|q} | \ ||z|| = 1\}$ and $B_1 = \{z \in \mathbb{R}^{p|q} \mid ||z|| < 1\}$. Notice that S^1 is a submanifold of the Banach manifold $\mathbb{R}^{p|q}$. This may be understood by realizing that as a Banach space $\mathbb{R}^{p|q}$ is essentially the space of all square summable sequences (often denoted l_2) and S^1 is a "sphere" in this space. It is not difficult to write explicit formulas for charts covering it whose transition functions are C^∞ maps defined on open subsets of $\mathbb{R}^{p|q} \sim l_2$. Now γ is a C^∞ mapping on $I \times S^1$ with a C^∞ inverse. Indeed its inverse is the mapping from $B_1^* = \{z | \ 0 < ||z|| < 1\}$ onto $I \times S^1$ defined by $z \to (||z||, z/||z||)$. It is easy to show that the mapping $z \to ||z||$ is a C^∞ mapping on $0 < ||z|| < 1$ directly as it is the square root of a "polynomial".

With these preliminaries out of the way define $u : I \times \mathcal{X}(U) \to \mathfrak{g}^0$ by

$$u(t, z) = \omega(d(\hat{s} \circ \mathcal{X}^{-1})(\dot{\gamma}(t, z))). \tag{8.1}$$

Now $\mathcal{X}(U)$ is open in the Banach space $\mathbb{R}^{p|q}$ and $u(I \times \mathcal{X}(U))$ is contained in the Banach space $\mathfrak{g}^0$. Thus for each $z \in \mathcal{X}(U_0)$, there is a unique solution $g : I \times \mathcal{X}(U)) \to \mathcal{G}$ of the initial value problem $g(0, z) = e, \quad \frac{d}{dt} g(t, z) = u(t, z)g(t, z)$. Since u depends smoothly on z, so does g (see [73] for a good treatment of these ideas). Consequently g is a C^∞ mapping. Now the special section $\mathfrak{s}$ is given by $\mathfrak{s}(\mathcal{X}^{-1}(\gamma(t, z)) = \hat{s}(\mathcal{X}^{-1}(\gamma(t, z))g(t, z)$. It follows that for $m \in \mathcal{X}^{-1}(B_1^*), \mathfrak{s}(m) = \hat{s}(m)g(\gamma^{-1}(\mathcal{X}(m))$ and consequently $\mathfrak{s}$ is a C^∞ mapping from $\mathcal{X}^{-1}(B_1^*)$ into $\mathcal{P}$.

Observe that $S = \mathfrak{s}(\mathcal{X}^{-1}(B_1^*))$ is a sub-supermanifold of $\mathcal{P}$. The charts are $\tilde{\mathcal{X}} \circ \tau_S$ where $\tilde{\mathcal{X}}$ is a chart in the G^∞ atlas defined on the open subset $\mathcal{X}^{-1}(B_1^*)$ of $\mathcal{M}$ and τ_S is the restriction of τ to S. Two such charts are G^∞ related, and the inclusion mapping is a G^∞ mapping relative to this choice of atlas.

Finally, to see that $\mathfrak{s}$ is a G^∞ mapping it suffices to show that it is of class G^1 (see [68]). Thus we have only to show that $d\mathfrak{s}$ is linear over Λ^0. Let $\alpha \in {}^0\Lambda$ and let X be a tangent vector of $\mathcal{M}$ tangent to a point $m \in \mathcal{X}^{-1}(B_1^*)$, then $d\tau(d\mathfrak{s}(\alpha X)) = \alpha X = d\tau(\alpha d\mathfrak{s}(X))$. Since τ is invertible on $\mathfrak{s}(\mathcal{X}^{-1}(B_1^*))$, $d\tau$ is invertible on tangents to $\mathfrak{s}(\mathcal{X}^{-1}(B_1^*))$ and consequently $d\mathfrak{s}(\alpha X) = \alpha d\mathfrak{s}(X)$. The proposition follows. □

Once again, following Lichernowicz [81], we show that if $x \in U$ and $v \in T^0_x U$ where

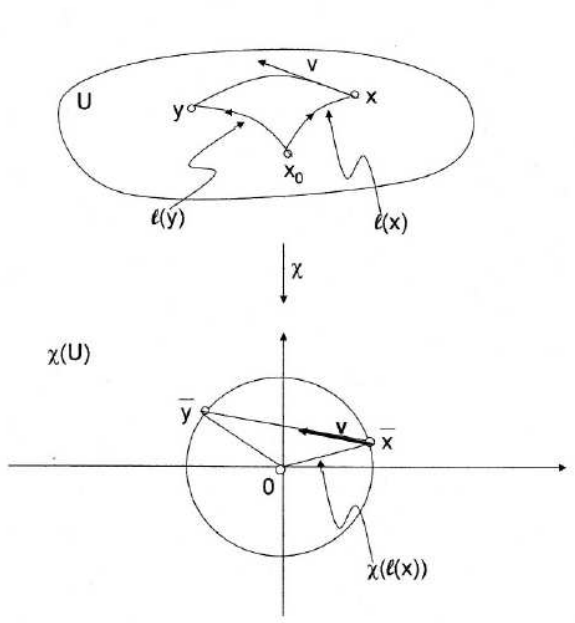

Figure 8.2: Surface in Chart Domain

U is the domain of a special section $\mathfrak{s}$ at $x_o \in U$, then $(\mathfrak{s}^*\omega)_x(v)$ can be written as an integral

$$\int_0^1 (\mathfrak{s}_x^*\Omega)\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right)(s,1)\, ds \tag{8.2}$$

where s, t are parameters of a surface in U determined by x. Actually we only need such a formula when v is transversal to the tangent to a ray in U so we can derive it only in this case. Let $x \in U \setminus \{x_o\}$ and assume that $v \in T_x^0 U$ is a vector such that v is not tangent to the curve $l(x)$ defined by $l(x)(t) = \mathcal{X}^{-1}(t\mathcal{X}(x))$, $0 \leq t \leq 1$. Since we choose $\mathcal{X}(U)$ to be the unit ball $B_1 \subseteq \mathbb{R}^{p|q}$ centered at 0, it is clear that there exists a path $\mu = l(x_o, x)$ from x_o to x in U such that $\mu^{-1}l(x)$ is the piecewise smooth boundary of a surface S contained in U. Moreover, μ can be chosen so that $\mu(0) = x_o$, $\mu(1) = x$, $\dot{\mu}(1) = v$.

In fact, we want a specific surface S such that a point $p \in S$ iff there exists $0 \leq t \leq 1$ such that p lies on the ray $s \mapsto l(\mu(t))(s)$ where $0 \leq s \leq 1$ (we use the term surface loosely as we are not claiming that it is a sub-super manifold; we only require it be the image of the mapping σ^{-1} defined below). Here $l(\mu(t))$ is the curve defined by $l(\mu(t))(s) = \mathcal{X}^{-1}(s\mathcal{X}(\mu(t)))$ for $0 \leq s \leq 1$. Define a chart σ on S by

$$\sigma^{-1} : [0,1] \times [0,1] \to S \quad \text{where} \quad \sigma^{-1}(s,t) = l(\mu(t))(s) \tag{8.3}$$

Thus we have vector fields on S defined by

$$\left.\frac{\partial}{\partial s}\right|_{(s,t)} = l(\mu(t))'(s) \qquad \left.\frac{\partial}{\partial t}\right|_{(s,t)} = \frac{d}{dt}\big(t \mapsto l(\mu(t))\big)(s) \tag{8.4}$$

Notice that both of these vectors are derivatives of curves in $\mathcal{M}$ and so are in $T^0_{(s,t)}\mathcal{M}$. Also since $l(\mu(t))$ is a ray in U emanating from x_o we see that $d\mathfrak{s}(\frac{\partial}{\partial s}|_{(s,t)})$ is horizontal.

Proposition 8.1.4. *If the set of pairs (s,t) parametrize a surface S as described above, and v is a tangent vector to U which is transversal to one of the "rays" of S then*

$$(\mathfrak{s}^*\omega)_x(v) = \int_0^1 (\mathfrak{s}^*\Omega)\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right)(s,1)ds$$

where Ω is the curvature of ω.

Proof. First note that $\mathfrak{s}^*\Omega = \mathfrak{s}^*(d\omega + [\omega,\omega]) = d(\mathfrak{s}^*\omega) + [\mathfrak{s}^*\omega, \mathfrak{s}^*\omega]$. Also notice that $(\mathfrak{s}^*\omega)(\frac{\partial}{\partial s}) = \omega(d\mathfrak{s}(\frac{\partial}{\partial s}))) = 0$. Thus, $\mathfrak{s}^*\Omega(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}) = d(\mathfrak{s}^*\omega)(\frac{\partial}{\partial s}, \frac{\partial}{\partial t})$. Consequently,

$$\begin{aligned} \mathfrak{s}^*\Omega(\tfrac{\partial}{\partial s}, \tfrac{\partial}{\partial t}) &= d(\mathfrak{s}^*\omega)(\tfrac{\partial}{\partial s}, \tfrac{\partial}{\partial t}) \\ &= \tfrac{\partial}{\partial s}((\mathfrak{s}^*\omega)(\tfrac{\partial}{\partial t})) - \tfrac{\partial}{\partial t}((\mathfrak{s}^*\omega)(\tfrac{\partial}{\partial s})) - (\mathfrak{s}^*\omega)([\tfrac{\partial}{\partial s}, \tfrac{\partial}{\partial t}]) \\ &= \tfrac{\partial}{\partial s}(\mathfrak{s}^*\omega)(\tfrac{\partial}{\partial t}) \end{aligned} \tag{8.5}$$

and

$$\begin{aligned} \textstyle\int_0^1 (\mathfrak{s}^*\Omega)(\frac{\partial}{\partial s}, \frac{\partial}{\partial t})(s,t)ds &= \textstyle\int_0^1 \frac{\partial}{\partial s}[(\mathfrak{s}^*\omega)(\frac{\partial}{\partial t})(s,t)]ds \\ &= (\mathfrak{s}^*\omega)(\tfrac{\partial}{\partial t})(1,t) - (\mathfrak{s}^*\omega)(\tfrac{\partial}{\partial t})(0,t) \\ &= (\mathfrak{s}^*\omega)(\tfrac{\partial}{\partial t})(1,t) \\ &= (\mathfrak{s}^*\omega)[\tfrac{d}{dt}[l(\mu(t))(1)]] \end{aligned} \tag{8.6}$$

But $\frac{d}{dt}[l(\mu(t))(1)]|_{t=1} = v$ and the proposition follows. □

Definition 8.1.5. *Denote by $\mathfrak{h}^\Omega(U)$ the Lie sub-superalgebra of $\mathfrak{g}$ generated by the set of elements $\Omega_z(v,w)$ for $z \in \tau^{-1}(U)$ and $v, w \in T^0_z(\tau^{-1}(U))$.*

Corollary 8.1.6. *If $\mathfrak{s} : U \to \mathcal{P}$ is a special local section of τ at $x_o \in U$ and $x \in U$ then $(\mathfrak{s}^*\omega)_x(v)$ is in the <u>closure</u> of $\mathfrak{h}^\Omega(U)$ for each $v \in T^0_x U$.*

Proof. If v is tangent to a ray then $(\mathfrak{s}^*\omega)_x(v) = 0 = (\mathfrak{s}^*\Omega)_x(v,v)$. If v is not tangent to a ray, then by Proposition 1.4

$$(\mathfrak{s}^*\omega)_x(v) = \int_0^1 (\mathfrak{s}^*\omega)\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right)(s,1)ds$$

which is a limit of finite sums of elements of $\mathfrak{h}^\Omega(U)$. □

Corollary 8.1.7. *If $\mathfrak{s} : U \to \mathcal{P}$ is a special section of τ at $x_o \in U$ and if $g : I \to \mathcal{G}$ is the development of an arbitrary loop $\gamma : I \to U$ in U relative to $\mathfrak{s}$, then $\dot{g}(t)g(t)^{-1} \in \mathfrak{h}^\Omega(U)$ for all $t \in I$.*

Proof. Recall that $\dot{g}(t)g(t)^{-1} = -(\mathfrak{s}^*\omega)(\dot{\gamma}(t))$ for all $t \in I$ and that $(\mathfrak{s}^*\omega)_x(v) \in \mathfrak{h}^\Omega(U)$ for all $x \in U$, $v \in T^0_x U$. □

Once more observe that if one used left actions of the structure group of a principle fiber bundle (instead of the right action we use here), then the mapping $g : I \to \mathcal{G}$ satisfies $g(t)^{-1}\dot{g}(t) = -(\mathfrak{s}^*\omega)(\dot{\gamma}(t))$.

Corollary 8.1.8. *If $\Omega \equiv 0$, then each development of each special section is trivial, i.e. $g(t) = e$ for all $t \in I$. Thus the horizontal lift of each loop in U is a loop in $\mathcal{P}$.*

Proof. Note that $\dot{g}(t)g(t)^{-1}$ is the zero vector in $T^0_e\mathcal{G}$ so $\dot{g}(t) = 0$ in $T^0_{g(t)}\mathcal{G}$ for each $t \in I$. Thus g is constant and $g(t) = e$ for all $t \in I$. Since $\tilde{\gamma}(t) = \mathfrak{s}(\gamma(t))g(t), t \in I$, we have $\tilde{\gamma}(1) = \tilde{\gamma}(0)g(1) = \tilde{\gamma}(0)$. □

Observation 8.1.9. *Clearly this is independent of whether left or right actions are used in the definition of a principle fiber bundle.*

If $\mathcal{M}$ is a $(p|q)$ dimensional supermanifold, then vector superbundles E over $\mathcal{M}$ are defined as in the usual case of ordinary Banach manifolds except that all relevant mappings are G^∞-mappings (see [80]) and [27]). If $E \hookrightarrow T^0\mathcal{M}$ is a sub-bundle of the even tangent bundle $T^0\mathcal{M} \to \mathcal{M}$, then we say that $E \to \mathcal{M}$ is integrable iff whenever X, Y are sections of $E \to \mathcal{M}$ then so is $[X, Y]$. Since $T^0\mathcal{M} = T\mathcal{BM}$, the sub-bundle $E \to \mathcal{M}$ is an integrable sub-bundle of the tangent bundle $T\mathcal{BM} \to \mathcal{BM}$ of the underlying Banach manifold $\mathcal{BM}$, and so by the Frobenius theorem for Banach manifolds [80], one has a foliation of $\mathcal{BM}$. The leaves of this foliation are initial submanifolds of $\mathcal{BM}$ and thus are G^∞ submanifolds as well (see [73] and [37]). Since $\mathcal{BM}$ is modelled on $\mathbb{K}^{p|q} = ({}^0\Lambda)^p \times ({}^1\Lambda)^q$, one has cubical neighborhoods $(U, (z^1, z^2, \ldots, z^{p+q})) = (U, z)$ at each point $p \in \mathcal{M}$. The cubical neighborhood chart z is chosen such that $z(p) = 0$, and the leaves are given by $z^{r+i} = c^i = constant$, $z^{p+s+j} = d^j = constant$ for each i, j with $1 \leq i \leq p-r$ and $1 \leq j \leq q-s$. This is in close analogy to the finite dimensional case except here the c^i are even supernumbers and the d^j are odd supernumbers (see [27]). The leaves are of dimension $(r|s)$ in this case. We say the foliation is <u>regular</u> iff each leaf is an immersed sub-supermanifold with fixed dimension $(r|s)$ for some r, s.

Recall that there exists cubical charts (U_o, z) of $\mathcal{M}$ at each point of $\mathcal{M}$ such that each leaf $\mathcal{L}$ of $\mathcal{F}_\mathcal{M}$ intersected with U_o is given by $z^{r+i} = c^i = constant$ and $z^{p+s+j} = d^j = constant$ where $c^i \in {}^0\Lambda$ and $d^j \in {}^1\Lambda$ for each i, j with $1 \leq i \leq p-r$ and $1 \leq j \leq q-s$. We choose U_o small enough so that there exists a local section $\mathfrak{s}_o : U_o \to P$ of τ. Here r, s are fixed and denote the dimension $(r|s)$ of each leaf. For each leaf $\mathcal{L}$ such that $\mathcal{L} \cap U_o \neq \emptyset$ define $x^\mathcal{L}_o \in \mathcal{L}$ by $z^L(x^\mathcal{L}_o) = 0$ for all L not in $I_{r|s} = \{i \mid 1 \leq i \leq r \text{ or } p+1 \leq i \leq p+s\}$. One can choose U small enough so that $x_o \in U \subseteq U_o$ and such that for some $\delta > 0$ one has for each leaf $\mathcal{L}$ such that $\mathcal{L} \cap U \neq \emptyset$ that, $x \in \mathcal{L} \cap U$ iff $|z^L(x)| < \delta$ for all $L \notin I_{r|s}$. Thus the image of $\mathcal{L} \cap U$ under z is a δ-ball about 0 in a slice determined by the constants c^i, d^j. We will refer to $\mathcal{L} \cap U$ as

the δ-ball about $x_o^{\mathcal{L}}$ in $\mathcal{L}$. Using the chart z we can define "curvilinear rays" in $\mathcal{L} \cap U$ emanating out of $x_o^{\mathcal{L}}$. These "curvilinear rays" map to actual "rays" in $z(U) \subseteq \mathbb{R}^{p|q}$ emanating out of $z(x_o^{\mathcal{L}}) = 0$. Now define a "special section" by horizontally lifting each curvilinear ray emanating out of $x_o^{\mathcal{L}}$ and terminating at $\mathfrak{s}_o(x_o^{\mathcal{L}})$ in $\tau^{-1}(\mathcal{L}) \subseteq \mathcal{P}$.

This construction defines a section $\mathfrak{s} : U \to \mathcal{P}$ such that for each leaf $\mathcal{L}$ such that $U \cap \mathcal{L} \neq \emptyset$, $\mathfrak{s}_{\mathcal{L}} = \mathfrak{s}|(U \cap \mathcal{L})$ is a special section of $\tau^{-1}(\mathcal{L}) \to \mathcal{L}$ in the sense of Lichernowicz [81].
We will refer to such a section $\mathfrak{s} : U \to \mathcal{P}$ as a special section of τ and to the points $x_o^{\mathcal{L}}$ as distinguished points of the section.

Proposition 8.1.10. *Let $\tau : \mathcal{P} \to \mathcal{M}$ be a super principal fiber bundle with structure group a super Lie group $\mathcal{G}$. Let $\mathcal{F}_{\mathcal{M}}$ be a regular foliation of $\mathcal{M}$ and $\omega : T\mathcal{P} \to \mathfrak{g}$ an even connection with values in the super Lie algebra $\mathfrak{g}$ of $\mathcal{G}$. Assume that at each point $x_o \in \mathcal{M}$ there exists a local section $\mathfrak{s}_o : U_o \to \mathcal{P}$ of τ at x_o such that the restriction of ${\mathfrak{s}_o}^*\Omega$ to $U_o \cap \mathcal{L}$ is zero for each leaf $\mathcal{L}$ of $\mathcal{F}_{\mathcal{M}}$ such that $U_o \cap \mathcal{L} \neq \emptyset$, then there exists a local section $\mathfrak{s}$ defined on $U \subseteq U_o$ along with distinguished points $x_o^{\mathcal{L}} \in \mathcal{L} \cap U$ such that the horizontal lift of each loop γ in $U \cap \mathcal{L}$ at $x_o^{\mathcal{L}}$ is $\mathfrak{s}_{\mathcal{L}} \circ \gamma$ which is a loop in $\tau^{-1}(U \cap \mathcal{L})$. Here $\mathfrak{s}_{\mathcal{L}} = \mathfrak{s}|(U \cap \mathcal{L})$.*

Proof. Let $\mathcal{L}$ be a leaf of $\mathcal{F}_{\mathcal{M}}$. We first show that the curvature Ω of ω restricted to $\tau^{-1}(\mathcal{L})$ is zero. Let $i_{\mathcal{L}} : \mathcal{L} \hookrightarrow \mathcal{M}$ and $\tilde{i} : \tau^{-1}(\mathcal{L}) \hookrightarrow \mathcal{P}$ be the inclusion mapping, then the curvature of $\tilde{i}_{\mathcal{L}}^*\omega$ is $\tilde{i}_{\mathcal{L}}^*\Omega$ so we must show that $\tilde{i}^*\Omega = 0$. Let $x_o \in \mathcal{L}$ be arbitrary and choose a local section $\mathfrak{s}_o : U_o \to \mathcal{P}$ of τ at x_o such that ${\mathfrak{s}_o}^*\Omega$ restricted to $U_o \cap \mathcal{L}$ is zero. Now ${\mathfrak{s}_o}^*(\tilde{i}^*\Omega) = i_{\mathcal{L}}^*({\mathfrak{s}_o}^*\Omega)$ which is zero by hypothesis, so ${\mathfrak{s}_o}^*(\tilde{i}^*\Omega) = 0$. But if z_o is any element of $\tau^{-1}(x_o)$ and $\tilde{\mathfrak{s}}$ is any local section of $\tau^{-1}(\mathcal{L}) \to \mathcal{L}$ through z_o, then there exists a mapping $g : dom(\tilde{\mathfrak{s}}) \to \mathcal{G}$ such that $\tilde{\mathfrak{s}}(x) = \mathfrak{s}(x)g(x)$ for all $x \in U_o \cap dom(\tilde{\mathfrak{s}})$. Now

$$\tilde{\mathfrak{s}}^*(\tilde{i}^*\Omega) = Ad(g^{-1})({\mathfrak{s}_o}^*(\tilde{i}^*\Omega)) = 0 \tag{8.7}$$

and since this is true for every such local section $\tilde{\mathfrak{s}}$ through z_o, we have $(\tilde{i}^*\Omega)_{z_o} = 0$. Since x_o is arbitrary and $z_o \in \tau^{-1}(x_o)$ is arbitrary, $\tilde{i}^*\Omega = 0$ as asserted.

Now by the construction just prior to the statement of Proposition 8.1.10 there exists, at each $x_o \in \mathcal{M}$, a local section $\mathfrak{s} : U \to \mathcal{P}$ and points $x_o^{\mathcal{L}}$ in each leaf $\mathcal{L}$ intersecting U such that $\mathfrak{s}_{\mathcal{L}} = \mathfrak{s}|(U \cap \mathcal{L})$ is a special section of $\tau^{-1}(\mathcal{L}) \to \mathcal{L}$ in the sense of Lichernowicz [81]. Consequently if γ is a loop at $x_o^{\mathcal{L}}$ in $U \cap \mathcal{L}$, then its development $g : I \to \mathcal{G}$ with respect to $\mathfrak{s}_{\mathcal{L}}$ is trivial (since the curvature $\tilde{i}_{\mathcal{L}}^*\Omega$ of $\tilde{i}_{\mathcal{L}}^*\omega$ is 0). Thus the horizontal lift $\tilde{\gamma} = (\mathfrak{s}_{\mathcal{L}} \circ \gamma)g$ of γ to $\mathfrak{s}_o(x_o^{\mathcal{L}}) = \mathfrak{s}(x_o^{\mathcal{L}})$ in $\tau^{-1}(\mathcal{L})$ is a loop. The proposition follows. □

Lemma 8.1.11. *If $u : I \to \mathfrak{g}^0$ is a smooth mapping then there is a smooth curve $g : I \to \mathcal{G}$ such that $g(0) = e$ and $\dot{g}(t)g(t)^{-1} = -u(t)$ for all $t \in I$. Moreover,*

$g(t) = \overrightarrow{P}exp(-\int_0^t u(\tau)d\tau)$ where the product integral is ordered opposite the usual ordering in [89] and [90].

Proof. It is asserted by Omori in [90] on page 65 that in a regular F-Lie group $\mathcal{G}$ with Lie algebra $\mathfrak{g}$ the differential equation

$$\dot{g}(t) = u(t)g(t) \qquad g(0) = e \tag{8.8}$$

has a unique solution which is given by the product or path ordered integral

$$g(t) = \overleftarrow{P}exp\int_0^t u(\tau)d\tau \tag{8.9}$$

If $h(t) = g(t)^{-1}$ we have $hg = e$ and $\dot{h}g + h\dot{g} = 0$ so that $\dot{g}(t)g(t)^{-1} = -h(t)^{-1}\dot{h}(t)$. Thus $\dot{g} = ug$ implies that $\dot{g}g^{-1} = u$ and $-h^{-1}\dot{h} = u$. Moreover if

$$g(t) = \overleftarrow{P}exp\int_0^t u(\tau)d\tau \tag{8.10}$$

then

$$g(t)^{-1} = \left[\overleftarrow{P}exp\int_0^t u(\tau)d\tau\right]^{-1} = \overrightarrow{P}exp\left(\int_0^t (-1)u(\tau)d\tau\right) \tag{8.11}$$

where the last integral reverses the path ordering used in [89] and [90]. So we have

$$\dot{h}(t) = -h(t)u(t) \tag{8.12}$$

and

$$h(t) = \overrightarrow{P}exp\left(-\int_0^t u(\tau)d\tau\right) \tag{8.13}$$

as required.

Now since a Banach Lie group is a regular F-Lie group, we see that this lemma holds in the Banach Lie group $\mathcal{BG}$ for any super Lie group $\mathcal{G}$ where we know $\mathcal{BG}$ can be identified with $\mathcal{G}$ relative to a suitably restricted atlas. Recall, however, that the Lie algebra of $\mathcal{BG}$ is the even part of of the Lie superalgebra $\mathfrak{g}$ (see [37]). □

Theorem 8.1.12. *Assume that $\tau : \mathcal{P} \to \mathcal{M}$ is a super principal fiber bundle with structure group a super Lie group $\mathcal{G}$ where $\mathcal{G}$ acts on the left of the bundle $\mathcal{P}$ (contrary to our convention up to this point). Let $\mathcal{F}_\mathcal{M}$ be a regular foliation of $\mathcal{M}$ whose leaves are supermanifolds of dimension $(r|s)$. Let $\omega : T\mathcal{P} \to \mathfrak{g}$ be an even connection on $\mathcal{P}$ whose curvature restricted to $\tau^{-1}(\mathcal{L})$ is zero for each leaf $\mathcal{L}$ of $\mathcal{F}_\mathcal{M}$. If $x_o \in \mathcal{M}$, then there exists an open subset U of $\mathcal{M}$ about x_o on which there is defined a local section $\mathfrak{s}$ of τ and a mapping $g : U \to \mathcal{G}$ such that $g(x_o) = e$. Moreover, this mapping has the property that if $p \in U$ and $\mathcal{L}$ is the leaf of $\mathcal{F}_\mathcal{M}$ containing p then*

$g(p)^{-1}d_pg(v) = -(\mathfrak{s}^*\omega)_p(v)$ *for every* $v \in T_p^0(U \cap \mathcal{L})$.

Proof. Let $x_o \in \mathcal{M}$. Choose an open set U about x_o and a section $\mathfrak{s} : U \to \mathcal{P}$ subject to the construction just prior to Proposition 8.1.10. Thus $\mathfrak{s}$ is what we have called a special section of τ and it has the property that if $\mathcal{L}$ of $\mathcal{F}_\mathcal{M}$ such that $U \cap \mathcal{L} \neq \emptyset$ and $\mathfrak{s}_\mathcal{L} = \mathfrak{s}|(U \cap \mathcal{L})$ then $\mathfrak{s}_\mathcal{L} : U \cap \mathcal{L} \to \tau^{-1}(\mathcal{L})$ is a special section of $\tau^{-1}(\mathcal{L}) \to \mathcal{L}$ in the sense of [81]. Choose points $x_o^\mathcal{L} \in U \cap \mathcal{L}$ for each leaf $\mathcal{L}$ which intersects U such that loops at $x_o^\mathcal{L}$ in $U \cap \mathcal{L}$ lift to horizontal loops at $\mathfrak{s}(x_o^\mathcal{L})$ in $\tau^{-1}(\mathcal{L})$. For each $p \in U$, let $\mathcal{L}$ be the leaf through p and choose an arbitrary path γ_p in $U \cap \mathcal{L}$ from $x_o^\mathcal{L}$ to p. Define

$$g(p) = \overrightarrow{P}exp\left(-\int_{\gamma_p}(\mathfrak{s}^*\omega)\right) \tag{8.14}$$

We now show that the construction of $g(p)$ to be independent of the choice of γ_p. To do this assume γ_1, γ_2 are paths in $U \cap \mathcal{L}$ such that $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = p = \gamma_2(1)$. By Corollary 8.1.8, $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$. But for $i = 1, 2$

$$\tilde{\gamma}_i(t) = g_i(t)\mathfrak{s}(\gamma_i(t)) \tag{8.15}$$

where

$$g_i(t)^{-1}\dot{g}_i(t) = -\mathfrak{s}^*\omega(\dot{\gamma}_i(t)) \tag{8.16}$$

and consequently using Lemma 8.1.11 we have

$$g_i(t) = \overrightarrow{P}exp\left[-\int_0^t(\mathfrak{s}^*\omega)(\dot{\gamma}_i(\tau))d\tau\right], \tag{8.17}$$

so that $g_1(1) = g_2(1)$. Thus

$$\overrightarrow{P}exp\left(-\int_{\gamma_1}(\mathfrak{s}^*\omega)\right) = \overrightarrow{P}exp\left(-\int_{\gamma_2}(\mathfrak{s}^*\omega)\right) \tag{8.18}$$

and it follows that the definition of $g(p)$ is independent of the choice of γ_p. We choose one such path γ_p for each $p \in U$. Now let $p \in U$ and let $\mathcal{L}$ be the leaf through p. Let $v \in T_p^0(U \cap \mathcal{L})$. We show that

$$dL_{g(p)^{-1}}(d_pg(v)) = -(\mathfrak{s}^*\omega)_p(v). \tag{8.19}$$

Let $\mu : I \to U \cap \mathcal{L}$ be a path such that $\mu(0) = p$ and $\dot{\mu}(0) = v$. We have chosen a path γ_p from $x_o^\mathcal{L}$ to p in $U \cap \mathcal{L}$ for each $p \in U \cap \mathcal{L}$, thus we have such a curve $\gamma_{\mu(\sigma)}$ from $x_o^\mathcal{L}$ to $\mu(\sigma)$ for each $\sigma \in I$. Let $\hat{\mu}_\sigma$ denote the path defined by $\hat{\mu}_\sigma(s) = \mu(s\sigma)$, $s \in I$. Now $\gamma_{\mu(\sigma)}$ and $\hat{\mu}_\sigma\gamma_{\mu(0)}$ both initiate at $x_o^\mathcal{L}$ and terminate at $\mu(\sigma)$.

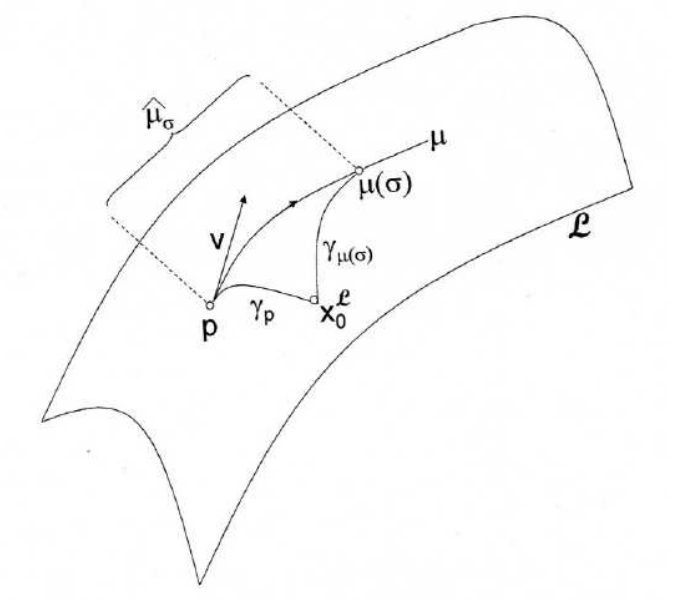

Figure 8.3: Path Independence

Moreover both paths lie in $U \cap \mathcal{L}$. Thus for $h = \overrightarrow{P}exp(-\int_{\gamma_{\mu(0)}}(\mathfrak{s}^*\omega)) = g(p)$ we find

$$\begin{aligned} g(\mu(\sigma)) &= \overrightarrow{P}exp\int_{\gamma_{\mu(\sigma)}}(-\mathfrak{s}^*\omega) \\ &= \overrightarrow{P}exp\int_{\hat{\mu}_\sigma\gamma_{\mu(0)}}(-\mathfrak{s}^*\omega) \\ &= [\overrightarrow{P}exp\int_{\gamma_{\mu(0)}}(-\mathfrak{s}^*\omega)][\overrightarrow{P}exp\int_{\hat{\mu}_\sigma}(-\mathfrak{s}^*\omega)] \\ &= h\overrightarrow{P}exp\big(-\int_0^\sigma(\mathfrak{s}^*\omega)(\dot{\mu}(\tau)d\tau\big) \end{aligned} \tag{8.20}$$

and

$$\begin{aligned} \tfrac{d}{d\sigma}\big(g(\mu(\sigma))\big) &= dL_h\tfrac{d}{d\sigma}\big[\overrightarrow{P}exp\big(-\int_0^\sigma(\mathfrak{s}^*\omega)(\dot{\mu}(\tau))d\tau\big)\big] \\ &= dL_h\big[\overrightarrow{P}exp\big(-\int_0^\sigma(\mathfrak{s}^*\omega)(\dot{\mu}(\tau))d\tau\big)\big(-(\mathfrak{s}^*\omega)(\dot{\mu}(\sigma))\big)\big] \end{aligned} \tag{8.21}$$

Thus

$$dL_{h^{-1}}\big(d_{\mu(\sigma)}g(\dot{\mu}(\sigma))\big) = \overrightarrow{P}exp\big(-\int_0^\sigma(\mathfrak{s}^*\omega)(\dot{\mu}(\tau))d\tau\big)\big(-(\mathfrak{s}^*\omega)(\dot{\mu}(\sigma))\big)$$

and for $\sigma = 0$,

$$dL_{g(p)^{-1}}\big(d_pg(v)\big) = -(\mathfrak{s}^*\omega)_p(v)$$

as required. □

Corollary 8.1.13. *If $g : U \to \mathcal{G}$ is defined as in Theorem 8.1.12 and if we utilize the convention that in a principal fiber bundle the group $\mathcal{G}$ acts on the left of the bundle,*

then

$$g(p)^{-1}d_pg(v) = -(\mathfrak{s}^*\omega)_p(v)$$

for every $p \in T_p(U \cap \mathcal{L})$ *where* $\mathcal{L}$ *is the leaf containing* p.

Proof. Let $v \in T_p(U \cap \mathcal{L})$ be an odd tangent vector. For each odd supernumber ζ, ζv is an even tangent vector in $T_p^0(U \cap \mathcal{L})$, consequently

$$dL_{g(p)^{-1}}(d_pg(\zeta v)) = -(\mathfrak{s}^*\omega)_p(\zeta v).$$

But this implies that

$$\zeta dL_{g(p)^{-1}}(d_pg(v)) = -\zeta(\mathfrak{s}^*\omega)_p(v).$$

for all $\zeta \in {}^1\Lambda$. Thus

$$dL_{g(p)^{-1}}(d_pg(v)) = -(\mathfrak{s}^*\omega)_p(v).$$

and the corollary follows. □

8.1.1 Choosing a Section which gives Gauge-Flat Leaves

In the remainder of this section we assume, once for all, that

- (1) $\tau : \mathcal{P} \to \mathcal{M}$ is a super principal fiber bundle over the supermanifold $\mathcal{M}$ with structure group a super Lie group $\mathcal{G}$,
- (2) $\mathcal{F}_\mathcal{M}$ is a regular foliation of $\mathcal{M}$ whose leaves are sub-supermanifolds of dimension $(r|s)$ and there is an induced foliation $\mathcal{F}_\mathcal{P}$ of $\mathcal{P}$ whose leaves are $\tau^{-1}(\mathcal{L})$ where $\mathcal{L}$ is a leaf of $\mathcal{F}_\mathcal{M}$,
- (3) ω is an even connection on $\mathcal{P}$ with values in the super Lie algebra $\mathfrak{g}$ such that the curvature Ω of ω vanishes on the tangents to the leaves of $\mathcal{F}_\mathcal{P}$. Here it suffices to assume that Ω vanishes on even tangent vectors.

It follows from Theorem 8.1.12 that there exists an open cover $\{U_\alpha\}_{\alpha\in\mathfrak{I}}$ of $\mathcal{M}$ along with local sections $\mathfrak{s}_\alpha : U_\alpha \to \mathcal{P}$ of τ and G^∞ maps $g_\alpha : U_\alpha \to \mathcal{G}$ such that if $\mathcal{L}$ is a leaf of $\mathcal{F}_\mathcal{M}$ such that $\mathcal{L} \cap U_\alpha \neq \emptyset$, then

$$g_\alpha(q)^{-1}d_qg_\alpha(v) = -({\mathfrak{s}_\alpha}^*\omega)_q(v) \tag{8.22}$$

for all $q \in U \cap \mathcal{L}$ and $v \in T_q(U \cap \mathcal{L})$ (even or odd).
Note that since $\mathcal{F}_\mathcal{M}$ is regular we may choose U_α such that $U_\alpha \cap \mathcal{L}$ is connected for each leaf $\mathcal{L}$ of $\mathcal{F}_\mathcal{M}$.
We select $U_\alpha, \mathfrak{s}_\alpha, g_\alpha$ subject to these properties and utilize this notation throughout this section. Next we introduce a notation to help describe leaf-dependent equations.

Definition 8.1.14. *If* η *is a differential form on an open subset* U *of* $\mathcal{M}$*, we write* $\eta \approx 0$ *iff for each leaf* $\mathcal{L}$ *of* $\mathcal{F}_\mathcal{M}$ *such that* $U \cap \mathcal{L} \neq \emptyset$, $\quad i_\mathcal{L}^*\eta = 0$ *where* $i_\mathcal{L} : U \cap \mathcal{L} \hookrightarrow U$

is the inclusion mapping. If η, ζ are both differential forms on U we write $\eta \approx \zeta$ iff $\eta - \zeta \approx 0$.

Proposition 8.1.15. *For each $\alpha \in \mathfrak{I}$ let $\tilde{\mathfrak{s}}_\alpha : U_\alpha \to \mathcal{P}$ be the local section of τ defined by $\tilde{\mathfrak{s}}_\alpha = g_\alpha \mathfrak{s}_\alpha$. Then $\tilde{\mathfrak{s}}_\alpha^* \omega \approx 0$.*

Proof. Note that

$$\tilde{\mathfrak{s}}_\alpha^* \omega = Ad(g_\alpha)[{\mathfrak{s}_\alpha}^* \omega + (dL_{g_\alpha}^{-1} \circ dg_\alpha)] \approx 0.$$

□

Proposition 8.1.16. *Let $\mathfrak{s} : U \to \mathcal{P}$ be a local section of τ and $g : U \to \mathcal{G}$ a C^∞ mapping which is constant on the leaves of $\mathcal{F}_\mathcal{M}$. Let $\tilde{\mathfrak{s}} = g\mathfrak{s}$, then*

1. *If $\mathfrak{s}^*\omega \approx -h^{-1}dh$ for some $h : U \to \mathcal{G}$ then $\mathfrak{s}^*\omega \approx -(hg^{-1})d(gh^{-1})$.*
2. *If $\mathfrak{s}^*\omega \approx 0$ then $\tilde{\mathfrak{s}}^*\omega \approx 0$*

Proof. Both parts follow from the fact that $\tilde{\mathfrak{s}}^*\omega = Ad(g)[\mathfrak{s}^*\omega + dL_g^{-1} \circ dg]$ and the fact that $dg \approx 0$ (since g is constant on the leaves of $\mathcal{F}_\mathcal{M}$). Thus $\tilde{\mathfrak{s}}^*\omega = Ad(g)[\mathfrak{s}^*\omega]$ and (2) follows immediately. The proof of (1) requires the additional observation that if $\mathfrak{s}^*\omega \approx -h^{-1}dh$, then

$$Ad(g)(\mathfrak{s}^*\omega) \approx -g(h^{-1}dh)g^{-1} = -(hg^{-1})^{-1}d(hg^{-1})$$

where the last equality uses the fact that $dg^{-1} \approx 0$. □

8.1.2 Quotients on $\mathcal{M}$ and the Lift to $\mathcal{P}$

Recall that the supermanifold $\mathcal{M}$ has an atlas $\mathcal{A}$ which has G^∞ transition functions. That same atlas gives $\mathcal{M}$ a Banach manifold structure, but it is not a maximal C^∞ atlas on $\mathcal{M}$ (see [37]).

Also recall that since the foliation $\mathcal{F}_\mathcal{M}$ is regular, it follows $\mathcal{M}/\mathcal{F}_\mathcal{M}$ is a manifold, and the quotient mapping is an open mapping (see [2]). In this section we impose the additional requirement that the foliation $\mathcal{F}_\mathcal{M}$ on $\mathcal{M}$ induces a Hausdorff Banach manifold structure on $\mathcal{M}/\mathcal{F}_\mathcal{M}$.

Now $\mathcal{M}$ is modeled on the Banach space $\mathbb{R}^{p|q}$, and the leaves of $\mathcal{F}_\mathcal{M}$ are submanifolds of dimension $(r|s)$ as supermanifolds. If one uses charts of $\mathcal{M}$ which respect the leaf structure of the foliation, then it follows that one obtains charts of $\mathcal{M}/\mathcal{F}_\mathcal{M}$ with values in $\mathbb{C}^{(p-r|q-s)}$, and consequently $\mathcal{M}/\mathcal{F}_\mathcal{M}$ is a supermanifold of dimension $(p-r|q-s)$. These remarks follow from the Frobenius theorem discussed earlier and the detailed proofs are left to the reader.

Notice that since the leaves of $\mathcal{F}_\mathcal{P}$ are of the form $\tau^{-1}(\mathcal{L})$ where $\mathcal{L}$ is a leaf of $\mathcal{F}_\mathcal{M}$, $\mathcal{P}/\mathcal{F}_\mathcal{P}$ is also a supermanifold. This is due to the fact that locally $\mathcal{P}$ has the

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\rho} & \mathcal{P}/\mathcal{F}_{\mathcal{P}} \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tilde{\tau}} \\ \mathcal{M} & \xrightarrow[q]{} & \mathcal{M}/\mathcal{F}_{\mathcal{M}} \end{array}$$

Figure 8.4: Quotient maps in Base and Bundle Space

form $U \times \mathcal{G}$ where U is an open subset of $\mathcal{M}$. If U is chosen appropriately, so that its image under the quotient mapping $q_{\mathcal{M}} : \mathcal{M} \to \mathcal{F}_{\mathcal{M}}$ is a chart domain in $\mathcal{M}/\mathcal{F}_{\mathcal{M}}$, then domains of charts of $\mathcal{P}/\mathcal{F}_{\mathcal{P}}$ will take the form $q_{\mathcal{M}}(U) \times V$ where V is a chart domain of the super Lie group $\mathcal{G}$. Moreover $\tilde{\tau} : \mathcal{P}/\mathcal{F}_{\mathcal{P}} \to \mathcal{M}/\mathcal{F}_{\mathcal{M}}$ is a super principal fiber bundle with structure group $\mathcal{G}$.

So by assuming that $\mathcal{F}_{\mathcal{M}}$ is regular and that $\mathcal{M}/\mathcal{F}_{\mathcal{M}}$ is a Hausdorff Banach manifold, we obtain supermanifold structures on $\mathcal{M}/\mathcal{F}_{\mathcal{M}}$ and $\mathcal{P}/\mathcal{F}_{\mathcal{P}}$. Let $q = q_{\mathcal{M}} : \mathcal{M} \to \mathcal{F}_{\mathcal{M}}$ and $\rho = \rho_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}/\mathcal{F}_{\mathcal{P}}$ denote the usual quotient mappings. By modifying the arguments on page 207 of Abraham, Marsden, and Ratiu [2], one can show that the quotient mappings q and ρ are G^∞ mappings. Moreover, by a similar analysis of the proof of Proposition 4.4.9 on page 334 of [2], one can show that the mappings q and ρ have local G^∞ sections. It is straightforward to show that the group $\mathcal{G}$ acts on $\mathcal{F}_{\mathcal{P}}$ via $g \cdot \rho(p) = \rho(g \cdot p)$ for all $p \in \mathcal{P}$ and $g \in \mathcal{G}$. Finally, the mapping $\tilde{\tau} : \mathcal{P}/\mathcal{F}_{\mathcal{P}} \to \mathcal{M}/\mathcal{F}_{\mathcal{M}}$ defined by $\tilde{\tau}(\rho(p)) = q(\tau(p))$ for all $p \in \mathcal{P}$ is well-defined and $\tilde{\tau} : \mathcal{P}/\mathcal{F}_{\mathcal{P}} \to \mathcal{M}/\mathcal{F}_{\mathcal{M}}$ is a super principal fiber bundle with structure group the super Lie group $\mathcal{G}$.

Recall that we have shown that at each point $x \in \mathcal{M}$ there exists a local section $\tilde{\mathfrak{s}} : U \to \mathcal{P}$ of τ such that $x \in U$ and $\tilde{\mathfrak{s}}^*\omega \approx 0$. This suggest that there ought to exist a connection $\tilde{\omega}$ on $\mathcal{P}/\mathcal{F}_{\mathcal{P}}$ which in an appropriate local section agrees with $\tilde{\mathfrak{s}}^*\omega$ in the directions transverse to the leaves of $\mathcal{F}_{\mathcal{P}}$. It appears that this is not generally true but we will determine conditions which insure that it is true in our context. First notice that given such a local section $\tilde{\mathfrak{s}}$ of τ we can define a local section $\hat{\mathfrak{s}}$ of $\tilde{\tau}$ as follows. Define $\hat{\mathfrak{s}} : q(U) \to \mathcal{P}/\mathcal{F}_{\mathcal{P}}$ by $\hat{\mathfrak{s}}(q(y)) = \rho(\tilde{\mathfrak{s}}(y))$ for each $y \in U$.

It is easy to show that $\hat{\mathfrak{s}}$ is well defined since leaves of $\mathcal{F}_{\mathcal{P}}$ are of the form $\tau^{-1}(\mathcal{L})$ where $\mathcal{L}$ is a leaf of $\mathcal{F}_{\mathcal{M}}$. Moreover $\hat{\mathfrak{s}}$ is a G^∞-mapping since in a neighborhood of each point in its domain one can factor $\hat{\mathfrak{s}}$ as a composite $\rho \circ s_\tau \circ s_q$ where s_τ, s_q are local G^∞ sections of τ and q, respectively. Thus we have $\hat{\mathfrak{s}} \circ q = \rho \circ \tilde{\mathfrak{s}}$ and we want to define a connection $\tilde{\omega}$ on $\mathcal{P}/\mathcal{F}_{\mathcal{P}}$ such that $(\hat{\mathfrak{s}}^*\tilde{\omega}) \circ dq = \hat{\mathfrak{s}}^*\omega$. This procedure does not always result in a well-defined mapping on $\hat{\mathfrak{s}}^*\tilde{\omega}$. To see this, first observe that for $y \in \mathcal{M}$ and $p \in \mathcal{P}$

$$T_{q(y)}(\mathcal{M}/\mathcal{F}_{\mathcal{M}}) = \{\ d_y q(v) \mid v \in T_y\mathcal{M}\ \} \qquad T_{\rho(p)}(\mathcal{P}/\mathcal{F}_{\mathcal{P}}) = \{\ d_p\rho(w) \mid w \in T_p\mathcal{P}\ \}$$

In order that $\hat{\mathfrak{s}}^*\tilde{\omega}$ be well defined, it must be the case that if $y_1, y_2 \in \mathcal{M}$ and $q(y_1) = q(y_2) \in \mathcal{M}/\mathcal{F}_\mathcal{M}$ and if v_1, v_2 are vectors such that $v_1 \in T_{y_1}\mathcal{M}$, $v_2 \in T_{y_2}\mathcal{M}$, $d_{y_1}q(v_1) = d_{y_2}q(v_2)$, then $\tilde{\mathfrak{s}}^*\tilde{\omega})_{y_1}(v_1) = \tilde{\mathfrak{s}}^*\tilde{\omega})_{y_2}(v_2)$. Now if $y_1 = y_2$, this is immediate since $d_{y_1}q(v_1) = d_{y_1}q(v_2)$ implies that $d_{y_1}q(v_1 - v_2) = 0$, and consequently $v_1 - v_2$ is tangent to the leaf of $\mathcal{F}_\mathcal{M}$ containing y_1. Since $\tilde{\mathfrak{s}}^*\omega \approx 0$, $(\tilde{\mathfrak{s}}^*\omega)_{y_1}(v_1 - v_2) = 0$ and consequently $(\tilde{\mathfrak{s}}^*\omega)_{y_1}(v_1) = (\tilde{\mathfrak{s}}^*\omega)_{y_1}(v_2)$.

Recall that if y_1, y_2 belong to the same leaf $\mathcal{L}$, there is a vector field X defined on an open subset of $\mathcal{M}$ containing y_1, y_2 such that X is tangent to the leaves of $\mathcal{F}_\mathcal{M}$ and whose flow $\{\phi_t\}$ takes y_1 to y_2, i.e., both y_1, y_2 lie on an integral curve of X (see [2] page 330). Since X is everywhere tangent to $\mathcal{L}$, $\phi_t(\mathcal{L}) \subseteq \mathcal{L}$ for all t. Choose a particular t such that $\phi_t(y_1) = y_2$ and notice that

$$d_{y_2}q(v_2) = d_{y_1}q(d_{y_2}\phi_{-t}(v_2)).$$

Now the fact that $d_{y_1}q(v_1) = d_{y_2}q(v_2)$ implies that $d_{y_1}q(v_1 - d_{y_2}\phi_{-t}(v_2)) = 0$ and consequently that

$$(\tilde{\mathfrak{s}}^*\omega)_{y_1}(v_1) = (\tilde{\mathfrak{s}}^*\omega)_{y_1}(d_{y_2}\phi_{-t}(v_2)).$$

In this case we see that in order to obtain $(\tilde{\mathfrak{s}}^*\omega)(v_1) = (\tilde{\mathfrak{s}}^*\omega)(v_2)$, it suffices to require that

$$(\tilde{\mathfrak{s}}^*\omega)(d_{y_2}\phi_{-t}(v_2)) = (\tilde{\mathfrak{s}}^*\omega)(v_2)$$

or that

$$\phi_{-t}^*(\tilde{\mathfrak{s}}^*\omega) = (\tilde{\mathfrak{s}}^*\omega).$$

This holds for all t iff $d/dt(\phi_{-t}^*(\tilde{\mathfrak{s}}^*\omega)) = 0$. Obviously requiring that $\mathcal{L}_X(\tilde{\mathfrak{s}}^*\omega) = 0$ is sufficient to obtain the desired result. The preceding discussion is a proof of the following theorem.

Theorem 8.1.17. *If $\tilde{\mathfrak{s}} : U \to \mathcal{P}$ is a local section of τ and $\hat{\mathfrak{s}} : q(U) \to \mathcal{P}/\mathcal{F}_\mathcal{P}$ is the local section of $\tilde{\tau}$ defined by $\hat{\mathfrak{s}} \circ q = \rho \circ \tilde{\mathfrak{s}}$, then in order that there be a well-defined gauge field given on $q(U)$ by*

$$d_y q(v) \mapsto (\tilde{\mathfrak{s}}^*\omega)_y(v)$$

it is sufficient that $\mathcal{L}_X(\tilde{\mathfrak{s}}^*\omega) = 0$ for each vector field X on $\mathcal{M}$ which is tangent to the leaves of $\mathcal{F}_\mathcal{M}$.

Remark 8.1.18. *Note that if $q : \mathcal{M} \to \mathcal{M}/\mathcal{F}_\mathcal{M}$ possesses a global slice, in the sense that there exists a G^∞ mapping $\sigma : \mathcal{M}/\mathcal{F}_\mathcal{M} \to \mathcal{M}$ such that $q \circ \sigma$ is the identity mapping on $\mathcal{M}/\mathcal{F}_\mathcal{M}$, then one also has a global slice $\tilde{\sigma} : \mathcal{P}/\mathcal{F}_\mathcal{P} \to \mathcal{P}$ of $\rho : \mathcal{P} \to \mathcal{P}/\mathcal{F}_\mathcal{P}$. One defines $\tilde{\sigma}$, by a minor abuse of notation, by first noting that elements of $\mathcal{P}/\mathcal{F}_\mathcal{P}$ may be identified with leaves $\tau^{-1}(\mathcal{L})$ of $\mathcal{P}$ and one simply requires that $\tilde{\sigma}(\tau^{-1}(\mathcal{L})) = \tau^{-1}(\sigma(\mathcal{L}))$. It is easy to show that $\tilde{\sigma}$ is well-defined, and it is also a G^∞-mapping since locally it may be shown to be G^∞ by the arguments similar to those*

of [2] referred to above. Also notice that $\tilde{\sigma}$ factors through τ; $\tau \circ \tilde{\sigma} = \sigma \circ q$. Given a section $\tilde{\mathfrak{s}} : U \to \mathcal{P}$ such that $\tilde{\mathfrak{s}}^*\omega \approx 0$ and its corresponding section $\hat{\mathfrak{s}} : q(U) \to \mathcal{P}/\mathcal{F}_{\mathcal{P}}$ of $\tilde{\tau}$ such that $\hat{\mathfrak{s}} \circ q = \rho \circ \tilde{\mathfrak{s}}$, one can define a connection $\tilde{\omega}$ on $\tilde{\tau}^{-1}(q(U)) \to q(U)$ by $\hat{\mathfrak{s}}^*\tilde{\omega} = (\hat{\mathfrak{s}}^*\omega) \circ d\sigma$ (here $\tilde{\omega}$ is defined in the gauge $\hat{\mathfrak{s}}$, it is easy to deduce the required properties of a connection as in the proof of the next theorem).

Now in general this definition is dependent on the choice of the slice σ. To clarify this dependence and to relate this definition of $\tilde{\omega}$ to the definition used in Theorem 8.1.17 and in Theorem 8.1.19 consider two such slices σ_1, σ_2 both defined on $U \subseteq \mathcal{M}/\mathcal{F}_{\mathcal{M}}$.

If $u \in U$, then $\sigma_1(u), \sigma_2(u)$ both belong to the same leaf $\mathcal{L}$ of $\mathcal{M}$ and thus there is a vector field X which is tangent to the leaves of $\mathcal{F}_{\mathcal{M}}$ and which is defined on an open subset of $\mathcal{M}$ whose flow $\{\phi_t\}$ takes $\sigma_1(u)$ to $\sigma_2(u)$ (see [2], page 330). If $\tilde{\omega}$ is to be independent of the choice σ_1, σ_2, then it must be independent of $\sigma_1(u), \sigma_2(u)$ for each $u \in U$ and so the flow $\{\phi_t\}$ of every local vector field X defined on U which is tangent to the leaves of $\mathcal{F}_{\mathcal{M}}$ and which takes points of $\sigma_1(U)$ to points of $\sigma_2(U)$ must satisfy the condition $\phi_t^*(\tilde{\mathfrak{s}}^*\omega) = \tilde{\mathfrak{s}}^*\omega$. Thus one must have $\mathcal{L}_X(\tilde{\mathfrak{s}}^*\omega) = 0$ for each such vector field X. It is in this sense that Theorem 8.1.17 has a converse.

Theorem 8.1.19. *Assume that ω is an even connection on the super principle fiber bundle $\tau : \mathcal{P} \to \mathcal{M}$ such that its curvature Ω satisfies $\Omega \approx 0$ and such that for every vector field X which is tangent to the leaves of $\mathcal{F}_{\mathcal{M}}$, it follows that $\mathcal{L}_{\tilde{X}}\omega = 0$ where $\tilde{X}$ is the ω-horizontal lift of X to $\mathcal{P}$. Then there is a smooth connection $\tilde{\omega}$ on $\tilde{\tau} : \mathcal{P}/\mathcal{F}_{\mathcal{P}} \to \mathcal{M}/\mathcal{F}_{\mathcal{M}}$ which is induced by ω in the sense that if $\tilde{\mathfrak{s}} : U \to \mathcal{P}$ is a local section of τ such that $\tilde{\mathfrak{s}}^*\omega \approx 0$ then $\hat{\mathfrak{s}}^*\tilde{\omega} \circ dq = \tilde{\mathfrak{s}}^*\omega$ where $\hat{\mathfrak{s}} : q(U) \to \mathcal{P}/\mathcal{F}_{\mathcal{P}}$ is the local section of $\tilde{\tau}$ defined by $\hat{\mathfrak{s}} \circ q = \rho \circ \tilde{\mathfrak{s}}$.*

Lemma 8.1.20. *Assume that $\omega, X, \tilde{X}, \tilde{\mathfrak{s}}, \hat{\mathfrak{s}}$ are subject to the hypothesis of Theorem 8.1.19. Then $(\mathcal{L}_{\tilde{X}}\omega)(\tilde{\mathfrak{s}}(p)) = 0$ iff $(\mathcal{L}_X(\hat{\mathfrak{s}}^*\omega)(p) = 0$ for every $p \in U$.*

Proof. Note that since X is tangent to the leaves of $\mathcal{F}_{\mathcal{M}}$ and $\tilde{\mathfrak{s}}^*\omega \approx 0$ we have

$$\begin{aligned} \omega(\tfrac{d}{dt}(\tilde{\mathfrak{s}}(\phi_t(p)))) &= \omega(d\tilde{\mathfrak{s}}(\tfrac{d}{dt}(\phi_t(p)))) \\ &= (\tilde{\mathfrak{s}}^*\omega)(X(\phi_t(p))) \\ &= 0 \end{aligned} \tag{8.23}$$

where $\{\phi_t\}$ is the flow of X. It follows that $t \mapsto \tilde{\mathfrak{s}}(\phi_t(p))$ is the horizontal lift of $t \mapsto \phi_t(p)$ to $\tilde{\mathfrak{s}}(p)$. It follows that if $\{\tilde{\phi}_t\}$ is the flow of $\tilde{X}$, then $\tilde{\phi}_t(\tilde{\mathfrak{s}}(p)) = \tilde{\mathfrak{s}}(\phi_t(p))$ and

$$\frac{d}{dt}(\tilde{\phi}_t^*\omega)_{\tilde{\mathfrak{s}}(p)} = 0 \iff \frac{d}{dt}((\tilde{\mathfrak{s}} \circ \phi_t)^*\omega|_p = 0 \iff \frac{d}{dt}(\phi_t^*(\tilde{\mathfrak{s}}^*\omega))|_p = 0.$$

Consequently $(\mathcal{L}_{\tilde{X}}\omega)(\tilde{\mathfrak{s}}(p)) = 0$ iff $\mathcal{L}_X(\hat{\mathfrak{s}}^*\omega)(p) = 0$ as required. The lemma follows. □

Proof. (of Theorem 8.1.19) Choose any point in $\mathcal{M}/\mathcal{F}_{\mathcal{M}}$ and write it as $q(y_o)$ for $y_o \in \mathcal{M}$. Let $\tilde{\mathfrak{s}} : U \to \mathcal{P}$ be a local section of τ such that $\tilde{\mathfrak{s}}^*\omega \approx 0$ and $y_o \in U$. By Theorem 8.1.17 there exists a well-defined mapping from $q(U)$ to $\mathfrak{g}$ defined by $d_y q(v) \mapsto (\tilde{\mathfrak{s}}^*\omega)_y(v)$ for $y \in U$, $v \in T_y\mathcal{M}$. Define $\hat{\mathfrak{s}} : q(U) \to \mathcal{P}/\mathcal{F}_{\mathcal{P}}$ by $\hat{\mathfrak{s}} \circ q = \rho \circ \tilde{\mathfrak{s}}$. Now define $\tilde{\omega}$ on $\hat{\mathfrak{s}}(q(U)$ by requiring that

$$\tilde{\omega}_{\hat{\mathfrak{s}}(q(y))}(d_{q(y)}\hat{\mathfrak{s}}(d_y q(v))) = (\tilde{\mathfrak{s}}^*\omega)_y(v) \qquad \tilde{\omega}_{\hat{\mathfrak{s}}(q(y))}(\zeta^{\#}) = \zeta$$

where $\zeta \in \mathfrak{g}$ and $\zeta^{\#}$ is the fundamental <u>vertical</u> vector field determined by ζ and the left action of $\mathcal{G}$ on $\mathcal{P}/\mathcal{F}_{\mathcal{P}}$. Now extend $\tilde{\omega}_{\hat{\mathfrak{s}}(q(y))}$ linearly to obtain a $\mathfrak{g}$-valued mapping on

$$T_{\hat{\mathfrak{s}}(q(y))}(\mathcal{P}/\mathcal{F}_{\mathcal{P}}) = d\hat{\mathfrak{s}}(T_{q(y)}(\mathcal{M}/\mathcal{F}_{\mathcal{M}}) \oplus T(\tilde{\tau}^{-1}(q(y))).$$

Thus $\tilde{\omega}$ is a well-defined $\mathfrak{g}$-valued one form on $T_{\hat{\mathfrak{s}}(q(y))}(\mathcal{P}/\mathcal{F}_{\mathcal{P}})$ such that $\tilde{\omega}(\zeta^{\#}) = \zeta$. One now defines $\tilde{\omega}$ at other points on $\tilde{\tau}^{-1}(q(y))$ by requiring that $L_g^*\tilde{\omega} = Ad(g)\omega$. It is well-known that this construction gives rise to a well-defined smooth connection form on all of the bundle $\tilde{\tau}^{-1}(q(U)) \to q(U)$. Now we know $\mathcal{M}/\mathcal{F}_{\mathcal{M}}$ is covered by open sets $q(U)$ on which all of this is valid so it remains to show that if $\tilde{\mathfrak{s}}_1$ and $\tilde{\mathfrak{s}}_2$ are local sections of τ defined on a common open set U_{12}, then the connection $\tilde{\omega}$ is independent of which of the two local sections $\tilde{\mathfrak{s}}_1$ and $\tilde{\mathfrak{s}}_2$ is used to construct it. To see this, observe that we assume that $\tilde{\mathfrak{s}}_1^*\omega \approx 0$, $\tilde{\mathfrak{s}}_2^*\omega \approx 0$ and there exists $g : U_{12} \to \mathcal{G}$ such that $\tilde{\mathfrak{s}}_2 = g\tilde{\mathfrak{s}}_1$. Since $\tilde{\mathfrak{s}}_2^*\omega = Ad(g)[\tilde{\mathfrak{s}}_1^*\omega + g^{-1}dg]$, we see that $g^{-1}dg \approx 0$ and so $dg \approx 0$. It follows that g is constant on leaves in $\mathcal{F}_{\mathcal{M}}$. However, $\hat{\mathfrak{s}}_1$ and $\hat{\mathfrak{s}}_2$ are presumed to be local sections of $\tilde{\tau}$ defined on $q(U_{12})$ such that $\hat{\mathfrak{s}}_i \circ q = \rho \circ \tilde{\mathfrak{s}}_i$, $i = 1, 2$. Thus,

$$\hat{\mathfrak{s}}_2^*\tilde{\omega} = (\tilde{\mathfrak{s}}_2^*\omega) \circ dq = Ad(g \circ q)[\tilde{\mathfrak{s}}_1^*\omega + (g \circ q)^{-1}d(g \circ q)]$$

and consequently $\tilde{\omega}$ is a well-defined connection on all of $\mathcal{P}/\mathcal{F}_{\mathcal{P}}$. Theorem 8.1.19 follows. □

8.2 Quotient Space Approach to Pregauge Transformations

Assume in this section that the supermanifold $\mathcal{M}$ is locally modelled on the Banach space $\mathbb{R}^{4|4}$. Thus at each point we have a chart whose components are $(x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$.

Additionally we assume the existence of four odd vector fields X_1, X_2, Y_1, Y_2 defined on $\mathcal{M}$ such that Y_i is (super)conjugate to X_i for $i = 1, 2$, and we assume that for each point $p \in \mathcal{M}$.

$$E_p^0 = \{aX_1(p) + bX_2(p) \mid a, b \in {}^1\Lambda\}$$

is a ${}^0\Lambda$-submodule of $T_p^0\mathcal{M}$ of dimension $(2|0)$. Moreover we assume that $E^0 \to \mathcal{M}$ is an integrable super sub-bundle of $T^0\mathcal{M} \to \mathcal{M}$ (see [27]) and that each point $p \in \mathcal{M}$

there exists a chart $(U, x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ such that at each $q \in U$,

$$X_i(q) = \sum\nolimits_{\alpha=1}^{2} M_i^\alpha(q) D_\alpha(q)$$

for some supermatrix $M_i^\alpha(q)$. Here $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^m}$ (we follow [116], see page 26)

Now we have two foliations $\mathcal{F}^{chiral}_{\mathcal{M}}, \mathcal{F}^{antichiral}_{\mathcal{M}}$ of $\mathcal{M}$. Leaves $\bar{\mathcal{L}}$ of $\mathcal{F}^{antichiral}_{\mathcal{M}}$ satisfy $T^0_q\bar{\mathcal{L}} = E^0_q$, while leaves $\mathcal{L}$ of $\mathcal{F}^{chiral}_{\mathcal{M}}$ satisfy $T^0_q\mathcal{L} = \bar{E}^0_q$. where $\bar{E}^0_q = \{cY_1(q) + dY_2(q) \mid c, d \in {}^1\Lambda\}$ is the conjugate of E^0_q, $q \in \bar{\mathcal{L}} \cap \mathcal{L}$. Here we assume Y_1, Y_2 are linear combinations of $\{\bar{D}_{\dot{\alpha}}\}$ locally where $\bar{D}_{\dot{\alpha}} = \frac{\partial}{\partial\theta^{\dot{\alpha}}} - i\bar{\theta}^{\dot{\alpha}}\sigma^m_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^m}$. As sub supermanifolds of $\mathcal{M}$ both $\bar{\mathcal{L}}$ and $\mathcal{L}$ are modelled on a Banach spaces which as supervector spaces are isomorphic to $T^0_q\bar{\mathcal{L}} = E^0_q$ and $T^0_q\mathcal{L} = \bar{E}^0_q$, respectively. Thus $\bar{\mathcal{L}}$ and $\mathcal{L}$ are $(0|2)$ dimensional supermanifolds.

Note that if $E_q = \{aX_1(q) + bX_2(q) \mid a, b \in \Lambda\}$ then E_q is a super vector space over ${}^0\Lambda$ of dimension $(2|2)$ where the even part is E_q. As a sub-supermanifold $\bar{\mathcal{L}}$ is modelled on a Banach space which as a super vector space is isomorphic to $T^0_q\bar{\mathcal{L}} = E^0_q$. Thus $\bar{\mathcal{L}}$ is a $(0|2)$ dimensional supermanifold.

Note that $\mathcal{E} = E + \bar{E}$ is a sub super vector bundle of $T\mathcal{M}$ but is not integrable since in general $\{D_\alpha, \bar{D}_{\dot{\alpha}}\}$ does not close in $\mathcal{E}$ under the brackets of supervector fields (for example, see [116], page 26).

Each of the foliations $\mathcal{F}^{chiral}_{\mathcal{M}}$, $\mathcal{F}^{antichiral}_{\mathcal{M}}$ give rise to principal fiber bundles,

$$\tau^{chiral} : \mathcal{P}/\mathcal{F}^{chiral}_{\mathcal{P}} \to \mathcal{M}/\mathcal{F}^{chiral}_{\mathcal{M}}$$

$$\tau^{antichiral} : \mathcal{P}/\mathcal{F}^{antichiral}_{\mathcal{P}} \to \mathcal{M}/\mathcal{F}^{antichiral}_{\mathcal{M}}$$

where $\tau : \mathcal{P} \to \mathcal{M}$ is any super principal bundle over $\mathcal{M}$ with super Lie group $\mathcal{G}$. Let $E_{\mathcal{P}}$ and $\bar{E}_{\mathcal{P}}$ denote the subbundles of $T\mathcal{P} \to \mathcal{P}$ corresponding to the foliations $\mathcal{F}^{chiral}_{\mathcal{P}}$ and $\mathcal{F}^{antichiral}_{\mathcal{P}}$ respectively.
If ω is any even connection on $\mathcal{P}$ such that its curvature Ω_p is zero on pairs of vectors from E_q and such that Ω_p is also zero on pairs of vectors from $\bar{E}_q$ for each $q \in \mathcal{P}$ and if the Lie derivative of ω is zero along horizontal lifts of vectors tangent to leaves of the two foliations, then there are induced connections ω^{chiral} and $\omega^{antichiral}$ on the corresponding quotient bundles defined above. We regard these connections as reformulations of the superconnections ϕ, $\tilde{\phi}$ defined by Gieres on page 64 of [47]. Our formulation encodes the chiral and antichiral "pregauge transformations", usually regarded as maps $\Sigma, \Pi : U \to \mathcal{G}$ such that $\bar{D}_{\dot{\alpha}}\Sigma = 0$ and $D_\alpha\Pi = 0$, as ordinary gauge transformations $\hat{\Sigma}, \hat{\Pi}$ on our quotient bundles. Indeed, the conditions $\bar{D}_{\dot{\alpha}}\Sigma = 0$ and $D_\alpha\Pi = 0$ may be regarded as requiring that $d\Sigma(\bar{D}_{\dot{\alpha}}) = 0$ and $d\Pi(D_\alpha) = 0$ which is

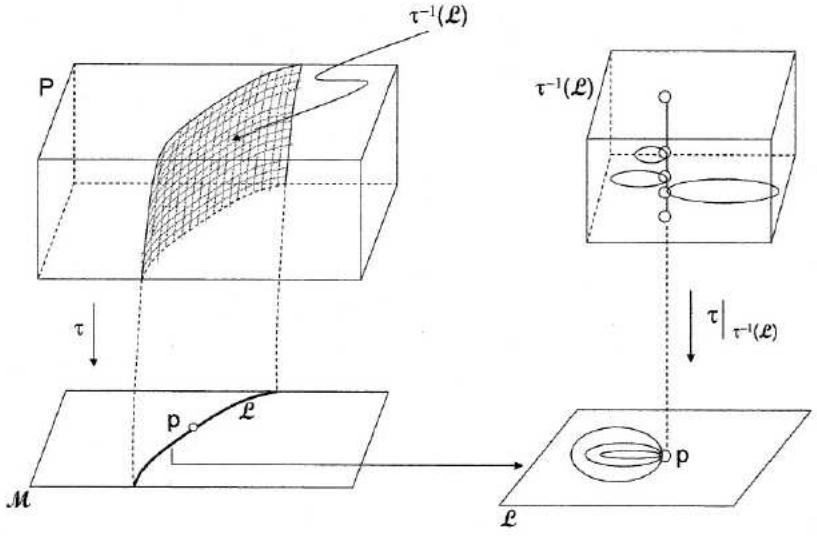

Figure 8.5: Loops Lift to Loops in Gauge Flat Leaf of Foliation

equivalent to saying that Σ and Π are constant on the leaves of $\mathcal{F}_{\mathcal{M}}^{chiral}$ and $\mathcal{F}_{\mathcal{M}}^{antichiral}$ respectively. Thus Σ, Π induce maps $\hat{\Sigma}, \hat{\Pi}$ on the space on which leaves are collapsed to points, and these become ordinary gauge transformations.
It should be emphasized that to make contact with the physics literature one must continue to work on the bundle $\tau : \mathcal{P} \to \mathcal{M}$; our quotient formalism merely provides a clearer conceptual framework at this point. The reason for this is that to obtain $N = 1$ super Yang-Mills theory, one must introduce additional constraints on Ω called the "conventional constraints". This constraint requires that for $q \in \mathcal{M}$ and $v \in E_q$, $w \in \bar{E}_q$, $\Omega_q(v, w) = 0$.

Remark 8.2.1. *We note that the constraints of super Yang-Mills theory have also been studied by Bartocci and Bruzzo in [10] where they related the constraints to Weil triviality. We would be interested in understanding better the connection of their result and the work we present in this chapter.*

We emphasize, however,that even before these extra constraints are imposed, we know that at each point of $\mathcal{M}$ there exists an open set U on which one has coordinates $(x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ and local sections $\mathfrak{s}_1$, $\mathfrak{s}_2$ of τ defined on U such that for maps $\mathcal{U}, \mathcal{V} : U \to \mathcal{G}$

$$(\mathfrak{s}_1^* \omega)_q(v) = -\mathcal{U}(q)^{-1} d_q \mathcal{U}(v)$$

for $q \in U$, $v \in \bar{E}_q$ and

$$(\mathfrak{s}_2^* \omega)_q(w) = -\mathcal{V}(q)^{-1} d_q \mathcal{V}(w)$$

for $q \in U$, $w \in E_q$. Moreover,

$$(\mathfrak{s}_1^*\Omega)(v_1, v_2) = 0$$

for $v_1, v_2 \in \bar{E}_q$ and

$$(\mathfrak{s}_2^*\Omega)(w_1, w_2) = 0$$

for $w_1, w_2 \in E_q$.

It is often inconvenient to have two gauges $\mathfrak{s}_1, \mathfrak{s}_2$ when one will do. Let $\bar{\mathfrak{s}} = \mathfrak{s}_1$ so that we have $\bar{\mathfrak{s}}^*\omega \overset{\text{chiral}}{\approx} -\mathcal{U}^{-1}d\mathcal{U}$ (on $\bar{E}$). Observe that there exists $g : U \to \mathcal{G}$ such that $\bar{\mathfrak{s}} = g\mathfrak{s}_2$ and consequently

$$\begin{aligned} \bar{\mathfrak{s}}^*\omega &= Ad(g)[(\mathfrak{s}_2^*\omega) + g^{-1}dg] \\ &= g(\mathfrak{s}_2^*\omega)g^{-1} + (dg)g^{-1} \end{aligned} \tag{8.24}$$

Thus on E

$$\begin{aligned} \bar{\mathfrak{s}}^*\omega &= -g\mathcal{V}^{-1}(d\mathcal{V})g^{-1} + (dg)g^{-1} \\ &= -(\mathcal{V}g^{-1})^{-1}d(\mathcal{V}g^{-1}) + g\mathcal{V}^{-1}\mathcal{V}dg^{-1} + (dg)g^{-1} \\ &= -(\mathcal{V}g^{-1})^{-1}d(\mathcal{V}g^{-1}) + d(gg^{-1}) \\ &= -(\mathcal{V}g^{-1})^{-1}d(\mathcal{V}g^{-1}). \end{aligned} \tag{8.25}$$

So if we replace $\mathcal{V}$ by $\mathcal{V}g^{-1}$ we have a single gauge $\bar{\mathfrak{s}}$ on U such that the connection ω pulls back to a pure gauge along both the chiral and antichiral leaves

$$(\bar{\mathfrak{s}}^*\omega)|_E = -\mathcal{V}^{-1}d\mathcal{V}|_E \qquad (\bar{\mathfrak{s}}^*\omega)|_{\bar{E}} = -\mathcal{U}^{-1}d\mathcal{U}|_{\bar{E}}$$

Moreover the pullback of the curvature Ω vanishes on pairs of chiral and antichiral vectors,

$$(\bar{\mathfrak{s}}^*\Omega)|_{E\times E} = 0 \qquad (\bar{\mathfrak{s}}^*\omega)|_{\bar{E}\times\bar{E}} = 0.$$

Recall that if $\mathbb{A}$ is a gauge field on an open subset of $\mathcal{M}$ then Giere's definition of a formal gauge transformation is

$${}^{\mathcal{X}}\mathbb{A} = \mathcal{X}^{-1}\mathbb{A}\mathcal{X} - \mathcal{X}^{-1}d\mathcal{X} \tag{8.26}$$

If $\mathfrak{s}$ is a local section of τ such that $\mathbb{A} = \mathfrak{s}^*\omega$ and if we define $\tilde{\mathfrak{s}} = \mathcal{X}^{-1}\mathfrak{s}$, we have

$$\begin{aligned} \tilde{\mathfrak{s}}^*\omega &= Ad(\mathcal{X}^{-1})[(\mathfrak{s}^*\omega) + (\mathcal{X}^{-1})^{-1}d\mathcal{X}^{-1}] \\ &= \mathcal{X}^{-1}\mathbb{A}\mathcal{X} + (d\mathcal{X}^{-1})\mathcal{X} \\ &= \mathcal{X}^{-1}\mathbb{A}\mathcal{X} - \mathcal{X}^{-1}d\mathcal{X}. \end{aligned} \tag{8.27}$$

Thus his gauge transformation requires us to use left actions on the principal bundle and to transform via $\mathcal{X}^{-1}$. This combination is equivalent to working with a right actions on principal bundles.

Throughout the remainder of this section $\mathfrak{s}$ will denote an arbitrary local section $\mathfrak{s} : U \to \mathcal{P}$ of τ such that the connection ω pulls back to a pure gauge along both the chiral and antichiral leaves

$$(\mathfrak{s}^*\omega)|_E = -\mathcal{V}^{-1}d\mathcal{V}|_E \qquad (\mathfrak{s}^*\omega)|_{\bar{E}} = -\mathcal{U}^{-1}d\mathcal{U}|_{\bar{E}}$$

And the pullback of the curvature Ω vanishes on pairs of chiral vectors and on pairs of antichiral vectors,

$$(\mathfrak{s}^*\Omega)|_{E\times E} = 0 \qquad (\mathfrak{s}^*\omega)|_{\bar{E}\times\bar{E}} = 0.$$

where $\mathcal{U}, \mathcal{V} : U \to \mathcal{G}$ are (super)smooth functions. We know such local sections exist at every point in $\mathcal{M}$.

Now define $\mathbb{A} = \mathfrak{s}^*\omega$ and $\mathbb{F} = \mathfrak{s}^*\Omega$. We see that

$$\mathbb{A}_\alpha = \mathbb{A}(D_\alpha) = (\mathfrak{s}^*\omega)(D_\alpha) = -\mathcal{V}^{-1}d\mathcal{V}(D_\alpha) = -\mathcal{V}^{-1}D_\alpha\mathcal{V}$$

and likewise

$$\mathbb{A}_{\dot\alpha} = \mathbb{A}(\bar{D}_{\dot\alpha}) = (\mathfrak{s}^*\omega)(\bar{D}_{\dot\alpha}) = -\mathcal{U}^{-1}d\mathcal{U}(\bar{D}_{\dot\alpha}) = -\mathcal{U}^{-1}\bar{D}_{\dot\alpha}\mathcal{U}$$

Moreover

$$\mathbb{F}_{\alpha\beta} = (\mathfrak{s}^*\omega)(D_\alpha, D_\beta), \qquad \mathbb{F}_{\dot\alpha\dot\beta} = (\mathfrak{s}^*\omega)(\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}), \qquad \mathbb{F}_{\alpha\dot\beta} = (\mathfrak{s}^*\omega)(D_\alpha, \bar{D}_{\dot\beta})$$

The constraints $\mathbb{F}_{\alpha\beta} = 0$, $\mathbb{F}_{\dot\alpha\dot\beta} = 0$ are precisely the conditions $(\mathfrak{s}^*\Omega)(E\times E) = 0$, $(\mathfrak{s}^*\omega)(\bar{E}\times\bar{E}) = 0$ respectively. The constraint $\mathbb{F}_{\alpha\dot\beta} = 0$ is called the "conventional constraint", and it can be stated in our bundle language as $(\mathfrak{s}^*\Omega)(E\times\bar{E}) = 0$.

We believe Giere's definitions of the superconnections ϕ, $\tilde\phi$ on page 62 of [47] are flawed. It seems certain what he wants are connections which are zero in chiral and antichiral directions (respectively) but which agree with $\mathbb{A}$ in transverse directions. If we are correct, then he should have

$$\begin{aligned} \phi &= \mathcal{U}\mathbb{A}\mathcal{U}^{-1} + (d\mathcal{U})\mathcal{U}^{-1} \\ \tilde\phi &= \mathcal{V}\mathbb{A}\mathcal{V}^{-1} + (d\mathcal{V})\mathcal{V}^{-1}. \end{aligned} \tag{8.28}$$

It will then follow, for example, that on $\bar{E}$

$$\phi = \mathcal{U}\mathbb{A}\mathcal{U}^{-1} + (d\mathcal{U})\mathcal{U}^{-1} \overset{\text{chiral}}{\approx} \mathcal{U}(-\mathcal{U}^{-1})d\mathcal{U}\mathcal{U}^{-1} + (d\mathcal{U})\mathcal{U}^{-1} \tag{8.29}$$

hence,

$$\phi \overset{\text{chiral}}{\approx} -(d\mathcal{U})\mathcal{U}^{-1} + (d\mathcal{U})\mathcal{U}^{-1} = 0. \tag{8.30}$$

A similar result applies for $\tilde{\phi}$ over E (replace $\overset{\text{chiral}}{\approx}$ with $\overset{\text{antichiral}}{\approx}$ etc...); $\tilde{\phi} \overset{\text{antichiral}}{\approx} 0$. If this is the case, then his ϕ, $\tilde{\phi}$ induce our connections ω^{chiral} on $\mathcal{P}/\mathcal{F}_{\mathcal{P}}^{chiral} \to \mathcal{M}/\mathcal{F}_{\mathcal{M}}^{chiral}$ and $\omega^{antichiral}$ on $\mathcal{P}/\mathcal{F}_{\mathcal{P}}^{antichiral} \to \mathcal{M}/\mathcal{F}_{\mathcal{M}}^{antichiral}$ as defined more generally in the previous sections of this chapter.

Now notice that if we define local sections of τ by $\mathfrak{s}_{chiral} = \mathcal{U}\mathfrak{s}$ and $\mathfrak{s}_{antichiral} = \mathcal{V}\mathfrak{s}$ then

$$\begin{aligned} \mathfrak{s}_{chiral}^{*}\omega &= Ad(\mathcal{U})[(\mathfrak{s}^{*}\omega) + \mathcal{U}^{-1}d\mathcal{U}] \\ &= \mathcal{U}\mathbb{A}\mathcal{U}^{-1} + (d\mathcal{U})\mathcal{U}^{-1} \\ &= \phi. \end{aligned} \tag{8.31}$$

Similarly we can derive $\mathfrak{s}_{antichiral}^{*}\omega = \tilde{\phi}$. Note also that we can connect both of these sections by the equation $\mathfrak{s}_{antichiral} = \mathcal{V}\mathcal{U}^{-1}\mathfrak{s}_{chiral}$ so that if $\mathcal{W} = \mathcal{V}\mathcal{U}^{-1}$, then

$$\begin{aligned} \tilde{\phi} = (\mathfrak{s}_{antichiral}^{*}\omega) &= Ad(\mathcal{W})[(\mathfrak{s}_{chiral}^{*}\omega) + \mathcal{W}^{-1}d\mathcal{W}] \\ &= \mathcal{W}\phi\mathcal{W}^{-1} + (d\mathcal{W})\mathcal{W}^{-1} \\ &= \mathcal{W}\phi\mathcal{W}^{-1} - \mathcal{W}d\mathcal{W}^{-1}. \end{aligned} \tag{8.32}$$

Remark 8.2.2. *In our view ϕ, $\tilde{\phi}$, and $\mathbb{A}$ are all on the same conceptual level; they are local gauge superfields which represent influence of the connection ω locally on the base supermanifold. The various special sections we have considered and constructed are chosen so that we can make contact with the physicists coordinate-dependent arguments. As is typical with physics, the choice of special coordinates enables a simple solution to otherwise intractable equations.*

Giere's Equation 2.58 in [47] also agrees with us that $\tilde{\phi} = \mathcal{W}\phi\mathcal{W}^{-1} - \mathcal{W}d\mathcal{W}^{-1}$ where $\mathcal{W} = \mathcal{V}\mathcal{U}^{-1}$ Also observe that since $\phi = \mathcal{U}\mathbb{A}\mathcal{U}^{-1} + (d\mathcal{U})\mathcal{U}^{-1}$, we may calculate that,

$$\begin{aligned} \phi_{\dot{\alpha}} = \phi(\bar{D}_{\dot{\alpha}}) &= \mathcal{U}\mathbb{A}_{\dot{\alpha}}\mathcal{U}^{-1} + (d\mathcal{U})(\bar{D}_{\dot{\alpha}})\mathcal{U}^{-1} \\ &= -\mathcal{U}(\mathcal{U}^{-1}\bar{D}_{\dot{\alpha}}\mathcal{U})\mathcal{U}^{-1} + (\bar{D}_{\dot{\alpha}}\mathcal{U})\mathcal{U}^{-1} \\ &= 0 \end{aligned} \tag{8.33}$$

and

$$\begin{aligned} \phi_{\alpha} = \phi(D_{\alpha}) &= \mathcal{U}\mathbb{A}_{\alpha}\mathcal{U}^{-1} + (d\mathcal{U})(D_{\alpha})\mathcal{U}^{-1} \\ &= -\mathcal{U}(\mathcal{V}^{-1}D_{\alpha}\mathcal{V})\mathcal{U}^{-1} + (D_{\alpha}\mathcal{U})\mathcal{U}^{-1} \\ &= -\mathcal{W}^{-1}(D_{\alpha}\mathcal{W}\mathcal{U})\mathcal{U}^{-1} + (D_{\alpha}\mathcal{U})\mathcal{U}^{-1} \\ &= -\mathcal{W}^{-1}(D_{\alpha}\mathcal{W})\mathcal{U}\mathcal{U}^{-1} - \mathcal{W}^{-1}\mathcal{W}(D_{\alpha}\mathcal{U})\mathcal{U}^{-1} + (D_{\alpha}\mathcal{U})\mathcal{U}^{-1} \\ &= -\mathcal{W}^{-1}D_{\alpha}\mathcal{W}. \end{aligned} \tag{8.34}$$

Similar equations hold for $\tilde{\phi}_{\alpha}, \tilde{\phi}_{\dot{\alpha}}$ and agree with Giere's equations (2.65).

Giere's denoted the curvatures of ϕ, $\tilde{\phi}$ by $\mathcal{F}$, $\bar{\mathcal{F}}$ respectively. In our notation

$$\mathcal{F} = \mathfrak{s}^*_{chiral}\Omega = \mathcal{U}(\mathfrak{s}^*\Omega)\mathcal{U}^{-1} = \mathcal{U}\mathbb{F}\mathcal{U}^{-1}$$

and

$$\bar{\mathcal{F}} = \mathfrak{s}^*_{antichiral}\Omega = \mathcal{V}(\mathfrak{s}^*\Omega)\mathcal{V}^{-1} = \mathcal{V}\mathbb{F}\mathcal{V}^{-1}$$

thus $\bar{\mathcal{F}} = \mathcal{W}\mathcal{F}\mathcal{W}^{-1}$.

8.3 Locally Supersymmetric Superspace

In this section we endeavor to describe a general class of supermanifolds which are locally diffeomorphic to $\mathbb{R}^{4|4}$. Let (M, g) be a Lorentzian spacetime and assume that M has a spin structure, then there is a double cover SM of the g-orthonormal frame bundle O_gM with structure group $SL(2,\mathbb{C})$ the double cover of $SO^+(1,3)$ (see [16]). Note that $SL(2,\mathbb{C})$ act on frames $\{e_i\} \in O_gM$ via $\{e_i\} \cdot S \equiv \{e_j \Lambda(S^{-1})^j_i\}$ and acts on $\mathbb{R}^{0|4}$ via $(\theta, \bar{\theta}) \cdot S \equiv (\theta^\beta S^\alpha_\beta, \bar{\theta}^{\dot{\beta}} \bar{S}^{\dot{\alpha}}_{\dot{\beta}})$. We then let $SL(2,\mathbb{C})$ act on the associated bundle $\mathcal{M} = O_gM \times_\rho \mathbb{R}^4$ in the standard way. The associated bundle constructed in this manner has the structure of a supermanifold locally modeled on $\mathbb{R}^{4|4}$.

We assume there exists an atlas $\mathcal{A}$ on $\mathcal{M}$ such that if $(x, \theta, \bar{\theta})$ and $(\tilde{x}, \tilde{\theta}, \tilde{\bar{\theta}})$ are two charts with intersecting domains, then

$$\begin{aligned} \tilde{x}^n &= g^n(x) \\ \tilde{\theta}^\alpha &= S^\alpha_\beta(x)\theta^\beta \\ \tilde{\bar{\theta}}^{\dot{\alpha}} &= \bar{S}^{\dot{\alpha}}_{\dot{\beta}}(x)\bar{\theta}^{\dot{\beta}} \end{aligned} \tag{8.35}$$

where $x = (x^0, x^1, x^2, x^3) \in \mathbb{R}^{4|0}$ and $S(x) \in SL(2,\mathbb{C})$. Thus $x \mapsto S^\alpha_\beta(x)$ are transition functions induced from those of the spin bundle SM as in Theorem 8.1.1 of [102].

As in the previous section we assume the existence of two integrable subbundles $\mathcal{E}$,$\dot{\mathcal{E}}$ of $T^0\mathcal{M} \to \mathcal{M}$ such that relative to a chart $(x, \theta, \bar{\theta})$ in $\mathcal{A}$ we have $\mathcal{E} =< D_\alpha >$, $\dot{\mathcal{E}} =< \bar{D}_{\dot{\alpha}} >$ where

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^m} \qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^m} \tag{8.36}$$

We consider a subbundle $\mathcal{Z}$ of the frame bundle of M defined by

$$\mathcal{Z} = \{(e_a, e_\alpha, e_{\dot{\alpha}}) \mid \{e_a\} \in O_gM, e_\alpha \in \mathcal{E}, e_{\dot{\alpha}} \in \dot{\mathcal{E}}\}$$

where we identify a frame $\{e_a\}$ of M with its corresponding frame in $\mathcal{M}$ defined as follows.

First recall that the spin connection on SM induces a spin connection on each

of its associated bundles, and thus it induces one on $\tau_B : \mathcal{M} \to M$. Let $\mathcal{H} \subseteq T^0\mathcal{M}$ be the subbundle of horizontal vectors relative to the induced connection. Now we identify $\{e_a\}$ with $(d\tau_B|\mathcal{H})^{-1}(e_a)$ a basis of the bundle $\mathcal{H} \to M$ at each point of the fiber $\tau_B^{-1}(p)$ of $\mathcal{M}$ over the point $p \in M$ at which $\{e_a\}$ is defined. Relative to a chart $(x, \theta, \bar{\theta})$ in $\mathcal{A}$, points of $\mathcal{Z}$ take the form,

$$(e^m{}_a \frac{\partial}{\partial x^m}, \lambda^\alpha D_\alpha, \mu^{\dot\beta} \bar{D}_{\dot\beta}) \tag{8.37}$$

where $e^m_a \in {}^0\Lambda$ and $\lambda^\alpha, \mu^{\dot\beta} \in {}^1\Lambda$. The local bases $\{D_\alpha\}$, $\{\bar{D}_{\dot\alpha}\}$ of $\mathcal{E}$, $\dot{\mathcal{E}}$ respectively can be reformulated in terms of the frame $\{e_a\}$ as follows. We denote the inverse of $e^m{}_a$ by $e_m{}^a$. Thus while $e_a = e^m{}_a \frac{\partial}{\partial x^m}$, we also have $\frac{\partial}{\partial x^m} = e_m{}^a e_a$. These matrices $e^m{}_a$ serve to convert "curved" indices to "Lorentz" indices. For example, on the entire chart domain we can define $\sigma^a_{\alpha\dot\alpha}$

$$\sigma^a_{\alpha\dot\alpha} \equiv e^a{}_m \sigma^m_{\alpha\dot\alpha}$$

so that $\sigma^a_{\alpha\dot\alpha}$ transforms via the Lorentz index a. In terms of the frames we can rewrite equation 8.36

$$\begin{aligned} D_\alpha &= \tfrac{\partial}{\partial\theta^\alpha} + i\sigma^a_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} e_a \\ \bar{D}_{\dot\alpha} &= -\tfrac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha \sigma^a_{\alpha\dot\alpha} e_a. \end{aligned} \tag{8.38}$$

If the frame $\{e_a\}$ is transformed to $\{\tilde{e}_b\}$ then $\tilde{e}_b = l^a_b e_a$ for some $l \in SO^+(1,3)$ and

$$\sigma^a_{\alpha\dot\alpha} e_a = \tilde\sigma^a_{\alpha\dot\alpha} \tilde{e}_a$$

is invariant. Notice that the structure group of $\mathcal{Z}$ is $SO^+(1,3) \times SL(2,\mathbb{C})$ since two frames $(\tilde{e}_a, \tilde{e}_\alpha, \tilde{e}_{\dot\alpha})$ and $(e_a, e_\alpha, e_{\dot\alpha})$ are related by the equations,

$$\tilde{e}_a = l^b_a e_b, \qquad \tilde{e}_\alpha = S^\beta_\alpha e_\beta, \qquad \tilde{e}_{\dot\alpha} = \bar{S}^{\dot\beta}_{\dot\alpha} e_{\dot\beta}$$

for $l \in SO^+(1,3)$, $S \in SL(2,\mathbb{C})$.

To understand how these relate to one another in local coordinates observe that if we denote the inverse of the matrix l by $\tilde{l}$, then $e_b = e^k{}_b \frac{\partial}{\partial x^k}$ and $e_b = \tilde{l}^a{}_b \tilde{e}_a = \tilde{l}^a{}_b \tilde{e}^m{}_a \frac{\partial}{\partial \tilde{x}^m} = \tilde{l}^a{}_b \tilde{e}^m{}_a \frac{\partial x^n}{\partial \tilde{x}^m} \frac{\partial}{\partial x^n}$ so $e^n{}_b = \tilde{l}^a{}_b \tilde{e}^m{}_a \frac{\partial x^n}{\partial \tilde{x}^m}$. Thus locally a change of frame in M is encoded as a tensor transformation law.

To proceed we require a reduction $\mathcal{Z}_0$ of $\mathcal{Z}$ to the group $SL(2,\mathbb{C})$. We <u>assume</u> that $\mathcal{Z}_0 \subseteq \mathcal{Z}$ such that if $(\tilde{e}_a, \tilde{e}_\alpha, \tilde{e}_{\dot\alpha})$ and $(e_a, e_\alpha, e_{\dot\alpha})$ both belong to $\mathcal{Z}_0$, then there exists $S \in SL(2,\mathbb{C})$ such that

$$\begin{aligned} \tilde{e}_a &= \Lambda(S^{-1})^b_a e_b \\ \tilde{e}_\beta &= S^\alpha_\beta e_\alpha \\ \tilde{e}_{\dot\alpha} &= \bar{S}^{\dot\beta}_{\dot\alpha} e_{\dot\beta} \end{aligned} \tag{8.39}$$

where $\Lambda(S^{-1})$ indicates the double cover map of $SL(2,\mathbb{C})$ onto $SO^+(1,3)$ (see [16]).

Remark 8.3.1. *We do not know when such reductions are possible generally. Clearly if $\mathcal{A}$ has one global chart as in the rigid superspace case, this reduction holds. Also if $\mathcal{M} = M \times \mathbb{R}^{0|4}$ is trivial this reduction is possible. In general it could depend on the foliations $\mathcal{E}$, $\dot{\mathcal{E}}$ and so there may be topological obstructions. We would be interested in knowing the answer to this question.*

We now briefly explore the local consequences of assuming the existence of such a reduction of $\mathcal{Z}_0$ of $\mathcal{Z}$ to the group $SL(2,\mathbb{C})$.
If there exists such a reduction $\mathcal{Z}_0$ and if one has $(e_a, D_\alpha, \bar{D}_{\dot\alpha}) \in \mathcal{Z}_0$ for $e_a = e^m{}_a \frac{\partial}{\partial x^m}$ relative to a chart $(x,\theta,\bar\theta) \in \mathcal{A}$ and if $(\tilde e_a, \tilde D_\alpha, \tilde{\bar D}_{\dot\alpha})$ is also in $\mathcal{Z}_0$ relative to another overlapping chart $(\tilde x, \tilde\theta, \tilde{\bar\theta})$, then one has

$$\tilde e_a = \tilde e^m{}_a \frac{\partial}{\partial \tilde x^m}, \qquad e_a = e^m{}_a \frac{\partial}{\partial x^m}, \qquad \tilde e_b = \Lambda(S^{-1})^b_a e_b \tag{8.40}$$

$$\tilde D_\alpha = S^\beta_\alpha D_\beta \qquad \tilde{\bar D}_{\dot\alpha} = \bar S^{\dot\beta}_{\dot\alpha} \bar D_{\dot\beta} \tag{8.41}$$

relative to these charts. Recall that

$$\begin{aligned} \tilde x^n &= g^n(x) \\ \tilde\theta^\alpha &= S^\alpha_\beta(x)\theta^\beta \\ \tilde{\bar\theta}^{\dot\alpha} &= \bar S^{\dot\alpha}_{\dot\beta}(x)\bar\theta^{\dot\beta} \end{aligned} \tag{8.42}$$

and observe that we may choose curved coordinates such that at a point they are Lorentzian, hence

$$S^\alpha_\beta \sigma^m_{\alpha\dot\alpha} \bar S^{\dot\alpha}_{\dot\beta} \Lambda(S^{-1})^n_m = \sigma^n_{\beta\dot\beta}.$$

Now multiply by $e_n{}^a$ to convert n to the Lorentz index a,

$$S^\alpha_\beta \sigma^m_{\alpha\dot\alpha} \bar S^{\dot\alpha}_{\dot\beta} \Lambda(S^{-1})^n_m e_n{}^a = \sigma^a_{\beta\dot\beta}$$

Then rewrite $\sigma^m_{\alpha\dot\alpha} = \sigma^b_{\alpha\dot\alpha} e^m{}_b$

$$S^\alpha_\beta \sigma^b_{\alpha\dot\alpha} \bar S^{\dot\alpha}_{\dot\beta} e^m{}_b \Lambda(S^{-1})^n_m e_n{}^a = \sigma^a_{\beta\dot\beta}$$

Observe that $e^m{}_b \Lambda(S^{-1})^n_m e_n{}^a = \Lambda(S^{-1})^a_b$. Thus,

$$S^\alpha_\beta \sigma^b_{\alpha\dot\alpha} \bar S^{\dot\alpha}_{\dot\beta} \Lambda(S^{-1})^a_b = \sigma^a_{\beta\dot\beta}$$

which is precisely the Lorentz index version of the identity.

Recall from the calculation above that a change in frame is locally encoded by the tensor transformation law $e^n{}_b = \tilde l^a{}_b \tilde e^m{}_a \frac{\partial x^n}{\partial \tilde x^m}$. In the reduced bundle $\mathcal{Z}_0$ the matrix

l takes the form $l = \Lambda(S)$. Consequently in the reduced bundle this transformation law becomes

$$\boxed{e^n{}_b = \Lambda(S^{-1})^a{}_b \tilde{e}^m{}_a \frac{\partial x^n}{\partial \tilde{x}^m}}$$

Thus $(e^n{}_b)$ is a tensor in both n, b indices except one has that $l = \Lambda(S)$ when we require a reduction. Notice that

$$\frac{\partial}{\partial \tilde{\theta}^\alpha} = \frac{\partial \theta^\beta}{\partial \tilde{\theta}^\alpha} \frac{\partial}{\partial \theta^\beta} = (S^{-1})^\beta_\alpha \frac{\partial}{\partial \theta^\beta}$$

where the matrix S is a function of x which is in the intersection of the domains of the two charts. We suppress this dependence below for simplicity. Consequently,

$$\begin{aligned} \tilde{D}_\alpha &= \tfrac{\partial}{\partial \tilde{\theta}^\alpha} + i\sigma^a_{\alpha\dot{\alpha}} \tilde{\bar{\theta}}^{\dot{\alpha}} \tilde{e}_a \\ &= (S^{-1})^\beta_\alpha \tfrac{\partial}{\partial \theta^\beta} + i\bar{S}^{\dot{\alpha}}_{\dot{\beta}} \bar{\theta}^{\dot{\beta}} \sigma^a_{\alpha\dot{\alpha}} \Lambda(S^{-1})^b_a e_b \\ &= (S^{-1})^\beta_\alpha \tfrac{\partial}{\partial \theta^\beta} + i\bar{\theta}^{\dot{\beta}} \bar{S}^{\dot{\alpha}}_{\dot{\beta}} \sigma^a_{\alpha\dot{\alpha}} \Lambda(S^{-1})^b_a e_b \\ &= (S^{-1})^\beta_\alpha \tfrac{\partial}{\partial \theta^\beta} + i\bar{\theta}^{\dot{\beta}} (S^{-1})^\beta_\alpha \sigma^b_{\beta\dot{\beta}} e_b \\ &= (S^{-1})^\beta_\alpha [\tfrac{\partial}{\partial \theta^\beta} + i\bar{\theta}^{\dot{\beta}} \sigma^b_{\beta\dot{\beta}} e_b] \\ &= (S^{-1})^\beta_\alpha D_\beta \end{aligned} \tag{8.43}$$

from which it follows that upon change of chart, D_α transforms like a spinor.

8.4 Consequences of the Bianchi identities

At this point we consider the implications of the Bianchi identity coupled with our restrictions on the curvature. Recall that the Bianchi identity is $0 = D\Omega = d\Omega + \Omega \wedge \omega$ which in a local gauge assumes the form $d\mathbb{F} + \mathbb{F} \wedge \mathbb{A} = 0$ (see page 97 of [116]). Wess and Bagger show on pages 104-105 of [116] that in coordinates the constraints

$$\mathbb{F}_{\alpha\beta} = \mathbb{F}_{\dot{\alpha}\dot{\beta}} = \mathbb{F}_{\alpha\dot{\beta}} = 0 \tag{8.44}$$

along with the Bianchi identities imply the existence of fields $W^\alpha, \overline{W}^{\dot{\alpha}}$ such that

$$\begin{aligned} \mathbb{F}_{a\alpha} &= -i\sigma_{a\alpha\dot{\beta}} \overline{W}^{\dot{\beta}} \\ \mathbb{F}_{a\dot{\alpha}} &= -iW^\beta \sigma_{a\beta\dot{\alpha}\dot{\beta}}. \end{aligned} \tag{8.45}$$

In our situation we have assumed the existence of a reduction $\mathcal{Z}_0$ of the bundle $\mathcal{Z}$ to an $Sl(2, \mathbb{C})$ subbundle. So locally there exist vector fields $e_a, e_\alpha, e_{\dot{\alpha}}$ such that at each point p of an open subset of $\mathcal{M}$,

$$(e_a(p), e_\alpha(p), e_{\dot{\alpha}}(p)) \in \mathcal{Z}_0.$$

If we define

$$\mathbb{F}_{a\alpha} = \mathbb{F}(e_a, e_\alpha), \mathbb{F}_{\alpha\beta} = \mathbb{F}(e_\alpha, e_\beta), \mathbb{F}_{\dot\alpha\dot\beta} = \mathbb{F}(e_{\dot\alpha}, e_{\dot\beta}), \mathbb{F}_{a,\dot\alpha} = \mathbb{F}(e_a, e_{\dot\alpha}), \mathbb{F}_{\dot\alpha,a} = \mathbb{F}(e_{\dot\alpha}, e_a)$$

and invoke the Bianchi identities along with the constraints in Equation 8.44, then calculations analogous to those in Wess and Bagger yield the identity

$$(-2i)\sigma^a_{\beta\dot\alpha}\mathbb{F}_{\alpha a} + (-2i)\sigma^a_{\alpha\dot\alpha}\mathbb{F}_{\alpha\beta} = 0 \tag{8.46}$$

along with an additional identity which arises since we work in a nonholonomic frame, namely

$$f^d_{bc}\mathbb{F}_{ad} - f^d_{ac}\mathbb{F}_{db} = 0 \tag{8.47}$$

where

$$f^a_{bc} = e_b{}^m de_m{}^a(e_c). \tag{8.48}$$

Notice that the identities are written in terms of the vector fields $e_a, e_\alpha, e_{\dot\alpha}$. It follows from Equation 8.44 and 8.46 that we can define spinor fields

$$\begin{aligned} W^\alpha &= (-i/4)\mathbb{F}_{a\dot\alpha}\overline{\sigma}^{a\dot\alpha\alpha} \\ \overline{W}^{\dot\alpha} &= (-i/4)\mathbb{F}_{a\alpha}\overline{\sigma}^{a\dot\alpha\alpha}. \end{aligned} \tag{8.49}$$

The only difference in these fields from those of Wess and Bagger is that our fields are defined in terms of local sections of the bundle $\mathcal{Z}_0 \to \mathcal{M}$ and so are independent of local data. To see this for W^α, let $(\tilde{x}^m, \tilde{\theta}^\alpha, \tilde{\overline{\theta}}^{\dot\alpha})$ and $(x^m, \theta^\alpha, \overline{\theta}^{\dot\alpha})$ be charts in our atlas $\mathcal{A}$ with domains $\tilde{U} \subseteq \mathcal{M}, U \subseteq \mathcal{M}$, respectively. On these domains we have local sections $(\tilde{e}_a, \tilde{e}_\alpha, \tilde{e}_{\dot\alpha})$ and $(e_a, e_\alpha, e_{\dot\alpha})$ of $\mathcal{Z}_0$ defined in terms of charts as in Equation 8.37. We have that

$$\tilde{e}_a = \Lambda(S^{-1})^b_a e_b \qquad \tilde{e}_\alpha = S_\alpha{}^\beta e_\beta \qquad \tilde{e}_{\dot\alpha} = \overline{S}_{\dot\alpha}{}^{\dot\beta} e_{\dot\beta}. \tag{8.50}$$

Now

$$\begin{aligned} \tilde{W}^\alpha &= (-i/4)\mathbb{F}(\tilde{e}_a, \tilde{e}_{\dot\alpha})\overline{\sigma}^{a\dot\alpha\alpha} \\ &= (-i/4)\Lambda(S^{-1})^b_a \overline{S}_{\dot\alpha}{}^{\dot\beta}\mathbb{F}(e_b, e_{\dot\beta})\overline{\sigma}^{a\dot\alpha\alpha} \\ &= (-i/4)\mathbb{F}(e_b, e_{\dot\beta})\Lambda(S^{-1})^b_a \overline{S}_{\dot\alpha}{}^{\dot\beta}\overline{\sigma}^{a\dot\alpha\alpha} \\ &= (-i/4)\mathbb{F}(e_b, e_{\dot\beta})\overline{\sigma}^{b\dot\beta\beta}(S^{-1})^\alpha_\beta \\ &= W^\beta(S^{-1})^\alpha_\beta. \end{aligned} \tag{8.51}$$

Since the two sets of frames of $\mathcal{Z}_0$ are related by $S \in Sl(2,\mathbb{C})$, this shows that one has a well-defined spinor field $\hat{W}$ defined on the bundle $\mathcal{Z}_0$ which in local sections $\tilde{s}, s$ are related to the fields $\tilde{W}, W$ by $\tilde{s}^*\hat{W} = \tilde{W}, s^*\hat{W} = W$. We see that $W^\alpha W_\alpha$ is an Lorentz invariant on $\mathcal{M}$.

We can show through very similar arguments that $\bar{W}_{\dot\alpha}$ transforms as a dotted-

Weyl spinor on an overlap. And $\bar{W}^{\dot{\alpha}}$ transforms inversely to $\bar{W}_{\dot{\alpha}}$ with respect to a Lorentz transformation. Thus $\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}$ forms a Lorentz scalar.

Finally define an action S_W in the usual way,

$$S_W = \int d^4x \; tr\Big[W^\alpha W_\alpha|_{\theta\theta} + \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}|_{\bar{\theta}\bar{\theta}}\Big]. \tag{8.52}$$

This action is a supersymmetric, real, gauge and Lorentz invariant. The "tr" is over the group representation which we have not discussed. We admit that some details are hidden in our current notation. See [47] for additional discussion concerning the reality conditions. Also see Equation 10.80 in Chapter 10 for how this action unfolds into many terms at the level of component fields.

Chapter 9

Supersymmetry and Superfields

9.1 Overview

A superfield in general is a supersmooth mapping from $\mathbb{K}^{p|q}$ to $\mathbb{K}^{r|s}$. The analysis of such superfields is straightforward (as we explored in depth in previous chapters) in view of the fact that superfields are mappings between Banach spaces. Our focus in this section will not be so general. Instead, we will focus on superfields that are used to construct models which have $N = 1$ supersymmetry.

We choose to study mappings from $\mathbb{R}^{4|4}$ to $\mathbb{C}_c$. This space is known as $N = 1$ rigid superspace. The label "rigid" derives from the fact that superspace is trivial as a supermanifold; it is parametrized by a single global chart into $\mathbb{R}^{4|4}$. The mappings we are studying in this section are termed *unconstrained scalar bosonic superfields* in the physics literature. Our goal here is simply to make explicit the connection between our real parametrization of $\mathbb{R}^{4|4}$ and the *Weyl spinor* based parametrizations of $\mathbb{R}^{4|4}$ prevalent throughout the physics literature on superfields on superspace.

Historically, it was Salam and Strathdee who made the idea of a superfield popular. Other physicists had made some preliminary studies of supersymmetry from what we would term the *component field* viewpoint. Such arguments can be made without any explicit reference to superspace. One can begin with a set of bosonic and fermionic fields over Minkowski space and then study supersymmetry operations which mix the fields together. Generally we define a supersymmetry transformation to be one which mixes fields of different Lorentz type; that is, supersymmetry transformations mix together fields with different spin. With those operations in mind, one can then construct actions so that they are invariant under those supersymmetry operations. Conversely, one could begin with an action and then discover it has supersymmetries. All of that could be done without superspace, but it is much more efficient and clear to use the superfield approach where the whole collection of fields (sometimes termed a supermultiplet) is described compactly by a single superfield. The parametrization of superspace that Salam and Strathdee used was the one we have labled the *Weyl parametrization*, as is usually the case in physics, all the calculations were done in

those coordinates (well almost all, there is also a notion of chiral coordinates). Only later did a geometric coordinate free formulation emerge. It should be noted that the mathematics of superspace was known to Berezin and others before its application by Salam and Strathdee. Grassmann variables have wide application throughout modern field theory. For example, they are used in path integrals involving fermionic fields and the BRST cohomology. Our interest here is quite narrow. We just want to understand explicitly what superspace is and how it encodes N=1 supersymmetry. This is an interesting question because N=1 supersymmetry forms the basis of what is known as the *MSSM*. That is the Minimal Supersymmetric Standard Model. This model has predictions which differ from the current Standard Model of particle physics. It is possible that the Large Hadron Collider (LHC) at CERN will detect supersymmetry as early as 2010. Of course, if it is not detected the theorists can always push off its discovery a few more TeV's (or in experimental terms a few decades).

In $\mathbb{R}^{4|4}$ there are four independent real Grassmann variables; we denote them ϕ^i for $i = 1, 2, 3, 4$. Here ϕ plays the role that θ assumed previously. Because $\phi^i\phi^i = 0$, there are finitely many terms in the fermionic Taylor series for the supersmooth function $F : \mathbb{R}^{4|4} \to \mathbb{C}_c$

$$F = F_0 + F_i\phi^i + F_{ij}\phi^i\phi^j + F_{ijk}\phi^i\phi^j\phi^k + F_{ijkl}\phi^i\phi^j\phi^k\phi^l \tag{9.1}$$

To be more careful we should mention that F depends on (x^m, ϕ^i) whereas $F_0, F_i, F_{ij}, F_{ijk}, F_{ijkl}$ are all $\mathbb{C}_c$-valued or $\mathbb{C}_a$-valued functions of the even coordinates x^m.

Alternatively, we can expand $F : \mathbb{R}^{4|4} \to \mathbb{C}_c$ in the *component field expansion* relative to the *Weyl parametrization* of superspace

$$F = f + \theta\phi + \bar{\theta}\bar{\mathcal{X}} + \theta\theta m + \bar{\theta}\bar{\theta}n + \theta\sigma^m\bar{\theta}v^m + \theta\theta\bar{\theta}\bar{\lambda} + \bar{\theta}\bar{\theta}\theta\psi + \theta\theta\bar{\theta}\bar{\theta}d. \tag{9.2}$$

Each of the component fields is an ordinary relativistic quantum field. However, there are several inequivalent representations of the Poincare group that appear here. Scalar fields f, m, n, d (spin zero), Weyl spinor fields $\phi, \psi, \bar{\lambda}, \bar{\mathcal{X}}$ (spin 1/2), and the vector field v^m (spin one). Contained in this single superfield we have all the necessary fields to construct known particle physics. Assembling them in this one superfield assumes an additional symmetry of physics which is called supersymmetry. Supersymmetry asserts that there is a balance between the number of bosons and the number of fermions in a theory. A representation of supersymmetry then necessarily has that property. From our analysis above we see that there are 8 bosonic degrees of freedom (4 scalars plus one 4-vector), and there are 8 fermionic degrees of freedom (4 Weyl spinors). Until we place further constraints on the system these are all complex degrees of freedom.

9.2 Poincare Algebra

The Poincare algebra is a Lie algebra that is formed by the four generators of spacetime translations (P_m) and the six generators of the Lorentz transformations ($J_{mn} = -J_{nm}$). For now we can view the Poincare algebra as an abstract Lie algebra over $\mathbb{C}$ defined by the following relations, note η_{ij} is the Minkowski metric tensor with $diag(\eta) = \{-1, 1, 1, 1\}$

$$\begin{array}{ll} [P_m, P_n] & = 0 \\ [P_m, J_{nk}] & = i(\eta_{mn}P_k - \eta_{mk}P_n) \\ [J_{mn}, J_{lk}] & = i(\eta_{nl}J_{mk} - \eta_{ml}J_{nk} + \eta_{mk}J_{nl} - \eta_{nk}J_{ml}) \end{array} \tag{9.3}$$

where $l, k, m, n = 0, 1, 2, 3$. Lorentz transformations include ordinary rotations in three dimensions as well as boosts. Boosts are transformations to moving frames of reference, they can be viewed as hyperbolic rotations of time and space. In particular,

$$\begin{array}{lll} J_{ij} = \epsilon_{ijk}J_k & i, j, k = 1, 2, 3 & \textit{generate rotations} \\ J_{i0} = -K_i & i = 1, 2, 3 & \textit{generate boosts.} \end{array} \tag{9.4}$$

To be careful, we should emphasize that the operators above are not the transformations. Instead they are the generators of the transformations. Mathematically they form the Lie algebra corresponding to the Lie group of transformations. Later on, we'll expand on the relation of the Lie algebra to the Lie group as it relates to the Poincare algebra and group.

For now we prefer to point out that the Poincare algebra has several interesting subalgebras,

$$\begin{array}{lll} [J_i, J_j] = \epsilon_{ijk}J_k & & su(2, \mathbb{C}) \\ [P_i, P_j] = 0 & & \textit{Abelian subalgebra} \end{array} \tag{9.5}$$

The existence of the $su(2, \mathbb{C})$ subalgebra was particularly striking in the 1950's and 1960's when much of the theoretical physics communities efforts were placed in understanding the role isospin played in fundamental interactions. Since isospin also has a $su(2, \mathbb{C})$ algebra structure it was (and is) tempting to try to identify the $su(2, \mathbb{C})$ of isospin with the $su(2, \mathbb{C})$ of the Poincare algebra. To be less naive, one might ask if there is a way to extend the Poincare algebra so that the enlarged version has subalgebras from which isospin could be derived. This would have been very beautiful in some sense as it would have placed fundamental nuclear interactions on the same foundation as momentum or energy (which are associated to P_m). However, this ambitious dream to enlarge the Poincare algebra was shot down by the famous paper by Coleman and Mandula (Physical Review 159,1251 (1967)). They proved a very important no-go theorem which stated that it was not possible to enlarge the Poincare algebra without violating important symmetries of the S-matrix. The dream of understanding isospin and other "external" symmetries in a more intrinsic geometric manner lives on. This theorem merely shows that it cannot be accomplished in a

strictly conventional way. The standard formalism of relativistic quantum field theory will not admit it. To give isospin a geometric (in the sense of real spatial origins) meaning will require a change in fundamental formalism like strings, twistors or perhaps noncommutative geometry.

Interestingly, the no-go theorem of Coleman and Mandula sparked a very different line of inquiry than one might have expected. Hagg, Lopuszanski and Sohnius (Nuclear Physics B 88 257 (1975)) noticed that the no-go theorem's proof assumed that the additional operators to the Poincare algebra should obey commutator brackets. Why should that be ? Why can't there be physical symmetries which are generated by anticommuting generators? Hagg, Lopuszanski and Sohnius argued that the no-go theorem was too narrow in its assumptions, that in fact it was possible to extend the Poincare algebra by adding generators which anticommute. They argued that for physical reasons (absence of higher spin states for example) that the anticommuting generators must obey the following algebraic structure,

$$\begin{array}{ll} \{Q^A_\alpha, Q^B_\beta\} & = Z^{AB} \\ \{\bar{Q}^A_\alpha, \bar{Q}^B_\beta\} & = \bar{Z}^{AB} \\ \{Q^A_\alpha, \bar{Q}^B_\beta\} & = 2\sigma^m_{\alpha\dot{\beta}} P_m \delta^{AB} \end{array} \tag{9.6}$$

where the anticommutator is $\{X, Y\} = XY + YX$, $\sigma^m_{\alpha\dot{\beta}}$ are the Pauli matrices for $m = 1, 2, 3$, and $A, B = 1, 2, 3, \ldots N$. Indices like α, β, γ are called "undotted indices" while indices like $\dot{\alpha}, \dot{\beta}, \dot{\gamma}$ are called "dotted indices", both types take values 1 or 2 hopefully without danger of confusion. The central charges Z^{AB} commute with everything and are antisymmetric in A and B. These relations plus the Poincare algebra form the N=1,2,3,4 super Poincare algebras.

The case of interest to us is $N = 1$ for which there are no central charges since indices A,B=1. We will call the generators $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ the supercharges. In total the super Poincare algebra is defined by the relations,

$$\begin{array}{l} [P_m, P_n] = 0 \\ [P_m, J_{nk}] = i(\eta_{mn} P_k - \eta_{mk} P_n) \\ [J_{mn}, J_{lk}] = i(\eta_{nl} J_{mk} - \eta_{ml} J_{nk} + \eta_{mk} J_{nl} - \eta_{nk} J_{ml}) \\ [Q_\alpha, P_m] = 0 \\ [\bar{Q}_{\dot{\alpha}}, P_m] = 0 \\ [J_{mn}, Q_\alpha] = -i(\sigma_{mn})_\alpha{}^\beta Q_\beta \\ [J_{mn}, \bar{Q}_{\dot{\alpha}}] = -i(\bar{\sigma}_{mn})_{\dot{\alpha}}{}^{\dot{\beta}} \bar{Q}_{\dot{\beta}} \\ \{Q_\alpha, Q_\beta\} = 0 \\ \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0 \\ \{Q_\alpha, \bar{Q}_\beta\} = 2\sigma^m_{\alpha\dot{\beta}} P_m \end{array} \tag{9.7}$$

The matrices σ_{mn} and $\bar{\sigma}_{mn}$ are formed from antisymmetrized products of the Pauli

matrices. The details need not concern us here (see Wess and Bagger [116] for many useful formulas dealing with such objects; generally we follow their conventions).

We should pause and note that the super Poincare algebra is a $\mathbb{Z}_2$-graded Lie algebra. It possesses both even and odd elements. The bracket is in fact a superbracket which is sometimes a commutator and other times an anticommutator. The Jacobi identity of Lie algebras is replaced by the graded Jacobi identity, but generally things look mostly the same as Lie algebras. To see a rather complete account of $\mathbb{Z}_2$-graded algebras over $\mathbb{C}$ we point the reader to the classic paper by Kac [69].

9.3 The Coset View of Spacetime

In the previous section we often referred to the objects of consideration as "operators", but that begs the question "operators on what?". The answer is not unique as there are many possible representations. One very fundamental type of operator is the linear differential operator on functions of space and time. Modulo time this is often what one encounters in the study of quantum mechanics. For example the formulas $P_x = i\frac{\partial}{\partial x}$ or $J_z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial y}$ ought to be familiar to the introductory student of quantum mechanics. In the discussion that follows, we will see how to derive the form of the differential operators from the starting point of just knowing the Lie algebra. We will explain how to use the algebra to construct the group. First, we derive the transformations that the Poincare algebra induces on spacetime, then once that is established we will explain how to choose a representation of the algebra in terms of linear differential operators acting on functions of spacetime.
Recall a nontrivial identity known as the Baker-Cambell-Hausdorff relation,

$$exp(A)exp(B) = exp(A + B + \frac{1}{2}[A, B] + \frac{1}{12}[[A, B], A] - \frac{1}{12}[[A, B], B] + \dots) \quad (9.8)$$

The higher order terms can again be formed by taking 3,4,5,... fold nested commutators. This relation allows us to reconstruct a neighborhood of the identity in the group from the algebra. The fact that we consider a Lie algebra means that we know how to calculate all the commutators from the very definition of a Lie algebra. This process is referred to as exponentiation; we exponentiate the algebra g to form the group G.

9.3.1 Momenta P_m Generate Translations in Space and Time

Consider then the following calculation, let $A = x_o^m P_m$ and let $B = x^m P_m$ where $x_o^m, x^m \in \mathbb{R}$ and where we utilize the Einstein conventions; summation over the index $m = 0, 1, 2, 3$ is implicit. Note that the momenta P_m commute so all the commutator terms vanish,

$$exp(ix_o^m P_m)exp(ix^m P_m) = exp(ix_o^m P_m + ix^m P_m) = exp(i(x_o^m + x^m)P_m). \quad (9.9)$$

Let us explain the meaning of the calculation above; multiplying on the right by $exp(ix^m P_m)$ has shifted the initial position x_o to the new position $x_o + x$.

$$exp(ix_o^m P_m)exp(ix^m P_m) \longrightarrow x_o^m \mapsto x_o^m + x^m. \tag{9.10}$$

Hence the enigmatic claim that momentum are the generators of translations. Note that to make this claim we have identified the parameters x_o^m as an event in spacetime. This identification is called the "coset" view of spacetime although it is not clear yet here why the term "coset" is warranted.

9.3.2 J_{mn} Generate Boosts and Rotations

Next let $\omega^{mn} \in \mathbb{R}$ and consider right multiplication by $exp(i\omega^{mn}J_{mn})$,

$$exp(ix_o^k P_k)exp(i\omega^{mn}J_{mn}) \quad = exp(ix_o^k P_k + i\omega^{mn}J_{mn} - \tfrac{1}{2}[x_o^k P_k, \omega^{mn}J_{mn}] + \dots). \tag{9.11}$$

Lets calculate the commutator separately, let $(x_o^m) = (t, r^j)$ for $j = 1, 2, 3$,

$$\begin{aligned}
[x_o^k P_k, \omega^{mn}J_{mn}] &= x_o^k \omega^{mn}[P_k, J_{mn}] \\
&= x_o^k \omega^{mn} i(\eta_{km}P_n - \eta_{kn}P_m) \\
&= i(t\omega^{mn}(\eta_{0m}P_n - \eta_{0n}P_m) + r^j\omega^{mn}(\eta_{jm}P_n - \eta_{jn}P_m)) \\
&= i(t\omega^{mn}(-\delta_{0m}P_n + \delta_{0n}P_m) + r^j\omega^{mn}(\delta_{jm}P_n - \delta_{jn}P_m)) \\
&= i(-t\omega^{0n}P_n + t\omega^{m0}P_m + r^j\omega^{jn}P_n - r^j\omega^{mj}P_m) \\
&= i(2t\omega^{m0}P_m - 2r^j\omega^{mj}P_m) \\
&= 2i(t\omega^{m0} - r^j\omega^{mj})P_m)
\end{aligned} \tag{9.12}$$

In the last couple of steps we have assumed that the parameters $\omega^{mn} = -\omega^{nm}$. To summarize we can see that the position x_o is shifted via right multiplication by $exp(\omega^{mn}J_{mn})$, as follows

$$x_o^m \quad \mapsto x_m^0 - t\omega^{m0} + r^j\omega^{mj}. \tag{9.13}$$

Now this is only to the first order in the parameters ω^{mn}. A short examination of the higher commutators will reveal that they also contribute to the transformation. We leave such calculations as an character building exercise for the reader. At this stage in the calculation it takes a little imagination to see why the motion above are rotations and boosts. To see this we break up the parameters ω^{mn} into parameters

of boosts and rotations,

$$\begin{array}{lll} \omega^{01} = \phi^1 & rapidity\ of\ the\ x-boost \\ \omega^{02} = \phi^2 & rapidity\ of\ the\ y-boost \\ \omega^{03} = \phi^3 & rapidity\ of\ the\ z-boost \\ \omega^{12} = \theta^3 & Euler\ angle\ w.r.t\ the\ z-axis \\ \omega^{31} = \theta^2 & Euler\ angle\ w.r.t\ the\ y-axis \\ \omega^{23} = \theta^1 & Euler\ angle\ w.r.t\ the\ x-axis \end{array} \tag{9.14}$$

In short, $\frac{1}{2}\omega^{mn}J^{mn} = \phi^i K_i + \theta^i J_i$ where i is summed over $i = 1, 2, 3$.

Now we specialize to the case that the parameter is zero for all components except $\omega^{12} = -\omega^{21} = \theta$, this will reduce the transformation since $\omega^{m0} = 0$ leaving just

$$\begin{array}{ll} t & \mapsto t \\ r^j & \mapsto r^j + r^k\omega^{jk} \end{array} \tag{9.15}$$

Expanding $(r^j) = (x, y, z)$ we find,

$$\begin{array}{ll} t & \mapsto t \\ x & \mapsto x + y\theta \\ y & \mapsto y - x\theta \\ z & \mapsto z. \end{array} \tag{9.16}$$

Behold, this is the first order approximation of a rotation around the z-axis by an angle θ. If you don't see it yet, then recall that such a rotation could be written in matrix form as follows (ignoring time t),

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} cos(\theta) & sin(\theta) & 0 \\ -sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} xcos(\theta) + ysin(\theta) \\ -xsin(\theta) + ycos(\theta) \\ z \end{pmatrix}. \tag{9.17}$$

Then to first order in θ we know that $cos(\theta) = 1$ and $sin(\theta) = \theta$. Thus we find that $J_{12} = J_3$ generates a rotation around the z-axis. Our proof is approximate here; we have only checked the first order. Higher orders are straightforward but tedious to check.

Next we specialize to the case that ω^{mn} is non-zero only for $\omega^{01} = -\omega^{10} = \phi$. We claim that this choice of parameter will generate a boost in the x-direction. Here our previous calculations specialize to,

$$\begin{array}{ll} t & \mapsto t + x\phi \\ x & \mapsto x + t\phi \\ y & \mapsto y \\ z & \mapsto z. \end{array} \tag{9.18}$$

This is the first order approximation to a boost in the x-direction by rapidity ϕ. For the reader unfamiliar with rapidity let us introduce the concept and explain how it relates to a Lorentz transformation. Recall first that a Lorentz transformation is a change of coordinates to a moving frame of reference. If the new frame has velocity v with respect to x in the old coordinates, and the two coordinate systems coincide at the origin, then we can relate the new moving coordinates (t', x', y', z') and the old coordinates (t, x, y, z) by the standard Lorentz transformation.

$$\begin{aligned} t' &= \gamma t + \gamma\beta x \\ x' &= \gamma x + \gamma\beta t \\ y' &= y \\ z' &= z \end{aligned} \tag{9.19}$$

We take the speed of light c=1 for convenience and have introduced the parameters $\beta = v/c = v$ and $\gamma = 1/(\sqrt{1-\beta^2})$. Define then the rapidity ϕ by the equation $tanh(\phi) = \beta$. Recall the Pythagorean theorem for hyperbolic functions,

$$cosh^2(\phi) - sinh^2(\phi) = 1 \implies tanh^2(\phi) = \frac{sinh^2(\phi)}{cosh^2(\phi)} = \frac{sinh^2(\phi)}{1 + sinh^2(\phi)} = \beta^2. \tag{9.20}$$

Solve for $sinh^2(\phi)$,

$$sinh^2(\phi) = \frac{\beta^2}{1-\beta^2} = \beta^2\gamma^2. \tag{9.21}$$

Thus we find that $sinh(\phi) = \beta\gamma$ and consequently $cosh(\phi) = \gamma$. Rewrite the standard Lorentz transformation in terms of rapidity, (this makes manifest the fact that boosts are hyperbolic rotations)

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} cosh(\phi) & sinh(\phi) & 0 & 0 \\ sinh(\phi) & cosh(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma t + \gamma\beta x \\ \gamma x + \gamma\beta t \\ y \\ z \end{pmatrix}. \tag{9.22}$$

To first order in ϕ one can show $cosh(\phi) = 1$ and $sinh(\phi) = \phi$. Thus, we find our claim was true; $J_{01} = K_1$ generates a boost in the x-direction. Again we emphasize that this proof is incomplete; we leave the higher orders to the reader to check.

9.3.3 Poincare/Lorentz = Spacetime

The Poincare group is generated by exponentiation of the Poincare algebra. An arbitrary element has the form,

$$exp(ix^m P_m + i\omega^{mn} J_{mn}) \tag{9.23}$$

By the Poincare algebra and the Baker Cambell Hausdorff relation we can prove that this factors,

$$exp(ix^m P_m + i\omega^{mn} J_{mn}) = exp(iy^m P_m) exp(ia^{mn} J_{mn}). \tag{9.24}$$

This is a long calculation. y^m are defined by infinite series of the parameters x^m and ω^{mn}. The details need not concern us. What is important is to see that we divide by elements of the form $exp(ia^{mn} J_{mn})$ that will leave just the coset $exp(iy^m P_m)$. As we have explored in depth in the previous sections it is natural to identify the coset with Minkowski space; indeed, multiplication in the Poincare group naturally induces the standard translations, rotations and Lorentz transformations. This is the coset view of space time.

9.3.4 Linear Differential Operators Represent the Poincare Algebra

We narrow our focus to infinitesimal motions of the Poincare group. Our first goal here is to find a linear differential operator $D(P_m)$ which represents momenta and generates translations in the following sense, for y small,

$$exp(iy^m D(P_m))x^k = (1 + iy^m D(P_m))x^k = x^k + y^k. \tag{9.25}$$

Notice here in our view we do not have to assume the far r.h.s. of the equation. Rather, it is derived from the coset view of spacetime and the Poincare algebra. This view-point takes the algebra as the starting point and attempts to derive other ideas from that foundation. Examining the equation above yields,

$$iy^m D(P_m)x^k = y^k \implies D(P_m) = -i\frac{\partial}{\partial x^m}. \tag{9.26}$$

As an abuse of notation we will hereafter identify P_m with $-i\frac{\partial}{\partial x^m}$. Notice that the fact that partial commutes shows that $P_m = -i\frac{\partial}{\partial x^m}$ satisfies $[P_m, P_n] = 0$. Now let's try to deduce the form of the operators that represent J_{mn}. Let ω^{mn} be small and write $(x^m) = (t, r^j)$,

$$exp(i\omega^{mn} D(J_{mn}))x^k = (1 + i\omega^{mn} D(J_{mn}))x^k = x^k - t\omega^{m0} + r^j \omega^{mj}. \tag{9.27}$$

Recall the right hand side followed from the Poincare group acting on the coset space. Also, there is an implicit summation over $j = 1, 2, 3$. Examining the equation above reveals that,

$$i\omega^{mn} D(J_{mn})x^k = -t\omega^{m0} + r^j \omega^{mj}. \tag{9.28}$$

Let us specialize to the particular case of $J_{12} = J_3$. Recall we derived that this operator generated a rotation around the z-axis by an angle θ,

$$i\theta D(J_{12})x^k = y\theta\delta_1^k - x\theta\delta_2^k. \tag{9.29}$$

It is not difficult to see that

$$D(J_{12}) = -i(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) \tag{9.30}$$

satisfies the condition. Since $J_{12} = J_3$ we can, by a slight abuse of notation, claim,

$$J_3 = -i(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}). \tag{9.31}$$

The abuse here is that the algebra and its representation are identified. By calculations very similar to those we have thus far done, one can calculate the following representation of the Poincare algebra by linear differential operators that act on space time,

$$\begin{array}{ll} P_0 & = -i\partial_t \\ P_1 & = -i\partial_x \\ P_2 & = -i\partial_y \\ P_3 & = -i\partial_z \\ J_1 & = -i(y\partial_z - z\partial_y) \\ J_2 & = -i(z\partial_x - x\partial_z) \\ J_3 & = -i(x\partial_y - y\partial_x) \\ K_1 & = i(t\partial_x - x\partial_t) \\ K_2 & = i(t\partial_y - y\partial_t) \\ K_3 & = i(t\partial_z - z\partial_t). \end{array} \tag{9.32}$$

This makes functions of spacetime states in this representation; That is, to represent the Poincare algebra, we can simply take functions of spacetime, then the operators above induce an action of the Poincare algebra onto such functions. An obvious question is whether this is possible for the super Poincare algebra, and if so what space plays the role of spacetime. This is the question that we really want to answer in this chapter.

9.3.5 Finding the Algebra Given the Group

The direction we have taken in this section is somewhat counter to what one usually finds in literature. It is much more natural to begin with the group and then derive the algebra. This really amounts to differentiation, whereas what we have attempted in this section is integration. For a good account of how to find the Poincare algebra from the group, we point the reader to chapter 2 of *Quantum Field Theory* [107]. He shows how the Lie algebra is found by looking at the derivatives of the group at the

identity. Specifically, if one considers curves through the identity generated by one-parameter groups, then one can form the vector space for the Lie algebra by taking the span of the tangents to those curves at the identity. The group multiplication near the identity will yield the bracket for the tangent vectors viewed as derivations. Ryder also provides more physical motivations for the topics in consideration here.

9.4 Looking Ahead to Superspace

We have gone to some trouble to explain how one may represent an algebra in terms of linear differential operators that act on functions of the parameter space. Actually, we did not need the whole parameter space, but rather a four-dimensional subspace which we found it natural to identify with Minkowksi space. The question that we consider in this section is whether it is possible to find some space on which we can form a linear representation of the super Poincare algebra. Unlike the usual Poincare algebra, we cannot begin with transformations of the group (which are rather intuitive for the Poincare group, certainly the group came before the algebra historically) because we only have a formal idea of the group at this point. The super Poincare algebra came from general algebraic reasoning, but now we try to understand what space it can be understood to act on naturally. For the Poincare algebra, that space is simply spacetime. We will find that with the proper motivations and parametrizations the space $\mathbb{R}^{4|4}$ turns out to be the natural space on which the super Poincare group acts.

9.5 Coset View of Superspace

9.5.1 Super Poincare / Lorentz = Superspace

First by analogy with the Poincare group, a typical element in the super Poincare group is,

$$exp(ix^m P_m + i\omega^{mn} J_{mn} + i\theta Q + i\bar{\theta}\bar{Q}). \tag{9.33}$$

We assume that the operators above satisfy the super Poincare algebra. This is a Grassmann generalization of the concept of exponentiation; Grassmann in the sense that we use Grassmann parameters which are essentially real. Although to be more precise, θ^α is complex with conjugate $\bar{\theta}^{\dot{\alpha}}$. All the products that appear in the exponential are even objects (but not the factors !) so the Baker-Cambell-Hausdorff relation still holds and we can factor out the Lorentz transformations,

$$exp(ix^m P_m + i\omega^{mn} J_{mn} + i\theta Q + i\bar{\theta}\bar{Q}) = exp(i(y^m P_m + \beta Q + \bar{\beta}\bar{Q}))exp(ia^{mn} J_{mn}). \tag{9.34}$$

This is not a trivial calculation. In fact $y^m, \beta, \bar{\beta}, a^{mn}$ are formed by infinite series of the parameters $x^m, \theta, \bar{\theta}, \omega^{mn}$. We will use $x^m, \theta, \bar{\theta}$ as the parameters to label a typical coset. The coset space will then give us a natural construction of superspace, the

analogue of spacetime to the super Poincare algebra.

9.5.2 Translations and Supertranslations in Superspace

We saw previously how the algebra of the momenta generated a translation on the parameter space of the group. Let us generalize that calculation to the case of superspace,

$$exp(i(x^m P_m + \theta Q + \bar{\theta}\bar{Q}))exp(i(a^m P_m + \epsilon Q + \bar{\epsilon}\bar{Q})) =$$

$$= exp(i(x^m + a^m)P_m + i(\theta + \epsilon)Q + i(\bar{\theta} + \bar{\epsilon})\bar{Q} - \tfrac{1}{2}[x^m P_m + \theta Q + \bar{\theta}\bar{Q}, a^m P_m + \epsilon Q + \bar{\epsilon}\bar{Q}])$$

Besides the commutator terms everything looks fairly similar to ordinary translations. The even coordinates get shifted by a^m, and the odd coordinates get shifted by ϵ and $\bar{\epsilon}$. Naively that might be all you would expect for a supertranslation, however, there is more hidden in the commutator. Let us calculate it in stages. Remember all we have to work with at this stage is the super Poincare algebra itself, along with the properties of Grassmann variables (we assume that the operators P_m are even as they satisfy a commutation relation, whereas we assume that the operators Q_α and $\bar{Q}_{\dot{\alpha}}$ are odd as they satisfy an anticommutation relation). Notice

$$\begin{array}{l} [x^m P_m, a^m P_m + \epsilon Q + \bar{\epsilon}\bar{Q}] = 0 \\ [x^m P_m + \theta Q + \bar{\theta}\bar{Q}, a^m P_m] = 0. \end{array} \tag{9.35}$$

We used linearity of the bracket to isolate Q and P and P with P, these commute by definition of super Poincare algebra. Next,

$$\begin{array}{rl} [\theta Q, \epsilon Q] & = \theta Q \epsilon Q - \epsilon Q \theta Q \\ & = \theta^\alpha Q_\alpha \epsilon^\beta Q_\beta - \epsilon^\beta Q_\beta \theta^\alpha Q_\alpha \\ & = -\theta^\alpha \epsilon^\beta Q_\alpha Q_\beta + \epsilon^\beta \theta^\alpha Q_\beta Q_\alpha \\ & = -\theta^\alpha \epsilon^\beta (Q_\alpha Q_\beta + Q_\beta Q_\alpha) \\ & = -\theta^\alpha \epsilon^\beta \{Q_\alpha, Q_\beta\} = 0 \end{array} \tag{9.36}$$

Since $\{Q_\alpha, Q_\beta\} = 0$. Next,

$$\begin{array}{rl} [\bar{\theta}\bar{Q}, \bar{\epsilon}\bar{Q}] & = \bar{\theta}\bar{Q}\bar{\epsilon}\bar{Q} - \bar{\epsilon}\bar{Q}\bar{\theta}\bar{Q} \\ & = \bar{\theta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}} \bar{\epsilon}_{\dot{\beta}} \bar{Q}^{\dot{\beta}} - \bar{\epsilon}_{\dot{\beta}} \bar{Q}^{\dot{\beta}} \bar{\theta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}} \\ & = -\bar{\theta}_{\dot{\alpha}} \bar{\epsilon}_{\dot{\beta}} (\bar{Q}^{\dot{\alpha}} \bar{Q}^{\dot{\beta}} + \bar{Q}^{\dot{\beta}} \bar{Q}^{\dot{\alpha}}) \\ & = -\bar{\theta}_{\dot{\alpha}} \bar{\epsilon}_{\dot{\beta}} \{\bar{Q}^{\dot{\alpha}}, \bar{Q}^{\dot{\beta}}\} = 0 \end{array} \tag{9.37}$$

Since $\{\bar{Q}^{\dot{\alpha}}, \bar{Q}^{\dot{\beta}}\} = 0$. Next,

$$\begin{aligned}
[\theta Q, \bar{\epsilon}\bar{Q}] &= \theta Q \bar{\epsilon}\bar{Q} - \bar{\epsilon}\bar{Q}\theta Q \\
&= \theta^{\alpha} Q_{\alpha} \bar{Q}_{\dot{\beta}} \bar{\epsilon}^{\dot{\beta}} - \bar{Q}_{\dot{\beta}} \bar{\epsilon}^{\dot{\beta}} \theta^{\alpha} Q_{\alpha} \\
&= \theta^{\alpha} \bar{\epsilon}^{\dot{\beta}} Q_{\alpha} \bar{Q}_{\dot{\beta}} - \bar{\epsilon}^{\dot{\beta}} \theta^{\alpha} \bar{Q}_{\dot{\beta}} Q_{\alpha} \\
&= \theta^{\alpha} \bar{\epsilon}^{\dot{\beta}} (Q_{\alpha} \bar{Q}_{\dot{\beta}} + \bar{Q}_{\dot{\beta}} Q_{\alpha}) \\
&= \theta^{\alpha} \bar{\epsilon}^{\dot{\beta}} \{Q_{\alpha}, \bar{Q}_{\dot{\beta}}\} \\
&= 2\theta^{\alpha} \bar{\epsilon}^{\dot{\beta}} \sigma^{m}_{\alpha\dot{\beta}} P_m \\
&= 2\theta \sigma^{m} \bar{\epsilon} P_m,
\end{aligned} \tag{9.38}$$

where we used that the supercharges are the "square root of momentum"; $\{Q_{\alpha}, \bar{Q}_{\beta}\} = 2\sigma^{m}_{\alpha\dot{\beta}} P_m$. Next, by the same calculation (just switching θ and ϵ and their conjugates) we find,

$$\begin{aligned}
[\bar{\theta}\bar{Q}, \epsilon Q] &= -[\epsilon Q, \bar{\theta}\bar{Q}] \\
&= -2\epsilon \sigma^{m} \bar{\theta} P_m.
\end{aligned} \tag{9.39}$$

We have completed the calculation of the commutator encountered at the beginning of this section. The reader should verify that in fact all the higher commutators vanish. Our result here is not approximate since the series terminates after the terms we've calculated. To summarize we found,

$$exp(i(x^m P_m + \theta Q + \bar{\theta}\bar{Q}))exp(i(a^m P_m + \epsilon Q + \bar{\epsilon}\bar{Q})) =$$

$$= exp(i(x^m + a^m)P_m + i(\theta + \epsilon)Q + i(\bar{\theta} + \bar{\epsilon})\bar{Q} - \tfrac{1}{2}(2\theta\sigma^m\bar{\epsilon}P_m - 2\epsilon\sigma^m\bar{\theta}P_m))$$

$$= exp(i(x^m + a^m + i\theta\sigma^m\bar{\epsilon} - i\epsilon\sigma^m\bar{\theta})P_m + i(\theta + \epsilon)Q + i(\bar{\theta} + \bar{\epsilon})\bar{Q})$$

Taking the case $a^m = 0$, we find that the supercharges generate the following motion on the parameter space,

$$\begin{aligned}
x^m &\mapsto x^m + i\theta\sigma^m\bar{\epsilon} - i\epsilon\sigma^m\bar{\theta} \\
\theta^{\alpha} &\mapsto \theta^{\alpha} + \epsilon^{\alpha} \\
\bar{\theta}^{\dot{\alpha}} &\mapsto \bar{\theta}^{\dot{\alpha}} + \bar{\epsilon}^{\dot{\alpha}}.
\end{aligned} \tag{9.40}$$

These are supertranslations on superspace (the parameter space is superspace).

9.5.3 Derivations on $\mathbb{R}^{4|4}$ Represent the Super Poincare Algebra

We have found the transformations induced by the super Poincare algebra on superspace. We may now try to find a linear representation of the super Poincare algebra in much the same way as we did before for the Poincare algebra. We wish to find linear differential operators $D(Q_{\alpha})$ which act on functions of $(x^m, \theta^{\alpha}, \bar{\theta}^{\dot{\alpha}})$ such that

for small ϵ,

$$exp(i\epsilon^\alpha D(Q_\alpha))(x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) = (1 + i\epsilon^\alpha D(Q_\alpha))(x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) = (x^m - i\epsilon\sigma^m\bar{\theta}, \theta^\alpha + \epsilon^\alpha, \bar{\theta}^{\dot{\alpha}}).$$

We require that

$$\begin{aligned} i\epsilon^\alpha D(Q_\alpha)x^m &= -i\epsilon\sigma^m\bar{\theta} \\ i\epsilon^\alpha D(Q_\alpha)\theta^\alpha &= \epsilon^\alpha \\ i\epsilon^\alpha D(Q_\alpha)\bar{\theta}^{\dot{\alpha}} &= 0. \end{aligned} \tag{9.41}$$

Hence, we find that $D(Q_\alpha) = -i\frac{\partial}{\partial\theta^\alpha} - \sigma^m_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\frac{\partial}{\partial x^m}$. You can check this is correct, just substitute it back into the last equation.

Chapter 10

Deformed Super Yang-Mills Theory

This chapter breaks from the mathematical rigor of most of the earlier chapters. We consider a particular physical model, and we make calculations without much regard for domains. Consistency is key in this chapter. It is likely that one could understand the mathematics that follows carefully with the help of a sheaf theoretic construction. We chose to calculate like a physicist in this chapter. Rigor aside, this work has recently found a place as perhaps a sort of counter example within an ongoing discussion initiated by E.A. Ivanov and A.V. Smilga (see [66]). Apparently the concept of *cryptoreality* may be a better physical assumption than rigid hermiticity. After all, we will see in this chapter that hermiticity on non(anti) commutative superspace essentially forces us to consider a nonassociative star product. Associativity has been a critical feature in much work done on Poisson manifolds. It seems it would be better to maintain that feature. Having said that, this chapter may help give the mathematical reader a better sense of how superfields are used to construct Lagrangians and actions. To obtain the usual theory on undeformed superspace one can simply remove the star products.

We develop a gauged Wess-Zumino model in noncommutative Minkowski superspace. This is the natural extension of the work of Carlson and Nazaryan, which extended $N = 1/2$ supersymmetry written over deformed Euclidean superspace to Minkowski superspace. We investigate the interaction of the vector and chiral superfields. Noncommutativity is implemented by replacing products with star products. Although, in general, our star product is nonassociative, we prove that it is associative to the first order in the deformation parameter C. We show that our model reproduces the $N = 1/2$ theory in the appropriate limit, namely when the deformation parameters $\bar{C}^{\dot{\alpha}\dot{\beta}} = 0$. Essentially, we find the $N = 1/2$ theory and a conjugate copy. As in the $N = 1/2$ theory, a reparametrization of the gauge parameter, vector superfield and chiral superfield are necessary to write standard C-independent gauge theory. However, our choice of parametrization differs from that used in the $N = 1/2$

supersymmetry, which leads to some unexpected new terms.

10.1 Introduction

There have been a number of papers concerning deformations of superspace in recent years (see [111], [33], [41], [42], [75], [71], [38], [38], [112], [54], [87], [32] and [72] for an undoubtedly a partial list). Of particular interest to this chapter is the deformed Euclidean superspace constructed by Seiberg in [112]. Generally, the literature following Seiberg has focused on superspace with a Euclidean signature. One exception is [87], in which Carlson and Nazaryan found how to construct a deformed Minkowski superspace

Remark 10.1.1. *After the original completion of [36] the author learned that the work of M. Chaichian and A. Kobakhidze in [32] and the work of Y. Kobayashi and S. Sasaki in [72] also studied the Wess-Zumino model on deformed Minkowski superspaces in some detail. Both of these works employ a star product which is associative but not hermitian. The star product studied here is hermitian but not associative in general. Also, note that [71] and [42] study some aspects of deformed Minkowski superspace that have relevance to this work.*

In their paper, they implemented superspace noncommutativity with a star product which was hermitian but not associative in general. Their star product reproduces the deformation of $N = \frac{1}{2}$ supersymmetry in a certain limit. Additionally, they studied the Wess-Zumino model (without gauge interactions) and found results similar to Seiberg's. Our goal is to construct the gauged Wess-Zumino model in this noncommutative Minkowski superspace.

Following the construction of Nazaryan and Carlson, we deform $N = 1$ rigid Minkowski superspace as follows:

$$\{\hat{\theta}^\alpha, \hat{\theta}^\beta\} = C^{\alpha\beta} \qquad \{\hat{\bar{\theta}}^{\dot{\alpha}}, \hat{\bar{\theta}}^{\dot{\beta}}\} = \bar{C}^{\dot{\alpha}\dot{\beta}} \tag{10.1}$$

where $(C^{\alpha\beta})^* = \bar{C}^{\dot{\alpha}\dot{\beta}}$. In this deformation, all of the fermionic dimensions of superspace are deformed. Here both Q and $\bar{Q}$ are broken symmetries, so we will say that this space has $N = 0$ supersymmetry. Despite this, the deformation still permits most of the usual superfield constructions.

In section 10.2.1 we explicitly define the noncommutative Minkowski superspace by summarizing the required structure of the deformed coordinate algebra found in [87]. The deformed coordinates have hats on them to distinguish them from the standard coordinates. The usual model is then deformed by simply putting a hat on all of the objects in the standard theory. In practice, we will not explicitly calculate anything in terms of these operators. Instead, we will find it useful to make the usual exchange of the operator product for the star product of ordinary functions of

superspace;

$$\hat{f}_1 \hat{f}_2 \mapsto f_1 * f_2. \tag{10.2}$$

This correspondence allows us to work out the details of noncommutative theory using ordinary calculus on superspace. In this sense we obtain the noncommutative Wess-Zumino model by simply replacing ordinary products with star products.

In sections 10.2.2 and 10.2.3, we continue our brief summary of the work of Carlson and Nazaryan in [87]. In [87], deformed Minkowski space was constructed to the second order in the deformation parameter. In this chapter, we primarily examine the first order extensions of their work. In section 10.3, we examine how to write a nonabelian supersymmetric gauge theory on noncommutative Minkowski superspace. Following the standard superfield construction(see [116] for example), we introduce the vector superfield (V) and calculate the star exponential (e^V) in section 10.3.1. We calculate the explicit modification these definitions imply for the component fields of the vector multiplet.

The gauge transformation itself will be discussed in section 10.4. In section 10.4.1, we find a parametrization of the vector superfield such that the standard gauge transformations are realized at the component field level. This procedure is similar to Seiberg's in [112]. We employ a modified Wess-Zumino gauge throughout the calculations. This is possible provided that we define the gauge parameter Λ with some carefully chosen deformation dependent shifts. We will find that reality uniquely affixes this construction. Next, in section 10.4.2, we introduce the spinor superfield W_α by making the natural modification to the standard definition.

Then, in section 10.5, we examine the gauge transformation on a chiral superfield. Again, we will find it necessary to shift the chiral superfield by a deformation dependent term in order to preserve the usual gauge theory. These shifts, similar to those found in [112] and [6], are derived in detail.

Finally, in section 10.6, we construct the Lagrangian of the gauged Wess-Zumino model. This construction closely resembles that of Wess and Bagger in [116] except that products have been replaced by star products. Also, the component field expansions of the superfields have some C-dependent shifts as derived in the previous sections. Overall, the gauge symmetry of the Lagrangian is established by arguments analogous to the standard arguments. We conclude the chapter by computing the Lagrangian written explicitly in terms of the component fields. Our result is similar to [6], however, there are some unexpected terms.

Remark 10.1.2. *This chapter contains some second order results for the star exponential. However, we do not complete the development of the theory to second order in this chapter. We do find some partial results at the second order of the deformation parameter and they agree with the $N = \frac{1}{2}$ in the limit $\bar{C}^{\dot{\alpha}\dot{\beta}} = 0$. The paper [36] contains only the first order results given here.*

10.2 Noncommutative Minkowski Superspace

10.2.1 Deformed Coordinate Algebra

We begin by considering $N = 1$ rigid Minkowski superspace where a typical point is $(x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$. In the commutative case, we have:

$$\begin{array}{ll} [x^m, x^n] = 0 & [x^m, \theta^\alpha] = 0 \\ \{\theta^\alpha, \theta^\beta\} = 0 & [x^m, \bar{\theta}^{\dot{\alpha}}] = 0 \\ \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = 0 & \{\theta^\alpha, \bar{\theta}^{\dot{\beta}}\} = 0 \end{array} \tag{10.3}$$

The coordinates x^m are identified with spacetime coordinates, whereas the θ^α and $\bar{\theta}^{\dot{\alpha}}$ are Grassmann variables. We then construct noncommutative Minkowski superspace by replacing coordinate functions (x^m, θ^α, $\bar{\theta}^{\dot{\alpha}}$) with operators ($\hat{x}^m$, $\hat{\theta}^\alpha$, $\hat{\bar{\theta}}^{\dot{\alpha}}$). In particular, we require that the deformed coordinates satisfy

$$\begin{array}{ll} \{\hat{\theta}^\alpha, \hat{\theta}^\beta\} = C^{\alpha\beta} & [\hat{x}^m, \hat{\theta}^\alpha] = iC^{\alpha\beta}\sigma^m_{\beta\dot{\beta}}\hat{\bar{\theta}}^{\dot{\beta}} \\ \{\hat{\bar{\theta}}^{\dot{\alpha}}, \hat{\bar{\theta}}^{\dot{\beta}}\} = \bar{C}^{\dot{\alpha}\dot{\beta}} & [\hat{x}^m, \hat{\bar{\theta}}^{\dot{\alpha}}] = i\bar{C}^{\dot{\alpha}\dot{\beta}}\hat{\theta}^\beta\sigma^m_{\beta\dot{\beta}} \\ \{\hat{\theta}^\alpha, \hat{\bar{\theta}}^{\dot{\beta}}\} = 0 & [\hat{x}^m, \hat{x}^n] = (C^{\alpha\beta}\hat{\bar{\theta}}^{\dot{\alpha}}\hat{\bar{\theta}}^{\dot{\beta}} - \bar{C}^{\dot{\alpha}\dot{\beta}}\hat{\theta}^\alpha\hat{\theta}^\beta)\sigma^m_{\alpha\dot{\alpha}}\sigma^n_{\beta\dot{\beta}}. \end{array} \tag{10.4}$$

This algebra was defined by Carlson and Nazaryan so that the deformed chiral coordinates $\hat{y}^m = \hat{x}^m + i\hat{\theta}\sigma^m\hat{\bar{\theta}}$ and $\hat{\bar{y}}^m = \hat{x}^m - i\hat{\theta}\sigma^m\hat{\bar{\theta}}$ satisfy

$$\begin{array}{ll} \{\hat{\theta}^\alpha, \hat{\theta}^\beta\} = C^{\alpha\beta} & [\hat{y}^m, \hat{\theta}^\alpha] = 0 \\ \{\hat{\bar{\theta}}^{\dot{\alpha}}, \hat{\bar{\theta}}^{\dot{\beta}}\} = \bar{C}^{\dot{\alpha}\dot{\beta}} & [\hat{\bar{y}}^m, \hat{\bar{\theta}}^{\dot{\alpha}}] = 0 \\ \{\hat{\theta}^\alpha, \hat{\bar{\theta}}^{\dot{\beta}}\} = 0. & \end{array} \tag{10.5}$$

These relations will allow us to develop chiral and antichiral superfields in much the same way as in the commutative theory. In addition, we have:

$$\begin{aligned} [\hat{\bar{y}}^m, \hat{\theta}^\alpha] &= 2iC^{\alpha\beta}\sigma^m_{\beta\dot{\beta}}\hat{\bar{\theta}}^{\dot{\beta}} \\ [\hat{y}^m, \hat{\bar{\theta}}^{\dot{\alpha}}] &= 2i\bar{C}^{\dot{\alpha}\dot{\beta}}\hat{\theta}^\beta\sigma^m_{\beta\dot{\beta}} \\ [\hat{y}^m, \hat{y}^n] &= \left(4\bar{C}^{\dot{\alpha}\dot{\beta}}\hat{\theta}^\alpha\hat{\theta}^\beta - 2C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}\right)\sigma^m_{\alpha\dot{\alpha}}\sigma^n_{\beta\dot{\beta}} \\ [\hat{\bar{y}}^m, \hat{\bar{y}}^n] &= \left(4C^{\alpha\beta}\hat{\bar{\theta}}^{\dot{\alpha}}\hat{\bar{\theta}}^{\dot{\beta}} - 2C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}\right)\sigma^m_{\alpha\dot{\alpha}}\sigma^n_{\beta\dot{\beta}} \\ [\hat{y}^m, \hat{\bar{y}}^n] &= 2C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^m_{\alpha\dot{\alpha}}\sigma^n_{\beta\dot{\beta}}. \end{aligned} \tag{10.6}$$

This choice of deformed coordinate is motivated by our desire to follow the same construction of chiral superfields as in the commutative theory.

10.2.2 Star Product

The star product on Minkowski superspace is defined by

$$f * g = f(1 + S)g \tag{10.7}$$

where S is formed using the supercharges Q_α and $\bar{Q}_{\dot{\alpha}}$,

$$\begin{aligned} S &= -\tfrac{1}{2}C^{\alpha\beta}\overleftarrow{Q}_\alpha\overrightarrow{Q}_\beta - \tfrac{1}{2}\bar{C}^{\dot{\alpha}\dot{\beta}}\overleftarrow{\bar{Q}}_{\dot{\alpha}}\overrightarrow{\bar{Q}}_{\dot{\beta}} \\ &\quad +\tfrac{1}{8}C^{\alpha\beta}C^{\gamma\delta}\overleftarrow{Q}_\alpha\overleftarrow{Q}_\gamma\overrightarrow{Q}_\delta\overrightarrow{Q}_\beta + \tfrac{1}{8}\bar{C}^{\dot{\alpha}\dot{\beta}}\bar{C}^{\dot{\gamma}\dot{\delta}}\overleftarrow{\bar{Q}}_{\dot{\alpha}}\overleftarrow{\bar{Q}}_{\dot{\gamma}}\overrightarrow{\bar{Q}}_{\dot{\delta}}\overrightarrow{\bar{Q}}_{\dot{\beta}} \\ &\quad +\tfrac{1}{4}C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}(\overleftarrow{\bar{Q}}_{\dot{\alpha}}\overleftarrow{Q}_\alpha\overrightarrow{\bar{Q}}_{\dot{\beta}}\overrightarrow{Q}_\beta - \overleftarrow{Q}_\alpha\overleftarrow{\bar{Q}}_{\dot{\alpha}}\overrightarrow{Q}_\beta\overrightarrow{\bar{Q}}_{\dot{\beta}}). \end{aligned}$$

We follow the conventions of Wess and Bagger in [116]. In the chiral coordinates $y^m = x^m + i\theta\sigma^m\bar{\theta}$, the supercharges have the familiar forms. Note that the derivatives of θ^α and $\bar{\theta}^{\dot{\alpha}}$ are taken at fixed y^m.

$$\begin{aligned} Q_\alpha &= \tfrac{\partial}{\partial\theta^\alpha}|_y \\ \bar{Q}_{\dot{\alpha}} &= -\tfrac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}|_y + 2i\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\tfrac{\partial}{\partial y^m} \end{aligned} \tag{10.8}$$

Whereas, when the derivatives are taken at fixed antichiral coordinates $\bar{y}^m = x^m - i\theta\sigma^m\bar{\theta}$, we have

$$\begin{aligned} Q_\alpha &= \tfrac{\partial}{\partial\theta^\alpha}|_y - 2i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\tfrac{\partial}{\partial\bar{y}^m} \\ \bar{Q}_{\dot{\alpha}} &= -\tfrac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}|_{\bar{y}}. \end{aligned} \tag{10.9}$$

We will not make explicit $|_y$ or $|_{\bar{y}}$ elsewhere since they are to be understood implicitly. Many other formulae can be found in [87]. Some properties of this star product on functions f, g, and h are

$$\begin{aligned} \overline{f * g} &= \overline{g} * \overline{f} \qquad & (f + g) * h &= f * h + g * h \\ f * g &\neq g * f \qquad & f * (g * h) &\neq (f * g) * h. \end{aligned} \tag{10.10}$$

The noncommutativity and nonassociativity will require some attention in general. However, to the first order in the deformation parameter, we note that

$$f * (g * h) = (f * g) * h \tag{10.11}$$

the star product is associative. A proof is given in the last section of the chapter.

10.2.3 $N = 0$ Supersymmetry

The formulae below are stated for the operators acting on functions of the deformed Minkowki superspace. In particular, they should be understood as statements about

how the operators act on star products of functions. We define the star brackets as

$$\{A, B\}_* = A * B + B * A \quad \text{and} \quad [A, B]_* = A * B - B * A. \tag{10.12}$$

Then calculate

$$\begin{array}{ll} \{\theta^\alpha, \theta^\beta\}_* & = \theta^\alpha * \theta^\beta + \theta^\beta * \theta^\alpha = C^{\alpha\beta} \\ \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\}_* & = \bar{\theta}^{\dot{\alpha}} * \bar{\theta}^{\dot{\beta}} + \bar{\theta}^{\dot{\beta}} * \bar{\theta}^{\dot{\alpha}} = \bar{C}^{\dot{\alpha}\dot{\beta}}. \end{array} \tag{10.13}$$

It is important to note that products of both θ^α and $\bar{\theta}^{\dot{\alpha}}$ are deformed. This has the consequence of breaking all of the supersymmetry. Starting with the canonical forms of the supercharges, we obtain

$$\begin{array}{ll} \{Q_\alpha, Q_\beta\}_* & = -4\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^m_{\alpha\dot{\alpha}}\sigma^n_{\beta\dot{\beta}}\frac{\partial^2}{\partial\bar{y}^m\partial\bar{y}^n} \\ \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\}_* & = -4C^{\alpha\beta}\sigma^m_{\alpha\dot{\alpha}}\sigma^n_{\beta\dot{\beta}}\frac{\partial^2}{\partial y^m\partial y^n} \\ \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\}_* & = 2i\sigma^m_{\alpha\dot{\alpha}}\frac{\partial}{\partial y^m}. \end{array} \tag{10.14}$$

Comparing this to [112], we note that when $\bar{C}^{\dot{\alpha}\dot{\beta}} = 0$, then Q_α is an unbroken symmetry, hence the label $N = \frac{1}{2}$ supersymmetry. The author proposes that we call the theory constructed by Carlson and Nazaryan $N = 0$ supersymmetry to be consistent. Now, although the supercharges are broken, we still have

$$\begin{array}{l} \{D_\alpha, Q_\beta\}_* = \ \{\bar{D}_{\dot{\beta}}, Q_\beta\}_* = \ \{D_\alpha, \bar{Q}_{\dot{\beta}}\}_* = \ \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\}_* = 0 \\ \{D_\alpha, D_\beta\}_* = \ \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\}_* = 0. \end{array} \tag{10.15}$$

These relations are crucial. We can still define the chiral (Φ) and antichiral ($\bar{\Phi}$) superfields by the constraints $\bar{D}_{\dot{\alpha}} * \Phi = 0$ and $D_\alpha * \bar{\Phi} = 0$ on noncommutative Minkowski superspace. Thus, most of the usual techniques in Wess and Bagger [116] still apply for our discussion. The primary difference is that products will be replaced with star products.

10.3 Vector Superfield

Our goal is to construct a nonabelian gauge on deformed Minkowski superspace. Thus, we consider a vector superfield V which carries some matrix representation of the gauge group and is subject to the usual constraint: $\overline{V} = V$. In the standard super Yang-Mills theory, it is convenient to use a reduced set of component fields called the Wess Zumino gauge. We will show in section 10.4.1 that the Wess Zumino gauge can be generalized to the current discussion provided we make some C dependent shifts. For now, we let V take the canonical parametrization of the Wess-Zumino gauge

$$V = -\theta\sigma^m\bar{\theta}v_m + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}(D - i\partial_m v^m) \tag{10.16}$$

where the above is in chiral coordinates y^m.

10.3.1 Star Exponential of Vector Superfield

We define the star exponential of the vector superfield in the natural way:

$$e^V = 1 + V + \frac{1}{2}V * V + \frac{1}{3!}V * V * V + ... \tag{10.17}$$

Our notation for the usual exponential will be $exp(V)$ and powers are to be understood as ordinary powers - for example $V^2 = VV$. In this chapter, star products will be explicitly indicated.

The vector superfield is even, thus no new signs arise from pushing the Q_α or $\bar{Q}_{\dot{\alpha}}$ past V in the star product. Thus, to first order in the deformation parameter,

$$\begin{aligned} V * V &= V(1+S)V \\ &= V^2 - \frac{1}{2}C^{\alpha\beta}(Q_\alpha V)(Q_\beta V) - \frac{1}{2}\bar{C}^{\dot{\alpha}\dot{\beta}}(\bar{Q}_{\dot{\alpha}} V)(\bar{Q}_{\dot{\beta}} V) \\ &\quad + \frac{1}{8}C^{\alpha\beta}C^{\gamma\delta}(Q_\alpha Q_\gamma V)(Q_\delta Q_\beta V) + \frac{1}{8}\bar{C}^{\dot{\alpha}\dot{\beta}}\bar{C}^{\dot{\gamma}\dot{\delta}}(\bar{Q}_{\dot{\alpha}}\bar{Q}_{\dot{\gamma}} V)(\bar{Q}_{\dot{\delta}}\bar{Q}_{\dot{\beta}} V) \\ &\quad + \frac{1}{4}C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}\left((\bar{Q}_{\dot{\alpha}}Q_\alpha V)(\bar{Q}_{\dot{\beta}}Q_\beta V) - (Q_\alpha\bar{Q}_{\dot{\alpha}} V)(Q_\beta\bar{Q}_{\dot{\beta}} V)\right) \end{aligned}$$

We will now calculate these terms in chiral coordinates starting with

$$\begin{aligned} Q_\alpha V &= \partial_\alpha\left[-\theta\sigma^m\bar{\theta}v_m + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}(D - i\partial_m v^m)\right] \\ &= -\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}v_m + 2i\theta_\alpha\bar{\theta}\bar{\lambda} + \bar{\theta}\bar{\theta}\big(-i\lambda_\alpha + \theta_\alpha(D - i\partial_m v^m)\big). \end{aligned} \tag{10.18}$$

Continuing, we find that

$$\begin{aligned} Q_\beta Q_\alpha V &= \partial_\beta\left[-\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}v_m + 2i\theta_\alpha\bar{\theta}\bar{\lambda} + \bar{\theta}\bar{\theta}(-i\lambda_\alpha + \theta_\alpha(D - i\partial_m v^m))\right] \\ &= -2i\epsilon_{\beta\alpha}\bar{\theta}\bar{\lambda} - \epsilon_{\beta\alpha}\bar{\theta}\bar{\theta}(D - i\partial_m v^m). \end{aligned} \tag{10.19}$$

Next we calculate $\bar{Q}_{\dot{\alpha}}V$.

$$\begin{aligned} \bar{Q}_{\dot{\alpha}}V &= (-\partial_{\dot{\alpha}} + 2i\theta^\alpha\sigma^n_{\alpha\dot{\alpha}}\partial_n)\left[-\theta\sigma^m\bar{\theta}v_m + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}(D - i\partial_m v^m)\right] \\ &= -\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}v_m + \big(-2i\bar{\theta}_{\dot{\alpha}} + 2\bar{\theta}\bar{\theta}\sigma^m_{\alpha\dot{\alpha}}\theta^\alpha\partial_m\big)\theta\lambda \\ &\quad + \theta\theta\left(i\bar{\lambda}_{\dot{\alpha}} + \bar{\theta}_{\dot{\alpha}}(D - i\partial_m v^m) + i\epsilon^{\alpha\beta}\sigma^m_{\alpha\dot{\alpha}}\sigma^n_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_m v_n\right) \end{aligned} \tag{10.20}$$

The next calculation is a bit longer.

$$\begin{aligned}\bar{Q}_{\dot{\alpha}}\bar{Q}_{\dot{\beta}}V &= (-\partial_{\dot{\alpha}} + 2i\theta^{\alpha}\sigma^{m}_{\alpha\dot{\alpha}}\partial_m)(\bar{Q}_{\dot{\beta}}V) \\ &= -2i\epsilon_{\dot{\alpha}\dot{\beta}}\theta\lambda + \theta\theta\Big(\epsilon_{\dot{\alpha}\dot{\beta}}(D - i\partial_m v^m) \\ &\quad + i\epsilon^{\alpha\beta}(\sigma^{m}_{\alpha\dot{\beta}}\sigma^{n}_{\beta\dot{\alpha}} - \sigma^{m}_{\alpha\dot{\alpha}}\sigma^{n}_{\beta\dot{\beta}})\partial_m v_n + 2(\sigma^{m}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\beta}} - \bar{\theta}^{\dot{\alpha}}\sigma^{m}_{\alpha\dot{\beta}})\partial_m\lambda^{\alpha}\Big)\end{aligned} \tag{10.21}$$

Now, for the mixed supercharges, using the results above, we find that

$$\begin{aligned}Q_{\alpha}\bar{Q}_{\dot{\alpha}}V &= \partial_{\alpha}(\bar{Q}_{\dot{\alpha}}V) \\ &= -\sigma^{m}_{\alpha\dot{\alpha}}v_m + 2i(\theta_{\alpha}\bar{\lambda}_{\dot{\alpha}} - \bar{\theta}_{\dot{\alpha}}\lambda_{\alpha}) \\ &\quad + \theta_{\alpha}\Big(2\bar{\theta}_{\dot{\alpha}}(D - i\partial_m v^m) + 2i\bar{\theta}^{\dot{\beta}}\epsilon^{\sigma\beta}\sigma^{m}_{\sigma\dot{\alpha}}\sigma^{n}_{\beta\dot{\beta}}\partial_m v_n + 2\bar{\theta}\bar{\theta}\sigma^{m}_{\beta\dot{\alpha}}\partial_m\lambda^{\beta}\Big)\end{aligned} \tag{10.22}$$

Similarly, we find that

$$\begin{aligned}\bar{Q}_{\dot{\alpha}}Q_{\alpha}V &= (-\partial_{\dot{\beta}} + 2i\theta^{\alpha}\sigma^{m}_{\alpha\dot{\alpha}}\partial_m)(Q_{\alpha}V) \\ &= \sigma^{m}_{\alpha\dot{\alpha}}v_m - 2i(\theta_{\alpha}\bar{\lambda}_{\dot{\alpha}} - \bar{\theta}_{\dot{\alpha}}\lambda_{\alpha}) \\ &\quad + 2\theta_{\alpha}\bar{\theta}_{\dot{\alpha}}(D - i\partial_m v^m) - 2i\theta^{\beta}\bar{\theta}^{\dot{\beta}}\sigma^{m}_{\beta\dot{\alpha}}\sigma^{n}_{\alpha\dot{\beta}}\partial_m v_n \\ &\quad - 2\theta\theta\sigma^{m}_{\alpha\dot{\alpha}}\partial_m(\bar{\theta}\bar{\lambda}) + \bar{\theta}\bar{\theta}\Big(2\theta^{\beta}\sigma^{m}_{\beta\dot{\alpha}}\partial_m\lambda_{\alpha} + i\theta\theta\sigma^{m}_{\alpha\dot{\alpha}}\partial_m(D - i\partial_m v^m)\Big)\end{aligned} \tag{10.23}$$

The next task is to calculate the products of the terms above. In the product below, we have omitted from the beginning those terms with $\bar{\theta}\bar{\theta}$ because there is a $\bar{\theta}$ in each term.

$$\begin{aligned}\frac{1}{2}C^{\alpha\beta}Q_{\alpha}VQ_{\beta}V &= \frac{1}{2}C^{\alpha\beta}\big[-\sigma^{m}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}v_m + 2i\theta_{\alpha}(\bar{\theta}\bar{\lambda})\big]\big[-\sigma^{n}_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}v_n + 2i\theta_{\beta}(\bar{\theta}\bar{\lambda})\big] \\ &= \frac{1}{4}C^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^{m}_{\alpha\dot{\alpha}}\sigma^{n}_{\beta\dot{\beta}}v_m v_n\bar{\theta}\bar{\theta} + \frac{i}{2}C^{\alpha\beta}\theta^{\beta}\sigma^{m}_{\alpha\dot{\alpha}}[v_m, \bar{\lambda}^{\dot{\alpha}}]\bar{\theta}\bar{\theta} \\ &= \Big(\frac{1}{2}C^{mn}v_m v_n - \frac{i}{2}C^{\alpha\beta}\sigma^{m}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\beta}}[v_m, \bar{\lambda}^{\dot{\alpha}}]\Big)\bar{\theta}\bar{\theta}\end{aligned} \tag{10.24}$$

where we have used the identity $C^{mn} = \frac{1}{2}C^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^{m}_{\alpha\dot{\alpha}}\sigma^{n}_{\beta\dot{\beta}}$, following the conventions of [112]. Continuing to compute the products, since every term has a θ this time, we

can ignore the $\theta\theta$ terms from the outset.

$$\begin{aligned}\frac{1}{2}\bar{C}^{\dot{\alpha}\dot{\beta}}\bar{Q}_{\dot{\alpha}}V\bar{Q}_{\dot{\beta}}V &= \frac{1}{2}\bar{C}^{\dot{\alpha}\dot{\beta}}\left[-\theta^{\alpha}\sigma^{m}_{\alpha\dot{\alpha}}v_m - 2i\bar{\theta}_{\dot{\alpha}}\theta\lambda\right]\left[-\theta^{\beta}\sigma^{n}_{\beta\dot{\beta}}v_n - 2i\bar{\theta}_{\dot{\beta}}\theta\lambda\right] \quad (10.25)\\ &= -\frac{1}{4}\bar{C}^{\dot{\alpha}\dot{\beta}}\epsilon^{\beta\alpha}\sigma^{m}_{\alpha\dot{\beta}}\sigma^{n}_{\beta\dot{\alpha}}v_m v_n\theta\theta - \frac{i}{2}\bar{C}^{\dot{\alpha}\dot{\beta}}\bar{\theta}_{\dot{\beta}}\sigma^{m}_{\alpha\dot{\alpha}}[v_m,\lambda_{\alpha}]\theta\theta\\ &= \left(\frac{1}{2}\bar{C}^{mn}v_m v_n + \frac{i}{2}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^{m}_{\alpha\dot{\alpha}}\bar{\theta}_{\dot{\beta}}[v_m,\lambda^{\alpha}]\right)\theta\theta\end{aligned}$$

where we identified $\bar{C}^{mn} = -\frac{1}{2}\bar{C}^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^{m}_{\alpha\dot{\alpha}}\sigma^{n}_{\beta\dot{\beta}}$ following [87]. Next, consider the second order in deformation parameter terms:

$$\begin{aligned}\frac{1}{8}C^{\alpha\beta}C^{\gamma\delta}(Q_{\alpha}Q_{\gamma}V)(Q_{\delta}Q_{\beta}V) &= \frac{1}{8}C^{\alpha\beta}C^{\gamma\delta}\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}[2i\bar{\theta}\bar{\lambda} + \bar{\theta}\bar{\theta}(D - i\partial_m v^m)]^2\\ &= -\frac{1}{8}|C|^2\bar{\lambda}\bar{\lambda}\bar{\theta}\bar{\theta}\end{aligned}$$

where we use $|C|^2 = 4C^{\alpha\beta}C^{\gamma\delta}\epsilon_{\alpha\gamma}\epsilon_{\delta\beta}$. Similarly, we find that the next term is easily calculated due to a sizeable cancellation since we may omit a $\theta\theta$ term from the start.

$$\begin{aligned}\frac{1}{8}\bar{C}^{\dot{\alpha}\dot{\beta}}\bar{C}^{\dot{\gamma}\dot{\delta}}(\bar{Q}_{\dot{\alpha}}\bar{Q}_{\dot{\gamma}}V)(\bar{Q}_{\dot{\delta}}\bar{Q}_{\dot{\beta}}V) &= \frac{1}{8}\bar{C}^{\dot{\alpha}\dot{\beta}}\bar{C}^{\dot{\gamma}\dot{\delta}}\epsilon_{\dot{\alpha}\dot{\gamma}}\epsilon_{\dot{\beta}\dot{\delta}}[-2i\theta\lambda]^2\\ &= -\frac{1}{8}|\bar{C}|^2\lambda\lambda\theta\theta\end{aligned}$$

where we use $|\bar{C}|^2 = 4\bar{C}^{\dot{\alpha}\dot{\beta}}\bar{C}^{\dot{\gamma}\dot{\delta}}\epsilon_{\dot{\alpha}\dot{\gamma}}\epsilon_{\dot{\beta}\dot{\delta}}$. The remaining term to consider in $V * V$ is

$\frac{1}{4}C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}[(\bar{Q}_{\dot{\alpha}}Q_\alpha V)(\bar{Q}_{\dot{\beta}}Q_\beta V) - (Q_\alpha\bar{Q}_{\dot{\alpha}}V)(Q_\beta\bar{Q}_{\dot{\beta}}V)]$. We calculate

$$\begin{aligned}
&\tfrac{1}{4}C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}\Big((\bar{Q}_{\dot{\alpha}}Q_\alpha V)(\bar{Q}_{\dot{\beta}}Q_\beta V) - (Q_\alpha\bar{Q}_{\dot{\alpha}}V)(Q_\beta\bar{Q}_{\dot{\beta}}V)\Big) = \\
&\quad = \tfrac{1}{4}C^{\alpha\beta}\bar{C}^{\dot{\alpha}\dot{\beta}}\Big(\sigma^m_{\alpha\dot{\alpha}}\{v_m, 4i(\bar{\theta}^{\dot{\beta}}\lambda_\beta - \theta^\beta\bar{\lambda}_{\dot{\beta}})\} \\
&\qquad - 2i\sigma^m_{\alpha\dot{\alpha}}\{v_m, \partial_l v_k(\theta^\gamma\sigma^l_{\gamma\dot{\beta}}\sigma^k_{\beta\dot{\gamma}}\bar{\theta}^{\dot{\gamma}} + \theta^\beta\epsilon^{\sigma\gamma}\sigma^l_{\delta\dot{\beta}}\sigma^k_{\gamma\dot{\gamma}}\bar{\theta}^{\dot{\gamma}})\} \\
&\qquad + 2\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}\bar{\theta}\sigma^l_{\gamma\dot{\beta}}\{v_m, \theta^\gamma\partial_l\lambda_\beta - \theta^\beta\partial_l\lambda^\gamma\} \\
&\qquad - 2\sigma^m_{\alpha\dot{\alpha}}\theta\theta\sigma^l_{\beta\dot{\beta}}\{v_m, \partial_l(\bar{\theta}\bar{\lambda})\} \\
&\qquad + i\sigma^m_{\alpha\dot{\alpha}}\theta\theta\bar{\theta}\bar{\theta}\sigma^l_{\beta\dot{\beta}}\{v_m, \partial_l(D - i\partial_m v^m)\} \\
&\qquad - 4\theta_\alpha\theta^\gamma\sigma^m_{\gamma\dot{\beta}}\sigma^n_{\beta\dot{\gamma}}\bar{\theta}^{\dot{\gamma}}\{\bar{\lambda}_{\dot{\alpha}}, \partial_m v_n\} \\
&\qquad + 4\bar{\theta}_{\dot{\alpha}}\theta^\gamma\sigma^m_{\gamma\dot{\beta}}\sigma^n_{\beta\dot{\gamma}}\bar{\theta}^{\dot{\gamma}}\{\lambda_\alpha, \partial_m v_n\} \\
&\qquad + 4i\theta_\alpha\bar{\theta}\bar{\theta}\theta^\gamma\sigma^m_{\gamma\dot{\beta}}\{\bar{\lambda}_{\dot{\alpha}}, \partial_m\lambda_\beta\} \\
&\qquad + 4i\theta_\alpha\theta\theta\theta^\gamma\sigma^m_{\beta\dot{\beta}}\{\lambda_\alpha, \partial_m\bar{\lambda}_{\dot{\gamma}}\} \\
&\qquad - 4i\theta_\alpha\bar{\theta}_{\dot{\alpha}}\theta^\gamma\sigma^m_{\gamma\dot{\beta}}\sigma^n_{\beta\dot{\gamma}}\bar{\theta}^{\dot{\gamma}}\{(D - i\partial_m v^m), \partial_m v_n\} \\
&\qquad - 4\theta^\gamma\bar{\theta}^{\dot{\gamma}}\theta^\sigma\bar{\theta}^{\dot{\sigma}}\sigma^k_{\gamma\dot{\sigma}}\sigma^l_{\alpha\dot{\gamma}}\sigma^m_{\sigma\dot{\beta}}\sigma^n_{\beta\dot{\sigma}}\partial_k v_l\partial_m v_n\Big).
\end{aligned} \tag{10.26}$$

We can see from the expression above the full second order calculations will be lengthy. Additionally, we would have to deal with the nonassociativity of the star product. At present, the author has only calculated portions of the theory to the second order, mostly for the purpose of comparing the present work with [112]. We leave the complete development of the second order deformed gauge theory to a later paper.

Expanding $V * V * V$

We shall now find the correction to $V * V * V$ to the first order in $C^{\alpha\beta}$. First, recall first that in the commutative theory, V^3 is zero in the Wess-Zumino gauge. Thus any nontrivial term in $V * V * V$ must arise from the deformation.

$$\begin{aligned}
&V * (V * V) \; = \\
&\quad = \; V(V * V) - \tfrac{1}{2}C^{\alpha\beta}(Q_\alpha V)Q_\beta(V * V) - \tfrac{1}{2}\bar{C}^{\dot{\alpha}\dot{\beta}}(\bar{Q}_{\dot{\alpha}}V)\bar{Q}_{\dot{\beta}}(V * V)
\end{aligned} \tag{10.27}$$

We can replace $V * V$ with V^2 as we are looking for the first order in $C^{\alpha\beta}$ terms.

$$\begin{aligned}
&V * (V * V) \; = \\
&\quad = \; V(V * V) - \tfrac{1}{2}C^{\alpha\beta}(Q_\alpha V)Q_\beta(V^2) - \tfrac{1}{2}\bar{C}^{\dot{\alpha}\dot{\beta}}(\bar{Q}_{\dot{\alpha}}V)\bar{Q}_{\dot{\beta}}(V^2) \\
&\quad = \; V(V * V)
\end{aligned} \tag{10.28}$$

The two terms vanish because $Q_\alpha V$ and $\bar{Q}_{\dot{\alpha}}V$ have a $\bar{\theta}$ and θ in each term respectively while $Q_\beta(V^2)$ and $\bar{Q}_{\dot{\beta}}(V^2)$ are proportional to $\bar{\theta}\bar{\theta}$ and $\theta\theta$ respectively. To the first

order, we have

$$\begin{aligned} V*(V*V) \;&= \\ &= \big(-\theta\sigma^m\bar{\theta}v_m + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}(D - i\partial_m v^m)\big)(V*V). \end{aligned} \tag{10.29}$$

Now, if we examine the first order terms in $V*V$, we notice that each term either has $\theta\theta$ or $\bar{\theta}\bar{\theta}$; thus, the product with V which is proportional to θ and $\bar{\theta}$ vanishes. Therefore, to the first order in the deformation parameter,

$$V*(V*V) \;= 0. \tag{10.30}$$

It is not hard to see that this extends to higher star products. Thus, $(V)_*^n = 0$ for $n \geq 3$ to the first order in the deformation parameter. That is, to the first order in C, we have $e^V = 1 + V + \frac{1}{2}V*V$. This is nice but it will clearly be spoiled if we include the second order terms. For example, if one examines the mixed second order term (10.26), the first few lines have only θ or $\bar{\theta}$. Hence, in the product with V they will not vanish like the first order case, thus generating a nontrivial term in $V*(V*V)$. We will not complete the development of e^V to the second order in this chapter. Next, we shall show that in the limit of $\bar{C}^{\dot{\alpha}\dot{\beta}} = 0$, we recover the terms found by Seiberg in [112].

Collecting the results of this section, we find that the star exponential of V in the canonical Wess-Zumino gauge is

$$\begin{aligned} e^V &= 1 + V + \tfrac{1}{2}V*V \\ &= 1 - \theta\sigma^m\bar{\theta}v_m + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta^\alpha\lambda_\alpha + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}(D - i\partial_m v^m) \\ &\quad - \big(\tfrac{1}{4}C^{mn}v_m v_n + \tfrac{i}{4}C^{\alpha\beta}\theta_\beta\sigma^m_{\alpha\dot{\alpha}}[\bar{\lambda}^{\dot{\alpha}}, v_m]\big)\bar{\theta}\bar{\theta} \\ &\quad - \big(\tfrac{1}{4}\bar{C}^{mn}v_m v_n + \tfrac{i}{4}\bar{C}^{\dot{\alpha}\dot{\beta}}\bar{\theta}_{\dot{\beta}}\sigma^m_{\alpha\dot{\alpha}}[v_m, \lambda^\alpha]\big)\theta\theta. \\ &\quad - \tfrac{1}{16}|C|^2\bar{\lambda}\bar{\lambda}\bar{\theta}\bar{\theta} \\ &\quad - \tfrac{1}{16}|\bar{C}|^2\lambda\lambda\theta\theta \\ &\quad + \textit{other } 2^{nd} \textit{ order terms containing } \bar{C}^{\dot{\alpha}\dot{\beta}}. \end{aligned} \tag{10.31}$$

10.3.2 $N = 0$ Verses $N = \frac{1}{2}$ Star Exponentials

To compare with the $N = \frac{1}{2}$ construction, we make the following dictionary:

$$\begin{aligned} &m \longmapsto \mu \\ &v_m \longmapsto A_\mu \\ &\bar{\lambda}_{\dot{\alpha}} \longmapsto \bar{\lambda}_{\dot{\alpha}} \\ &\lambda_\alpha \longmapsto \lambda_\alpha + \tfrac{1}{4}\epsilon_{\alpha\beta}C^{\beta\gamma}\sigma^\mu_{\gamma\dot{\gamma}}\{\bar{\lambda}^{\dot{\gamma}}, A_\mu\} \\ &(D - i\partial_m v^m) \longmapsto D - i\partial_\mu A^\mu. \end{aligned} \tag{10.32}$$

We use Greek indices for Euclidean spacetime and Latin indices for Minkowski spacetime. In [112], only products of θ were deformed. It is clear that we can recover

this deformation by setting $\bar{C}^{\dot{\alpha}\dot{\beta}}$ to zero wherever it occurs. Using the dictionary and setting $\bar{C}^{\dot{\alpha}\dot{\beta}} = 0$, we have

$$\begin{aligned} e^V &= 1 + V + \tfrac{1}{2} V * V \\ &= 1 - \theta\sigma^\mu\bar{\theta}A_\mu + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta^\alpha(\lambda_\alpha + \tfrac{1}{4}\epsilon_{\alpha\beta}C^{\beta\gamma}\sigma^\mu_{\gamma\dot{\gamma}}\{\bar{\lambda}^{\dot{\gamma}}, A_\mu\}) \\ &\quad + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}(D - i\partial_\mu A^\mu) - \tfrac{1}{4}C^{\mu\nu}A_\mu A_\nu\bar{\theta}\bar{\theta} \\ &\quad - \tfrac{i}{4}C^{\alpha\beta}\theta_\beta\sigma^\mu_{\alpha\dot{\alpha}}[A_\mu, \bar{\lambda}^{\dot{\alpha}}]\bar{\theta}\bar{\theta} - \tfrac{1}{16}|C|^2\bar{\lambda}\bar{\lambda}\bar{\theta}\bar{\theta}. \end{aligned} \tag{10.33}$$

This is precisely the exponential that Seiberg found on noncommutative Euclidean superspace in [112].

10.4 Gauge Theory on $N = 0$ Minkowski Superspace

In this section, we generalize super Yang-Mills theory to deformed Minkowski superspace. Most of the usual constructions hold and the approach is similar to Sieberg's $N = \frac{1}{2}$ super Yang Mills theory in [112]. We simply replace products in [116] with star products. The main subtlety is finding the correct parametrization of the vector superfield.

10.4.1 Gauge Transformations

Our goal is to find a way to embed the usual C-independent gauge transformations into superfield equations on noncommutative Minkowski superspace. Since our spinors are built on Minkowski space, we are forced to relate θ and $\bar{\theta}$ by conjugation. This means that we cannot directly follow the construction of [112]. In [112], we can see that $\overline{(\theta^\alpha)} \neq \bar{\theta}^{\dot{\alpha}}$, $\overline{V} \neq V$ and $\overline{(\Lambda + \bar{\Lambda})} \neq \Lambda + \bar{\Lambda}$. These relations are sensible for Seiberg, who wrote them over noncommutative Euclidean superspace. On Minkowski space, these inequalities must become equalities. We will find that these reality conditions and the requirement that we recover $N = \frac{1}{2}$ theory in the $\bar{C}^{\dot{\alpha}\dot{\beta}} = 0$ limit almost uniquely fixes this construction.

Nonabelian gauge transformations on the vector superfield are embedded into the following superfield equation on noncommutative Minkowski superspace.

$$e^V \longmapsto e^{V\prime} = e^{-i\bar{\Lambda}} * e^V * e^{i\Lambda} \tag{10.34}$$

This is the natural modification of [116]. Infinitesimally, we have

$$\delta e^V = -i\bar{\Lambda} * e^V + i e^V * \Lambda. \tag{10.35}$$

The component fields of the vector superfield should transform in the adjoint representation of the gauge group as in the standard gauge theory. That is, under an

infinitesimal gauge transformation, we should have

$$\begin{aligned}\delta v_m &= -2\partial_m\phi + i[\phi, v_m]\\ \delta\lambda_\alpha &= i[\phi,\lambda_\alpha]\\ \delta D &= i[\phi, D].\end{aligned} \tag{10.36}$$

Our goal now is to find a suitable parametrization of the gauge parameter Λ and the vector superfield V such that (10.36) are embedded into (10.35). It is not surprising that the canonical Wess-Zumino gauge (10.31) does not work in the $N = 0$ case, since it was also necessary for [112] to shift the λ component in the $N = \frac{1}{2}$ case. The reality of V requires that we cannot shift only λ; we must also shift $\bar\lambda$. To be precise, $\lambda \mapsto \lambda + A$ and $\bar\lambda \mapsto \bar\lambda + B$. We now determine what choice of A and B will preserve the reality of V while concurrently embedding (10.36). To the first order in C, we find under the above redefinitions that (10.31) becomes,

$$\begin{aligned}e^V = 1 &- \theta\sigma^m\bar\theta v_m - \tfrac{1}{4}\bar C^{mn}v_m v_n\theta\theta + \tfrac{1}{4}C^{mn}v_m v_n\bar\theta\bar\theta + \tfrac{1}{2}(D - i\partial_m v^m)\theta\theta\bar\theta\bar\theta\\ &+ \bar\theta\bar\theta\theta^\alpha(-i\lambda_\alpha - iA + \tfrac{i}{4}\epsilon_{\alpha\beta}C^{\beta\gamma}\sigma^m_{\gamma\dot\alpha}[v_m, \bar\lambda^{\dot\alpha}])\\ &+ \theta\theta\bar\theta^{\dot\alpha}(-i\bar\lambda_{\dot\alpha} - iB - \tfrac{i}{4}\epsilon_{\dot\alpha\dot\beta}\bar C^{\dot\beta\dot\gamma}\sigma^m_{\alpha\dot\gamma}[v_m,\lambda^\alpha]).\end{aligned} \tag{10.37}$$

Additionally, we make a C-dependent shift of the gauge parameter Λ similar to that of [112]. For the moment, let us make a reasonably general ansatz for the gauge parameter in terms of a variable p.

$$\begin{aligned}\Lambda_p &= -\phi + ip\theta\sigma^m\bar\theta\partial_m\phi + \tfrac{i}{2}\theta\theta\bar C^{mn}\{v_n, \partial_m\phi\} - (p+1)\theta\theta\bar\theta\bar\theta\partial^2\phi\\ \bar\Lambda_p &= -\phi + i(2-p)\theta\sigma^m\bar\theta\partial_m\phi - \tfrac{i}{2}\bar\theta\bar\theta C^{mn}\{\partial_m\phi, v_n\} - (p+1)\theta\theta\bar\theta\bar\theta\partial^2\phi\end{aligned} \tag{10.38}$$

where everything is a function of y in the above. Notice that modulo the higher θ components in Λ, this reduces to the choice of gauge parameter in [112] when $p = 0$. We now determine which choice of p will embed (10.36) in (10.35). We calculate that the $\bar\theta\bar\theta\theta^\alpha$ term in the RHS of (10.35) is

$$[\phi,\lambda_\alpha] + [\phi, A] - \tfrac{1}{4}\epsilon_{\alpha\beta}C^{\beta\gamma}\sigma^m_{\gamma\dot\alpha}\big([\phi,[\bar\lambda^{\dot\alpha}, v_m]] - 2i(p\bar\lambda^{\dot\alpha}\partial_m\phi + (2-p)\partial_m\phi\bar\lambda^{\dot\alpha})\big). \tag{10.39}$$

Similarly, the $\theta\theta\bar\theta^{\dot\alpha}$ term in the RHS of (10.35) is

$$[\phi,\bar\lambda_{\dot\alpha}] + [\phi, B] + \tfrac{1}{4}\bar\epsilon_{\dot\alpha\dot\beta}\bar C^{\dot\beta\dot\gamma}\sigma^m_{\alpha\dot\gamma}\big([\phi,[\lambda^\alpha, v_m]] + 2i(p\lambda^\alpha\partial_m\phi + (2-p)\partial_m\phi\lambda^\alpha)\big). \tag{10.40}$$

The $\bar\theta\bar\theta\theta^\alpha$ component of the LHS of (10.35) is

$$-i\delta\lambda_\alpha - i\delta A + \tfrac{i}{4}\epsilon_{\alpha\beta}C^{\beta\gamma}\sigma^m_{\gamma\dot\alpha}\delta[\bar\lambda^{\dot\alpha}, v_m]. \tag{10.41}$$

Similarly, the $\theta\theta\bar\theta^{\dot\alpha}$ component of the LHS of (10.35) is

$$-i\delta\bar\lambda_{\dot\alpha} - i\delta B - \tfrac{i}{4}\bar\epsilon_{\dot\alpha\dot\beta}\bar C^{\dot\beta\dot\gamma}\sigma^m_{\alpha\dot\gamma}\delta[\lambda^\alpha, v_m]. \tag{10.42}$$

It is not difficult to show (applying (10.36)) that

$$\begin{array}{c} i\delta[\lambda^\alpha, v_m] + [\phi, [\lambda^\alpha, v_m]] = -2i[\lambda^\alpha, \partial_m\phi] \\ i\delta[\bar\lambda^{\dot\alpha}, v_m] + [\phi, [\bar\lambda^{\dot\alpha}, v_m]] = 2i[\bar\lambda^{\dot\alpha}, \partial_m\phi] \\ i\delta\{\bar\lambda^{\dot\alpha}, v_m\} + [\phi, \{\bar\lambda^{\dot\alpha}, v_m\}] = 2i\{\bar\lambda^{\dot\alpha}, \partial_m\phi\} \\ i\delta(v_m\lambda^\alpha) + [\phi, v_m\lambda^\alpha] = 2i\partial_m\phi\lambda^\alpha \\ i\delta(\bar\lambda^{\dot\alpha}v_m) + [\phi, \bar\lambda^{\dot\alpha}v_m] = 2i\bar\lambda^{\dot\alpha}\partial_m\phi. \end{array} \tag{10.43}$$

Next, equate (10.41) and (10.39). Then require that $\delta\lambda^\alpha = i[\phi, \lambda^\alpha]$ so that (10.43) holds. Some terms cancel and we find that

$$-i\delta A - [\phi, A] = \tfrac{i}{2}\epsilon_{\alpha\beta}C^{\beta\gamma}\sigma^m_{\gamma\dot\alpha}\big((p+1)\bar\lambda^{\dot\alpha}\partial_m\phi + (1-p)\partial_m\phi\bar\lambda^{\dot\alpha}\big). \tag{10.44}$$

Likewise, equate (10.42) and (10.40). Then require that $\delta\bar\lambda^{\dot\alpha} = i[\phi, \bar\lambda^{\dot\alpha}]$ so that (10.43) holds. Some terms cancel and we find that

$$-i\delta B - [\phi, B] = \tfrac{i}{2}\epsilon_{\alpha\beta}C^{\beta\gamma}\sigma^m_{\gamma\dot\alpha}\big((p-1)\lambda^\alpha\partial_m\phi + (3-p)\partial_m\phi\lambda^\alpha\big). \tag{10.45}$$

When $p = 0$, we find that (10.44) becomes

$$-i\delta A - [\phi, A] = \tfrac{i}{2}\epsilon_{\alpha\beta}C^{\beta\gamma}\sigma^m_{\gamma\dot\alpha}\{\bar\lambda^{\dot\alpha}, \partial_m\phi\}. \tag{10.46}$$

Hence, in view of (10.43), we can see why [112] shifted the λ_α component of the vector multiplet by $A = \frac{1}{4}\epsilon_{\alpha\beta}C^{\beta\gamma}\sigma^m_{\gamma\dot\alpha}\{\bar\lambda^{\dot\alpha}, v_m\}$. If we tried to use this choice of gauge parameter, we would destroy the reality of V because (10.45) would lead us to choose $B = \frac{1}{4}\bar\epsilon_{\dot\alpha\dot\beta}\bar C^{\dot\beta\dot\gamma}\sigma^m_{\alpha\dot\gamma}(-\lambda^\alpha v_m + 3v_m\lambda^\alpha)$. The correct choice is $p = 1$. With this choice of gauge parameter, we find the following conditions for A and B from (10.44) and (10.45):

$$\begin{array}{l} -i\delta A - [\phi, A] = i\epsilon_{\alpha\beta}C^{\beta\gamma}\sigma^m_{\gamma\dot\alpha}\bar\lambda^{\dot\alpha}\partial_m\phi \\ -i\delta B - [\phi, B] = i\bar\epsilon_{\dot\alpha\dot\beta}\bar C^{\dot\beta\dot\gamma}\sigma^m_{\alpha\dot\gamma}\partial_m\phi\lambda^\alpha. \end{array} \tag{10.47}$$

These conditions are satisfied by

$$\begin{array}{l} A = \frac{1}{2}\epsilon_{\alpha\beta}C^{\beta\gamma}\sigma^m_{\gamma\dot\alpha}\bar\lambda^{\dot\alpha}v_m \\ B = \frac{1}{2}\bar\epsilon_{\dot\alpha\dot\beta}\bar C^{\dot\beta\dot\gamma}\sigma^m_{\alpha\dot\gamma}v_m\lambda^\alpha. \end{array} \tag{10.48}$$

It is easy to see that $\bar A = B$ and $\bar B = A$, which is necessary in order to preserve $\bar V = V$. This is the only parametrization of the vector superfield and gauge parameter for noncommutative Minkowski superspace if we wish to stay in a generalized Wess Zumino gauge. In principle, we could use the other lower θ components of the vector superfield to do more complicated shifts. Fortunately, we will not need to do that.

Define the vector superfield to be

$$\begin{aligned} V(y) = &-\theta\sigma^m\bar{\theta}v_m + \theta\theta\bar{\theta}\dot{\alpha}(-i\bar{\lambda}_{\dot{\alpha}} + \tfrac{i}{2}\bar{\epsilon}_{\dot{\alpha}\dot{\beta}}\bar{C}^{\dot{\beta}\dot{\gamma}}\sigma^m_{\alpha\dot{\gamma}}v_m\lambda^\alpha) \\ &+\bar{\theta}\bar{\theta}\theta^\alpha(-i\lambda_\alpha - \tfrac{i}{2}\epsilon_{\alpha\beta}C^{\beta\gamma}\sigma^m_{\gamma\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}v_m) + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}(D - i\partial_m v^m). \end{aligned} \tag{10.49}$$

It should be evident from the calculations in this section that this parametrization of V embeds (10.36) in (10.35) while maintaining the reality of V. This, of course, requires that we define the gauge parameters as functions of y to be

$$\begin{aligned} \Lambda(y) &= -\phi + i\theta\sigma^m\bar{\theta}\partial_m\phi + \tfrac{i}{2}\theta\theta\bar{C}^{mn}\{v_n, \partial_m\phi\} - 2\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi \\ \bar{\Lambda}(y) &= -\phi + i\theta\sigma^m\bar{\theta}\partial_m\phi - \tfrac{i}{2}\bar{\theta}\bar{\theta}C^{mn}\{\partial_m\phi, v_n\} - 2\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi. \end{aligned} \tag{10.50}$$

For the remainder of this chapter, we will assume that the vector superfield is parametrized as in (10.49) and that the gauge parameter is parametrized as in (10.50). Explicitly in this parametrization, to the first order in C, (10.31) becomes:

$$\begin{aligned} e^V = 1 &- \theta\sigma^m\bar{\theta}v_m - \tfrac{1}{4}\bar{C}^{mn}v_m v_n\theta\theta + \tfrac{1}{4}C^{mn}v_m v_n\bar{\theta}\bar{\theta} + \tfrac{1}{2}(D - i\partial_m v^m)\theta\theta\bar{\theta}\bar{\theta} \\ &+ \bar{\theta}\bar{\theta}\theta^\alpha(-i\lambda_\alpha - \tfrac{i}{4}\epsilon_{\alpha\beta}C^{\beta\gamma}\sigma^m_{\gamma\dot{\alpha}}\{\bar{\lambda}^{\dot{\alpha}}, v_m\}) \\ &+ \theta\theta\bar{\theta}\dot{\alpha}(-i\bar{\lambda}_{\dot{\alpha}} - \tfrac{i}{4}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{C}^{\dot{\beta}\dot{\gamma}}\sigma^m_{\alpha\dot{\gamma}}\{\lambda^\alpha, v_m\}). \end{aligned} \tag{10.51}$$

10.4.2 Spinor Superfields

Again, we will construct these as in the commutative theory except that everywhere that we had a product in the commutative theory, we place a star product here. Define

$$W_\alpha = -\tfrac{1}{4}\bar{D}_{\dot{\alpha}} * \bar{D}^{\dot{\alpha}} * e^{-V} * D_\alpha * e^V. \tag{10.52}$$

Conveniently, in chiral coordinates $y^m = x^m + i\theta\sigma^m\bar{\theta}$, several of the star products in the above are ordinary products. Thus,

$$W_\alpha = -\tfrac{1}{4}\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}e^{-V} * D^\alpha * e^V. \tag{10.53}$$

Likewise define

$$\overline{W}_{\dot{\alpha}} = -\tfrac{1}{4}D^\alpha * D_\alpha * e^{-V} * \bar{D}_{\dot{\alpha}} * e^V. \tag{10.54}$$

Similarly, in antichiral coordinates $\bar{y}^m = x^m - i\theta\sigma^m\bar{\theta}$, the above simplifies to

$$\overline{W}_{\dot{\alpha}} = -\tfrac{1}{4}D^\alpha D_\alpha e^{-V} * \bar{D}_{\dot{\alpha}} * e^V. \tag{10.55}$$

We must determine the component field content of W_α and $\overline{W}_{\dot{\alpha}}$. Referring to (10.51) and keeping only up to the first order in C, we obtain

$$\begin{aligned} W_\alpha = &W_\alpha(C = 0) \\ &+\theta\theta(\tfrac{1}{2}\bar{C}^{mn}\{F_{mn}, \lambda_\alpha\} + \bar{C}^{mn}\{v_n, \mathcal{D}_m\lambda_\alpha - \tfrac{i}{4}[v_m, \lambda_\alpha]\}) \\ &+C^{\gamma\beta}\epsilon_{\beta\alpha}\theta_\gamma\bar{\lambda}\bar{\lambda}, \end{aligned} \tag{10.56}$$

where following Wess and Bagger's conventions in [116], the field strength and covariant derivative of the gaugino are

$$\begin{aligned} F_{mn} &= \partial_m v_n - \partial_n v_m + \tfrac{i}{2}[v_m, v_n] \\ \mathcal{D}_m \lambda_\alpha &= \partial_m \lambda_\alpha + \tfrac{i}{2}[v_m, \lambda_\alpha]. \end{aligned} \tag{10.57}$$

Additionally, the spinor superfield of ordinary superspace is

$$W_\alpha(C=0) = -i\lambda_\alpha + \theta_\alpha D - \sigma_\alpha^{mn\beta}\theta_\beta F_{mn} + \theta\theta\sigma^m_{\alpha\dot\beta}\mathcal{D}_m\bar\lambda^{\dot\beta}. \tag{10.58}$$

Notice that when we set $\bar C^{\dot\alpha\dot\beta} = 0$, we recover the result of Seiberg [112] for W_α. Likewise, we find that

$$\begin{aligned} \overline{W}_{\dot\alpha} = \overline{W}_{\dot\alpha}&(C=0) \\ &+\bar\theta\bar\theta(\tfrac{1}{2}C^{mn}\{F_{mn}, \bar\lambda_{\dot\alpha}\} + C^{mn}\{v_n, \mathcal{D}_m\bar\lambda_{\dot\alpha} - \tfrac{i}{4}[v_m, \bar\lambda_{\dot\alpha}]\}) \\ &+\bar C^{\dot\gamma\dot\beta}\epsilon^{\dot\beta\dot\alpha}\bar\theta_{\dot\gamma}\lambda\lambda \end{aligned} \tag{10.59}$$

where

$$\overline{W}_{\dot\alpha}(C=0) = i\bar\lambda_{\dot\alpha} + \bar\theta_{\dot\alpha}D - \sigma_{\dot\alpha}^{mn\dot\beta}\bar\theta_{\dot\beta}F_{mn} + \bar\theta\bar\theta\bar\sigma^{m\dot\alpha\beta}\mathcal{D}_m\lambda^\beta. \tag{10.60}$$

Again, we reproduce the result of [112] upon setting $\bar C^{\dot\alpha\dot\beta} = 0$.

Gauge Transformation of Spinor Superfields

The spinor superfield transforms as in the commutative theory. From the nonabelian gauge transformation (10.34), it follows that

$$W_\alpha \mapsto W'_\alpha = e^{-i\bar\Lambda} * W_\alpha * e^{i\Lambda}. \tag{10.61}$$

This can be shown by modifying the calculation used in the commutative theory. We simply change products to star products and utilize the algebra given in (10.15).

10.5 Chiral and Antichiral Superfields

Chiral (Φ) and antichiral($\bar\Phi$) superfields are defined as usual.

$$\bar D_{\dot\alpha} * \Phi = 0 \qquad D_\alpha * \bar\Phi = 0 \tag{10.62}$$

The stars deform any multiplications that result. However, as $D_\alpha = \partial_\alpha$ in the chiral coordinates $y^\mu = x^\mu + i\theta\sigma^m\bar\theta$ and $\bar D_{\dot\alpha} = \partial_{\dot\alpha}$ in the antichiral coordinates $\bar y^\mu = x^\mu - i\theta\sigma^m\bar\theta$, we find that the star products are ordinary products. Consequently, we find

the well-known solutions

$$\begin{aligned}\Phi(y,\theta) &= A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)\\ \bar{\Phi}(\bar{y},\bar{\theta}) &= \bar{A}(\bar{y}) + \sqrt{2}\bar{\theta}\bar{\psi}(\bar{y}) + \theta\theta\bar{F}(\bar{y}).\end{aligned} \tag{10.63}$$

These solutions follow from the chain rule as in the standard commutative theory. This construction need not be modified on noncommutative Minkowski superspace because the anticommutation relations given in (10.15) are uneffected by the deformation.

10.5.1 Parametrizing the Chiral and Antichiral Superfields

The matter fields in the Wess-Zumino model should transform in the fundamental and antifundamental representations of the gauge group. This is naturally embedded into the following superfield equation written on noncommutative Minkowski superspace, (as T. Araki, K. Ito and A. Ohtsuka did for Euclidean case in [6]),

$$\Phi \mapsto \Phi' = e^{-i\Lambda} * \Phi \qquad \bar{\Phi} \mapsto \bar{\Phi}' = \bar{\Phi} * e^{i\bar{\Lambda}}. \tag{10.64}$$

Infinitesimally, we have

$$\delta\Phi = -i\Lambda * \Phi \qquad \delta\bar{\Phi} = i\bar{\Phi} * \bar{\Lambda}. \tag{10.65}$$

At the level of component fields, (10.65) should embed

$$\begin{aligned}\delta A(y) &= i\phi A(y) & \delta\bar{A}(\bar{y}) &= -i\bar{A}\phi(\bar{y})\\ \delta\psi(y) &= i\phi\psi(y) & \delta\bar{\psi}(\bar{y}) &= -i\bar{\psi}\phi(\bar{y})\\ \delta F(y) &= i\phi F(y) & \delta\bar{F}(\bar{y}) &= -i\bar{F}\phi(\bar{y}).\end{aligned} \tag{10.66}$$

It was necessary for [6] to shift the $\bar{F}$-term in $\bar{\Phi}$ to maintain the usual C-independent gauge transformations on the component fields. Similarly, we must modify both Φ and $\bar{\Phi}$ from the canonical form given in (10.63).

$$\begin{aligned}\Phi(y) &= A + \sqrt{2}\theta\psi + \theta\theta(F+\eta)\\ \bar{\Phi}(\bar{y}) &= \bar{A} + \sqrt{2}\bar{\theta}\bar{\psi} + \bar{\theta}\bar{\theta}(\bar{F}+\beta)\end{aligned} \tag{10.67}$$

where the shifts η and β must be chosen as to embed (10.66) in (10.65). Now Λ and $\bar{\Lambda}$ were given in (10.50), however, it will be convenient to view $\bar{\Lambda}$ as a function of $\bar{y}$ for this section.

$$\begin{aligned}\Lambda(y) &= -\phi + i\theta\sigma^m\bar{\theta}\partial_m\phi + \tfrac{i}{2}\theta\theta\bar{C}^{mn}\{v_n, \partial_m\phi\} - 2\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi\\ \bar{\Lambda}(\bar{y}) &= -\phi - i\theta\sigma^m\bar{\theta}\partial_m\phi - \tfrac{i}{2}\bar{\theta}\bar{\theta}C^{mn}\{\partial_m\phi, v_n\} - 2\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi\end{aligned} \tag{10.68}$$

The $\theta\theta$ coefficient in (10.65) yields

$$\delta F + \delta\eta = i\phi F + i\phi\eta - 2i\bar{C}^{mn}\partial_m\phi\partial_n A + \tfrac{1}{2}\bar{C}^{mn}\{v_n, \partial_m\phi\}A. \tag{10.69}$$

Likewise, the $\bar{\theta}\bar{\theta}$ coefficient in (10.65) yields

$$\delta\bar{F} + \delta\beta = -i\bar{F}\phi - i\beta\phi - 2iC^{mn}\partial_n\bar{A}\partial_m\phi + \tfrac{1}{2}C^{mn}\bar{A}\{\partial_m\phi, v_n\}. \tag{10.70}$$

If we require that (10.66) holds, then we then find that the following condition on β from (10.70) is

$$\delta\beta - i\phi\beta = -2iC^{mn}\partial_n\bar{A}\partial_m\phi + \tfrac{1}{2}C^{mn}\bar{A}\{\partial_m\phi, v_m\}. \tag{10.71}$$

Similarly, we find that the following condition on η from (10.69) is

$$\delta\eta - i\phi\eta = -2i\bar{C}^{mn}\partial_m\phi\partial_n A + \tfrac{1}{2}\bar{C}^{mn}\{v_n, \partial_m\phi\}A. \tag{10.72}$$

Following [6], we notice that

$$\begin{aligned}&\delta[iC^{mn}\partial_m(\bar{A}v_n) - \tfrac{1}{4}C^{mn}\bar{A}v_m v_n] + i[iC^{mn}(\partial_m\bar{A}v_n) - \tfrac{1}{4}C^{mn}\bar{A}v_m v_n]\phi = \\ &= -2iC^{mn}(\partial_m\bar{A})(\partial_n\phi) + \tfrac{1}{2}C^{mn}\bar{A}\{\partial_m\phi, v_n\}.\end{aligned} \tag{10.73}$$

Additionally, we note that

$$\begin{aligned}&\delta[-i\bar{C}^{mn}\partial_m v_n A + \tfrac{1}{4}\bar{C}^{mn}v_m v_n A] - i\phi[-i\bar{C}^{mn}\partial_m(v_n A) + \tfrac{1}{4}\bar{C}^{mn}v_m v_n A] = \\ &= 2i\bar{C}^{mn}(\partial_n\phi)(\partial_m\bar{A}) + \tfrac{1}{2}\bar{C}^{mn}\{v_n, \partial_m\phi\}A.\end{aligned} \tag{10.74}$$

Then, observe that (10.74) and (10.72) indicate that

$$\eta = -i\bar{C}^{mn}\partial_m(v_n A) + \tfrac{1}{4}\bar{C}^{mn}v_m v_n A. \tag{10.75}$$

Then, observe that (10.73) and (10.71) indicate that

$$\beta = iC^{mn}\partial_m(\bar{A}v_n) - \tfrac{1}{4}C^{mn}\bar{A}v_m v_n. \tag{10.76}$$

Thus, we define the chiral and antichiral superfields with respect to (10.50) as

$$\begin{aligned}\Phi &= A + \sqrt{2}\theta\psi + \theta\theta(F - i\bar{C}^{mn}\partial_m(v_n A) + \tfrac{1}{4}\bar{C}^{mn}v_m v_n A) \\ \bar{\Phi} &= \bar{A} + \sqrt{2}\bar{\theta}\bar{\psi} + \bar{\theta}\bar{\theta}(\bar{F} + iC^{mn}\partial_m(\bar{A}v_n) - \tfrac{1}{4}C^{mn}\bar{A}v_m v_n).\end{aligned} \tag{10.77}$$

It should be clear from this section that this is the correct parametrization of the anti(chiral) superfields. This definition embeds (10.66) in (10.65). This parametrization gives the component fields the standard C-independent gauge transformations.

10.6 Gauged Wess-Zumino Model

We construct the gauge invariant Lagrangian of the Wess-Zumino model on noncommutative Minkowski superspace:

$$\mathcal{L} = \frac{1}{16kg^2}\left(\int d^2\theta \mathrm{tr} W * W + \int d^2\bar{\theta}\mathrm{tr}\overline{W} * \overline{W}\right) + \int \; d^2\theta d^2\bar{\theta}\bar{\Phi} * e^V * \Phi. \tag{10.78}$$

Gauge invariance of $\mathcal{L}$ follows directly from the cyclicity of the trace and equations (10.34), (10.61) , and (10.64). Also, note that this Lagrangian is real as the star product has the property $\overline{f * g} = \overline{g} * \overline{f}$. To first order in the deformation parameter, we can calculate

$$\begin{array}{l} \mathrm{tr} W * W|_{\theta\theta} = \mathrm{tr} W * W(C=0)|_{\theta\theta} - iC^{mn}\mathrm{tr}F_{mn}\bar{\lambda}\bar{\lambda} + i\bar{C}^{mn}\mathrm{tr}\lambda\lambda F_{mn} \\ \mathrm{tr}\overline{W} * \overline{W}|_{\bar{\theta}\bar{\theta}} = \mathrm{tr}\overline{W} * \overline{W}(C=0)|_{\bar{\theta}\bar{\theta}} - iC^{mn}\mathrm{tr}F_{mn}\bar{\lambda}\bar{\lambda} + i\bar{C}^{mn}\mathrm{tr}\lambda\lambda F_{mn} \end{array} \tag{10.79}$$

where

$$\begin{array}{l} W * W(C=0)|_{\theta\theta} = -2i\bar{\lambda}\bar{\sigma}^m\mathcal{D}_m\lambda - \frac{1}{2}F^{mn}F_{mn} + D^2 + \frac{i}{4}F^{mn}F^{lk}\epsilon_{mnlk} \\ \overline{W} * \overline{W}(C=0)|_{\bar{\theta}\bar{\theta}} = -2i\bar{\lambda}\bar{\sigma}^m\mathcal{D}_m\lambda - \frac{1}{2}F^{mn}F_{mn} + D^2 - \frac{i}{4}F^{mn}F^{lk}\epsilon_{mnlk}. \end{array} \tag{10.80}$$

To the first order, these terms match those found by [112] if we set the $\bar{C}^{mn} = 0$. Next, consider the coupling of the vector and chiral multiplets. After some calculation, we find

$$\begin{array}{rl} \bar{\Phi} * e^V * \Phi|_{\theta\theta\bar{\theta}\bar{\theta}} = & \bar{F}F + i\sigma^m_{\alpha\dot{\alpha}}(\partial_m\bar{\psi}^{\dot{\alpha}})\psi^\alpha + \frac{1}{2}\bar{\psi}^{\dot{\alpha}}\sigma^m_{\alpha\dot{\alpha}}v_m\psi^\alpha \\ & + \frac{1}{2}\bar{A}(D - i\partial_m v^m)A - \frac{1}{4}\bar{A}v^m v_m A + (\partial^2\bar{A})A \\ & - i(\partial_m\bar{A})v^m A + i\frac{\sqrt{2}}{2}\bar{A}\lambda\psi - i\frac{\sqrt{2}}{2}\bar{\psi}\bar{\lambda}A \\ & + iC^{mn}\partial_m(\bar{A}v_n)F - iC^{mn}(\partial_m\bar{A})v_n F \\ & - i\bar{C}^{mn}\bar{F}\partial_m(v_n A) + i\bar{C}^{mn}\bar{F}v_n\partial_m A \\ & - \frac{1}{2}C^{mn}\bar{A}v_m v_n F + \frac{1}{2}\bar{C}^{mn}\bar{F}v_m v_n A \\ & - i\frac{\sqrt{2}}{8}C^{\alpha\beta}\sigma^m_{\alpha\dot{\alpha}}\bar{A}\{\bar{\lambda}^{\dot{\alpha}}, v_m\}\psi_\beta \\ & - i\frac{\sqrt{2}}{8}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^m_{\alpha\dot{\alpha}}\bar{\psi}_{\dot{\beta}}\{\lambda^\alpha, v_m\}A \\ & - \frac{\sqrt{2}}{2}C^{\alpha\beta}\sigma^m_{\alpha\dot{\alpha}}(\partial_m\bar{A})\bar{\lambda}^{\dot{\alpha}}\psi_\beta \\ & - \frac{\sqrt{2}}{2}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^m_{\alpha\dot{\alpha}}\bar{\psi}_{\dot{\beta}}\lambda^\alpha\partial_m A. \end{array} \tag{10.81}$$

We identify the terms without deformation parameters as the usual terms in the Wess Zumino model; that is, up to a total derivative we have

$$\begin{array}{rl} \bar{\Phi} * e^V * \Phi(C=0)|_{\theta\theta\bar{\theta}\bar{\theta}} = & \bar{F}F - i\bar{\psi}\bar{\sigma}^m\mathcal{D}_m\psi - (\mathcal{D}_m\bar{A})(\mathcal{D}^m A) \\ & + \frac{1}{2}\bar{A}DA + \frac{i}{\sqrt{2}}(\bar{A}\lambda\psi - \bar{\psi}\bar{\lambda}A) \end{array} \tag{10.82}$$

where ψ and A are in the fundamental representation of the gauge group

$$\mathcal{D}_m\psi = \partial_m\psi + \tfrac{i}{2}v_m\psi \qquad \mathcal{D}_m A = \partial_m A + \tfrac{i}{2}v_m A. \tag{10.83}$$

In (10.81), we recover most of the terms found by [6] plus their conjugates. However, in comparison to the $N = \frac{1}{2}$ theory, terms that are linear in λ and $\bar{\lambda}$ are notably modified. The new shifts in the gauge parameters (10.50) lead to the modification of the λ and $\bar{\lambda}$ components of the vector superfield V which in turn give rise to the following terms in the Lagrangian $\mathcal{L}$:

$$\begin{array}{l} -i\frac{\sqrt{2}}{8}C^{\alpha\beta}\sigma^m_{\alpha\dot{\alpha}}\bar{A}\{\bar{\lambda}^{\dot{\alpha}}, v_m\}\psi_\beta - \frac{\sqrt{2}}{2}C^{\alpha\beta}\sigma^m_{\alpha\dot{\alpha}}(\partial_m\bar{A})\bar{\lambda}^{\dot{\alpha}}\psi_\beta \\ -i\frac{\sqrt{2}}{8}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^m_{\alpha\dot{\alpha}}\bar{\psi}_{\dot{\beta}}\{\lambda^\alpha, v_m\}A - \frac{\sqrt{2}}{2}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^m_{\alpha\dot{\alpha}}\bar{\psi}_{\dot{\beta}}\lambda^\alpha\partial_m A. \end{array} \tag{10.84}$$

Using covariant derivatives, these terms become

$$\begin{array}{l} -i\frac{\sqrt{2}}{8}C^{\alpha\beta}\sigma^m_{\alpha\dot{\alpha}}\bar{A}[\bar{\lambda}^{\dot{\alpha}}, v_m]\psi_\beta - \frac{\sqrt{2}}{2}C^{\alpha\beta}\sigma^m_{\alpha\dot{\alpha}}(\mathcal{D}_m\bar{A})\bar{\lambda}^{\dot{\alpha}}\psi_\beta \\ +i\frac{\sqrt{2}}{8}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^m_{\alpha\dot{\alpha}}\bar{\psi}_{\dot{\beta}}[\lambda^\alpha, v_m]A - \frac{\sqrt{2}}{2}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^m_{\alpha\dot{\alpha}}\bar{\psi}_{\dot{\beta}}\lambda^\alpha\mathcal{D}_m A. \end{array} \tag{10.85}$$

The term $-\frac{\sqrt{2}}{2}C^{\alpha\beta}\sigma^m_{\alpha\dot{\alpha}}(\mathcal{D}_m\bar{A})\bar{\lambda}^{\dot{\alpha}}\psi_\beta$ was also found in [6]. However, the commutator terms result from the choice of gauge parameter we made in (10.50). We might naively have expected only the terms without the commutators. Let us summarize:

$$\begin{array}{rl} \mathcal{L} & = \frac{1}{16kg^2}\mathrm{tr}\big(-4i\bar{\lambda}\bar{\sigma}^m\mathcal{D}_m\lambda - F^{mn}F_{mn} + 2D^2\big) \\ & +\bar{F}F - i\bar{\psi}\bar{\sigma}^m\mathcal{D}_m\psi - \mathcal{D}_m\bar{A}\mathcal{D}^m A + \frac{1}{2}\bar{A}DA + \frac{i}{\sqrt{2}}(\bar{A}\lambda\psi - \bar{\psi}\bar{\lambda}A) \\ & +\frac{1}{16kg^2}\mathrm{tr}\big(-2iC^{mn}F_{mn}\lambda\lambda + 2i\bar{C}^{mn}\bar{\lambda}\bar{\lambda}F_{mn}\big) \\ & +\frac{i}{2}C^{mn}\bar{A}F_{mn}F - \frac{i}{2}\bar{C}^{mn}\bar{F}F_{mn}A \\ & -i\frac{\sqrt{2}}{8}C^{\alpha\beta}\sigma^m_{\alpha\dot{\alpha}}\bar{A}[\bar{\lambda}^{\dot{\alpha}}, v_m]\psi_\beta - \frac{\sqrt{2}}{2}C^{\alpha\beta}\sigma^m_{\alpha\dot{\alpha}}(\mathcal{D}_m\bar{A})\bar{\lambda}^{\dot{\alpha}}\psi_\beta \\ & +i\frac{\sqrt{2}}{8}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^m_{\alpha\dot{\alpha}}\bar{\psi}_{\dot{\beta}}[\lambda^\alpha, v_m]A - \frac{\sqrt{2}}{2}\bar{C}^{\dot{\alpha}\dot{\beta}}\sigma^m_{\alpha\dot{\alpha}}\bar{\psi}_{\dot{\beta}}\lambda^\alpha\mathcal{D}_m A. \end{array} \tag{10.86}$$

10.7 Summary

We have developed a nonabelian gauge theory over deformed Minkowski superspace. In this deformation, all of the fermionic dimensions are deformed and as a result, all of the supersymmetry is broken. To be consistent with the $N = \frac{1}{2}$ terminology, we say that this deformed superspace has $N = 0$ supersymmetry. Many of the results directly mirror the results of $N = \frac{1}{2}$ from [112] or [6]. This is due to the fact that the deformation we consider in this chapter reduces to the deformation of $N = 1/2$ supersymmetry upon setting $\bar{C}^{\dot{\alpha}\dot{\beta}} = 0$. It is not surprising that we recover the same gauge theoretic results as [112] in the limit $\bar{C}^{\dot{\alpha}\dot{\beta}} = 0$. The exception to this rule is the choice of gauge parameter introduced by Seiberg in [112]. We found that it was not possible to use the same construction because it violated the hermiticity of the vector superfield. We fixed this by introducing a new gauge parameter which served

to maintain both hermiticity and the C-independent gauge transformations on the component fields.

Next, we introduced the chiral superfield Φ. Again, we found it necessary to modify the canonical component field expansion in order to maintain the standard gauge transformations on the component fields. The modification is similar in spirit to that of [6]. Essentially, what we found is the $N = \frac{1}{2}$ theory and conjugate copy where all of the usual $N = \frac{1}{2}$ terms are accompanied by their conjugates due to the hermiticity properties of the star product used in this construction.

Finally, we constructed the Lagrangian which coupled the gauge and matter fields. The gauge invariance of $\mathcal{L}$ follows for reasons similar to the commutative theory. We simply modified the standard arguments for the gauged Wess-Zumino model by replacing products with star products. The primary obstacle to this construction was the task of finding the correct parametrization for the superfields. The Lagrangian is similar to that found by [6], however, there are several new terms. Most new terms come directly from the added deformation $\{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\}_* = \bar{C}^{\dot{\alpha}\dot{\beta}}$ (which should have been expected from the outset). However, the reparametrization of the gauge parameter also led us to some terms which were not immediately obvious from the $N = \frac{1}{2}$ theory.

There is much work left to do. First, we should complete the program begun in this work to the second order in the deformation parameter. Nonassociativity will have to be addressed. It is likely that, the constructions of this chapter will need modification at the second order. Secondly, there are numerous papers investigating $N = 1/2$ supersymmetry [6], [60], [61], [50], [46], [19], [14], [20], [83], [103], [21], [12], [49], [5], [67], [110], [85], [115], [1], [109], [63], [62], [9], [48], [3], [93], [8], [53], [15], [7], [108], [34] and it would be interesting to find complementary results for the $N = 0$ case where possible. We could try to find the dual results for, instantons as in [60], [61], [50], [46], [19], [14], or renormalization as in [20], [83], [103], [21], [12], [49], [5], [67], or the possibility of residual supersymmetry as in [110], or the Seiberg Witten map as in [85]. We do not attempt to give a complete account of the $N = 1/2$ developments, we just wish to point out the variety of novel directions future research might take. Finally, it would be interesting to derive the $N = 0$ deformation from a string theoretical argument.

10.8 Star Product Approximately Associative

Define the parity of F to be ϵ^F. If F is even, then $\epsilon^F = 1$. If F is odd, then $\epsilon^F = -1$. We can express the star product to the first order as:

$$F * G = FG - \tfrac{1}{2} C^{\alpha\beta} \epsilon^F (Q_\alpha F)(Q_\beta G) - \tfrac{1}{2} \bar{C}^{\dot{\alpha}\dot{\beta}} \epsilon^F (\bar{Q}_{\dot{\alpha}} F)(\bar{Q}_{\dot{\beta}} G).$$

Let us then prove that the first order star product is associative. Consider:

$$\begin{aligned}
(F*G)*H &= (FG - \tfrac{1}{2}C^{\alpha\beta}\epsilon^F(Q_\alpha F)(Q_\beta G) - \tfrac{1}{2}\bar{C}^{\dot\alpha\dot\beta}\epsilon^F(\bar{Q}_{\dot\alpha}F)(\bar{Q}_{\dot\beta}G))*H \\
&= FGH - \tfrac{1}{2}C^{\alpha\beta}\epsilon^F(Q_\alpha F)(Q_\beta G)H - \tfrac{1}{2}\bar{C}^{\dot\alpha\dot\beta}\epsilon^F(\bar{Q}_{\dot\alpha}F)(\bar{Q}_{\dot\beta}G)H \\
&\quad -\tfrac{1}{2}C^{\alpha\beta}\epsilon^{FG}(Q_\alpha FG)(Q_\beta H) - \tfrac{1}{2}\bar{C}^{\dot\alpha\dot\beta}\epsilon^{FG}(\bar{Q}_{\dot\alpha}FG)(\bar{Q}_{\dot\beta}H) \\
&= FGH - \tfrac{1}{2}C^{\alpha\beta}\epsilon^F(Q_\alpha F)(Q_\beta G)H - \tfrac{1}{2}\bar{C}^{\dot\alpha\dot\beta}\epsilon^F(\bar{Q}_{\dot\alpha}F)(\bar{Q}_{\dot\beta}G)H \\
&\quad -\tfrac{1}{2}C^{\alpha\beta}\epsilon^{FG}[(Q_\alpha F)G + \epsilon^F F(Q_\alpha G)]Q_\beta H \\
&\quad -\tfrac{1}{2}\bar{C}^{\dot\alpha\dot\beta}\epsilon^{FG}[(\bar{Q}_{\dot\alpha}F)G + \epsilon^F F(\bar{Q}_{\dot\alpha}G)]\bar{Q}_{\dot\beta}H \\
&= FGH - \tfrac{1}{2}C^{\alpha\beta}\epsilon^F(Q_\alpha F)(Q_\beta G)H - \tfrac{1}{2}\bar{C}^{\dot\alpha\dot\beta}\epsilon^F(\bar{Q}_{\dot\alpha}F)(\bar{Q}_{\dot\beta}G)H \\
&\quad -\tfrac{1}{2}C^{\alpha\beta}[\epsilon^F\epsilon^G(Q_\alpha F)G(Q_\beta H) + \epsilon^G F(Q_\alpha G)(Q_\beta H)] \\
&\quad -\tfrac{1}{2}\bar{C}^{\dot\alpha\dot\beta}[\epsilon^F\epsilon^G(\bar{Q}_{\dot\alpha}F)G(\bar{Q}_{\dot\beta}H) + \epsilon^G F(Q_\alpha G)(\bar{Q}_{\dot\beta}H)].
\end{aligned}$$

Notice that we have used $\epsilon^{FG} = \epsilon^F\epsilon^G$ and $\epsilon^F\epsilon^F = 1$ to complete the calculation above. Likewise consider:

$$\begin{aligned}
F*(G*H) &= F*(GH - \tfrac{1}{2}C^{\alpha\beta}\epsilon^G(Q_\alpha G)(Q_\beta H) - \tfrac{1}{2}\bar{C}^{\dot\alpha\dot\beta}\epsilon^G(\bar{Q}_{\dot\alpha}G)(\bar{Q}_{\dot\beta}H)) \\
&= FGH - \tfrac{1}{2}C^{\alpha\beta}\epsilon^G F(Q_\alpha G)(Q_\beta H) - \tfrac{1}{2}\bar{C}^{\dot\alpha\dot\beta}\epsilon^G F(\bar{Q}_{\dot\alpha}G)(\bar{Q}_{\dot\beta}H) \\
&\quad -\tfrac{1}{2}C^{\alpha\beta}\epsilon^F(Q_\alpha F)(Q_\beta GH) - \tfrac{1}{2}\bar{C}^{\dot\alpha\dot\beta}\epsilon^F(\bar{Q}_{\dot\alpha}F)(\bar{Q}_{\dot\beta}GH) \\
&= FGH - \tfrac{1}{2}C^{\alpha\beta}\epsilon^G F(Q_\alpha G)(Q_\beta H) - \tfrac{1}{2}\bar{C}^{\dot\alpha\dot\beta}\epsilon^G F(\bar{Q}_{\dot\alpha}G)(\bar{Q}_{\dot\beta}H) \\
&\quad -\tfrac{1}{2}C^{\alpha\beta}\epsilon^F(Q_\alpha F)[(Q_\beta G)H + \epsilon^G G(Q_\beta H)] \\
&\quad -\tfrac{1}{2}\bar{C}^{\dot\alpha\dot\beta}\epsilon^F(\bar{Q}_{\dot\alpha}F)[(\bar{Q}_{\dot\beta}G)H + \epsilon^G G(\bar{Q}_{\dot\beta}H)] \\
&= FGH - \tfrac{1}{2}C^{\alpha\beta}\epsilon^G F(Q_\alpha G)(Q_\beta H) - \tfrac{1}{2}\bar{C}^{\dot\alpha\dot\beta}\epsilon^G F(\bar{Q}_{\dot\alpha}G)(\bar{Q}_{\dot\beta}H) \\
&\quad -\tfrac{1}{2}C^{\alpha\beta}[\epsilon^F(Q_\alpha F)(Q_\beta G)H + \epsilon^F\epsilon^G(Q_\alpha F)G(Q_\beta H)] \\
&\quad -\tfrac{1}{2}\bar{C}^{\dot\alpha\dot\beta}[\epsilon^F(\bar{Q}_{\dot\alpha}F)(\bar{Q}_{\dot\beta}G)H + \epsilon^F\epsilon^G(Q_\alpha F)G(\bar{Q}_{\dot\beta}H)].
\end{aligned}$$

Therefore, $F*(G*H) = (F*G)*H$ to the first order in the deformation parameter.

References

[1] R. Abbaspur, Scalar Solitons in Non(anti)commutative Superspace, [hep-th/0308050].

[2] R. Abraham, J. Marsden, T. Ratiu, Manifolds, Tensor Analysis, and Applications, Springer Verlag, 1988.

[3] L.G. Aldrovandi, D.H. Correa, F.A. Schaposnik, G.A. Silva, BPS Analysis of Gauge Field-Higgs Models in Non-Anticommutative Superspace, Phys.Rev. **D71**, 025015 (2005) [hep-th/0410256].

[4] L.G. Aldrovandi, F.A. Schaposnik, G.A. Silva, Non(anti)commutative superspace with coordinate-dependent deformation, Phys. Rev. D72 (2005) 045005 [hep-th/0505217].

[5] M. Alishahiha, A. Ghodsi, N. Sadooghi, One-Loop Perturbative Corrections to non(anti)commutativity Parameter of N=1/2 Supersymmetric U(N) Gauge Theory, Nucl.Phys. **B691**, 111-128 (2004) [hep-th/0309037].

[6] T. Araki, K. Ito and A. Ohtsuka, Supersymmetric Gauge Theories on Noncommutative Superspace, Phys.Lett. **B573** 209 (2003) [hep-th/0307076].

[7] T. Araki, K. Ito, A. Ohtsuka, Non(anti)commutative $N = (1, 1/2)$ Supersymmetric U(1) Gauge Theory, JHEP **0505**, 074 (2005) [hep-th/0503224].

[8] O.D. Azorkina, A.T. Banin, I.L. Buchbinder, N.G. Pletnev, Generic chiral superfield model on nonanticommutative N=1/2 superspace, Mod. Phys. Lett. A20 (2005) 1423-1436 [hep-th/0502008].

[9] A.T. Banin, I.L. Buchbinder, N.G. Pletnev, Chiral effective potential in $\mathcal{N} = 1/2$ non-commutative Wess-Zumino model, JHEP **0407**, 011 (2004) [hep-th/0405063].

[10] C. Bartocci, U. Bruzzo, Geometry of Standard Constraints and Weil Triviality in Supersymmetric Gauge Theories, Lett. Math. Phys. **18** (1989) 235-245.

[11] M. Batchelor, Two approaches to supermanifolds, Trans. Am. Math. Soc. 258 (1980) 257.

[12] D. Berenstein, S.J. Rey, Wilsonian Proof for Renormalizability of N=1/2 Supersymmetric Field Theories, Phys.Rev. **D68**, 121701 (2003) [hep-th/0308049].

[13] F.A. Berezin, D.A. Leĭtes, Supermanifolds, Sov. Math, Dokl. 16 (1975) 1218.

[14] M. Billo, M. Frau, I. Pesando, A. Lerda, N=1/2 gauge theory and its instanton moduli space from open strings in R-R background, JHEP **0405**, 023 (2004) [hep-th/0402160].

[15] M. Billo, M. Frau, F. Lonegro, A. Lerda, N=1/2 quiver gauge theories from open strings with R-R fluxes, JHEP **0505**, 047 (2005) [hep-th/0502084]

[16] D.D. Bleecker, Gauge theory and variational principles, Addison-Wesley Publishing Company Inc. 1981.

[17] L. Bonora, P. Pasti, M. Tonin Supermanifolds and BRS transformations J. Math. Phys. **23** (5) (May 1982) 839-845.

[18] C.P. Boyer, S. Gitler, The Theory of G^∞-Supermanifolds, Transactions of the American Mathematical Society, Vol. 285. No.1. (September 1984) 241-267.

[19] R. Britto, B. Feng, O. Lunin, S.J. Rey, U(N) Instantons on N=1/2 superspace – exact solution and geometry of moduli space, Phys.Rev. **D69**, 126004 (2004) [hep-th/0311275].

[20] R. Britto, B. Feng, S.J. Rey, Non(anti)commutative Superspace, UV/IR Mixing and Open Wilson Lines, JHEP **0308**, 001 (2003) [hep-th/0307091].

[21] R. Britto, B. Feng, N=1/2 Wess-Zumino model is renormalizable, Phys.Rev.Lett. **91**, 201601 (2003) [hep-th/0307165].

[22] T. Bröcker, T. tom Dieck, Representations of Compact Groups, Springer Verlag, 1985.

[23] U. Bruzzo, Supermanifolds modeled over finite-dimensional exterior algebras, Class. Quantum Grav. 9 (1992) 13-17.

[24] U. Bruzzo, R. Cianci, Mathematical theory of super fiber bundles, Class. Quantum Grav. 1 (1984) 213-226.

[25] U. Bruzzo, R. Cianci, An existence result of super Lie groups, Letters in Math. Phys. 8 (1984) 279-288.

[26] U. Bruzzo, R. Cianci, Structure of Supermanifolds and Supersymmetry Transformations, Commun. Math. Phys. 95. 393-400 (1984).

[27] U. Bruzzo, R. Cianci, Differential equations, Frobenius theorem and local flows on supermanifolds, J. Phys. A 18 (1985) 417-423.

[28] U. Bruzzo, R. Cianci, On the Structure of Superfields in a Field Theory on a Supermanifold, Letters in Math. Phys. 11 (1986) 21-26.

[29] J. Buchbinder, S. Kuzenko, Ideas and methods of supersymmetry and supergravity, Taylor and Francis group, 1998.

[30] P. Cartier, C. DeWitt-Morette, M. Ihl, C. Samann, Supermanifolds-Application to Supersymmetry, "Multiple facets of quantization and supersymmetry", Eds. M. Olshanetsky and A. Vainshtein, World Scientific, 2002 [math-ph/0202026v1].

[31] R. Catenacci, Cesare Reina, Paolo Teofilatto, On the body of supermanifolds, J. Math. Phys. **26** (4), April 1986.

[32] M. Chaichian, A. Kobakhidze, Deformed N=1 supersymmetry, Phys.Rev. **D71**, 047501 (2005) [hep-th/0307243].

[33] C.S. Chu, F. Zamora, Manifest Supersymmetry in Non-Commutative Geometry, JHEP **0002**, 022 (2000) [hep-th/9912153].

[34] C.S. Chu, T. Inami, Konishi Anomaly and Central Extension in N=1/2 Supersymmetry, Nucl. Phys. B725 (2005) 327-351 [hep-th/0505141].

[35] R. Cianci, Supermanifolds and automorphism of super Lie groups J. Math. Phys. **25** (3), March 1984.

[36] J.S. Cook, Gauged Wess-Zumino Model in Noncommutative Minkowski Superspace, J. Math. Phys. **47** (2006) 012304.

[37] J.S. Cook, R.O. Fulp, Infinite Dimensional Super Lie Groups, to appear in Journal of Differential Geometry and its Applications.

[38] J. de Boer, P. A. Grassi and P. Van Niewenhuizen, Non-commutative superspace from string theory, Phys. Lett. **B574**, 98-104 (2003) [hep-th/0302078]

[39] B. DeWitt, Supermanifolds, Cambridge University Press, 1984.

[40] J. Duistermaat, J. Kolk, Lie Groups, Springer Verlag, 2000.

[41] S. Ferrara, M. A. Lledo, Some Aspects of Deformations of Supersymmetric Field Theories, JHEP **0005**, 008 (2000) [hep-th/0002084].

[42] S. Ferrara, M. A. Lledo and O. Macia, Supersymmetry in noncommutative superspaces, JHEP **0309**, 068 (2003) [hep-th/0307039].

[43] P. Freund, Introduction to supersymmetry, Cambridge University Press, 1986.

[44] A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky, E.S. Sokatchev, Harmonic Superspace, Cambridge University Press 2001.

[45] S.J. Gates, Jr. Ectoplasm Has No Topology: The Prelude, Presentation at the International Seminar on "Supersymmteries and Quantum Symmetries", Dubna Russia, July 22-26, 1997 [hep-th/9709104v1].

[46] S. Giombi, R. Ricci, D. Robles-Llana, D. Trancanelli, Instantons and Matter in N=1/2 Supersymmetric Gauge Theory, JHEP 0510 (2005) 021 [hep-th/0505077].

[47] Francois Gieres, Geometry of Supersymmetric Gauge Theories, Lecture Notes in Physics, 302, Springer-Verlag 1988.

[48] A. Gorsky, M. Shifman, Spectral Degeneracy in Supersymmetric Gluodynamics and One-Flavor QCD related to N=1/2 SUSY, Phys.Rev. **D71**, 025009 (2005) [hep-th/0410099].

[49] M. T. Grisaru, S. Penati, A. Romagnoni, Two-loop Renormalization for Nonanticommutative N=1/2 Supersymmetric WZ Model, JHEP **0308**, 003 (2003) [hep-th/0307099].

[50] P.A. Grassi, R. Ricci, D. Robles-Llana, Instanton Calculations for N=1/ 2 super Yang-Mills Theory, JHEP **0407**, 065 (2004) [hep-th/0311155].

[51] M. Grasso, P. Teofilatto, Gauge Theories, Flat Superforms and Reduction of Super Fiber Bundles Reports on Mathematical Physics Vol. 25 (1987) 53-71.

[52] J. Harnad, J. Hurtubise, S. Shnider, Supersymmetric Yang-Mills Equations and Supertwistors, Annals of Physics **193** 40-79 (1989).

[53] T. Hatanaka, S.V. Ketov, Y. Kobayashi, S. Sasaki, Non-Anti-Commutative deformation of effective potentials in supersymmetric gauge theories, Nucl.Phys. **B716**, 88-104 (2005) [hep-th/0502026].

[54] M. Hatsuda, S. Iso and H. Umetsu, Noncommutative superspace, supermatrix and the lowest Landau level, Nucl.Phys. **B671**, 217-242 (2003) [hep-th/0306251].

[55] M. Henneaux, C. Teitelboim, Quantization of Gauge Systems, Princeton University Press, 1992.

[56] J. Hoyos, M. Quiros, J.R. Mittelbrunn, F.J. de Urries Generalized supermanifolds. I. Superspaces and linear operators J. Math. Phys. **25** (4) (April 1984) 833-845.

[57] J. Hoyos, M. Quiros, J.R. Mittelbrunn, F.J. de Urries Generalized supermanifolds. II. Analysis on superspaces J. Math. Phys. **25** (4) (April 1984) 841-846.

[58] J. Hoyos, M. Quiros, J.R. Mittelbrunn, F.J. de Urries Generalized supermanifolds. III. ρ-supermanifolds J. Math. Phys. **25** (4) (April 1984) 847-854.

[59] J. Hoyos, M. Quiros, J.R. Mittelbrunn, F.J. de Urries Superfiber bundle structure of gauge theories with Fadeev-Popov fields J. Math. Phys. **23** (8) (August 1982) 1504-1510.

[60] A. Imaanpur, On Instantons and Zero Modes of N=1/2 SYM Theory, JHEP **0309**, 077 (2003) [hep-th/0308171].

[61] A. Imaanpur, Comments on Gluino Condensates in N=1/2 SYM Theory, JHEP **0312**, 009 (2003) [hep-th/0311137].

[62] A. Imaanpur, S. Parvizi, N=1/2 Super Yang-Mills Theory on Euclidean AdS2xS2, JHEP **0407** 010 (2004) [hep-th/0403174].

[63] T. Inami, H. Nakajima, Supersymmetric CP^N Sigma Model on Noncommutative Superspace, Prog.Theor.Phys. **111**, 961-966 (2004) [hep-th/0402137].

[65] E.A. Ivanov, Intrinsic geometry of the N=1 supersymmetric Yang-Mills theory, J. Phys. A: Math. Gen. **16** (1983) 2571-2586.

[65] E.A. Ivanov, Supersymmetry at BLTP: how it started and where we are, September 2006, [hep-th/0609176v1].

[66] E.A. Ivanov, A.V. Smilga, Cryptoreality of nonanticommutative Hamiltonians March 2007, [hep-th/0703038v1].

[67] I. Jack, D.R.T. Jones, L.A. Worthy, One-loop renormalisation of N=1/2 supersymmetric gauge theory, Phys.Lett. **B611**, 199-206 (2005) [hep-th/0412009].

[68] A.Jadczyk, K. Pilch, Superspaces and Supersymmetries, Commun. Math. Phys. 78 (1981) 373-390.

[69] V.G. Kac, Lie superalgebras, Advances in Mathematics 26 (1977) 8-96.

[70] A. Khrennikov, Superanalysis, Kluwer Academic Publishers, 1999.

[71] D. Klemm, S. Pinati and L. Tammasia, Non(anti)commutative Superspace, Class. Quantum Grav. **20** , 2905 (2003) [hep-th/0104190];

[72] Y. Kobayashi, S. Sasaki, Non-local Wess-Zumino Model on Nilpotent Noncommutative Superspace, Phys. Rev. D72 (2005) 065015 [hep-th/0505011].

[73] I. Kolar, P. Michor, J. Slovak, Natural Operations in Differential Geometry, Springer Verlag, 1993

[74] A. Konechny, A. Schwarz Geometry of N=1 Super Yang-Mills Theory in Curved Superspace, J. Geom. Phys. 23 (1997) 97-110.

[75] P. Kosinski, J. Lukierski, P. Maslanka Quantum Deformations of Space-Time SUSY and Noncommutative Superfield Theory, [hep-th/0011053].

[76] B. Kostant, Graded Manifolds, graded Lie theory, and prequantization, Differential geometric methods in mathematical physics, in: K. Bleuler and A. Reetz, (Ed.), Lecture Notes in Mathematics 570, Springer, Berlin, 1977, pp. 177-306.

[77] V.A. Kostelecky, M.M. Nieto, D.R. Truax, Baker-Campbell-Hausdorff relations for the supergroups, J. Math. Phys. **27** (5) (May 1986) 1419-1429.

[78] V.A. Kostelecky, D.R. Truax, Baker-Campbell-Hausdorff relations for the super-Poincare group, J. Math. Phys. **28** (10) (October 1987) 2480-2487.

[79] A. Kriegl, P.W. Michor, Regular Infinite Dimensional Lie Groups, Jour. of Lie Theory, Vol. 7 (1997) 61-99.

[80] S. Lang, Differential Manifolds, Springer Verlag, 1985.

[81] A. Lichernowicz, Global Theory of Connections, English Ed., Noordhoff International Publishing, 1976.

[82] L.H. Loomis, S. Sternberg, Advanced Calculus, Addison-Wesley Publishing Company, 1968.

[83] O. Lunin, S.J. Rey, Renormalizability of Non(anti)commutative Gauge Theories with N=1/2 Supersymmetry, JHEP **0309**, 045 (2003) [hep-th/0307275].

[84] S.P. Martin, A Supersymmetry Primer, (2006), [hep-ph/9709356v4].

[85] D. Mikulovic, Seiberg-Witten Map for Superfields on $N = (1/2, 0)$ and $N = (1/2, 1/2)$ Deformed Superspace, JHEP **0405**, 077 (2004) [hep-th/0403290].

[86] M. Nakahara, Geometry, Topology and Physics, IOP Publishing Ltd 1990

[87] V. Nazaryan and C.E. Carlson, Field Theory in Noncommutative Minkowski Superspace, Phys.Rev. **D71**, 025019 (2005) [hep-th/0410056].

[88] K.H. Neeb, Infinite-dimensional Lie groups and their representations, in: J.P. Anker, B Orsted, (Ed.), In Lie Theory: Lie Algebras and Representations, Progress in Math. 228, Birkhauser Verlag, 2004, pp. 213-328.

[89] E. Nelson, Topics in Dynamics I: Flows, Princeton University Press, 1970.

[90] H. Omori, Infinite-Dimensional Lie Groups, Translations of Mathematical Monographs, Vol. 158, American Mathematical Society (1997).

[91] H. Ooguri and C. Vafa, The C-deformation of Gluino and Non-planar Diagrams, Adv. Theor. Math Phys. **7**, 53 (2003) [hep-th/0302109].

[92] L. O'Raifeartaigh, The Dawning of Gauge Theory, Princeton University Press, 1997.

[93] S.Penati, A.Romagnoni, Covariant quantization of N=1/2 SYM theories and supergauge invariance, JHEP **0502**, 064 (2005) [hep-th/0412041].

[94] V. Pestov, On a "super" version of Lie's third theorem, Letters in Math. Phys. 18 (1989) 27-33.

[95] J. Rabin, L. Crane, Global Properties of Supermanifolds, Commun. Math. Phys. 100, 141-160 (1985).

[96] J.M. Rabin, L. Crane, How Different are the Supermanifolds of Rogers and DeWitt? Commun. Math. Phys. 102, 123-137 (1985).

[97] R. Remmert, Theory of Complex Functions, Springer-Verlag, 1989.

[98] A. Rogers, A global theory of supermanifolds, J. Math. Phys. **21** (6) (June 1980) 1352-1365.

[99] A. Rogers, Some examples of compact supermanifolds with non-Abelian funadamental group, J. Math. Phys. **22** (3) (March 1981) 443-444.

[100] A. Rogers, Super Lie groups: global topology and local structure, J. Math. Phys. **22** (5) (May 1981) 939-945.

[101] A. Rogers, Graded Manifolds, supermanifolds and infinite-dimensional Grassman Algebras, Commun. Math. Phys. 105 275, 1986.

[102] A. Rogers, Supermanifolds: Theory and Applications, World Scientific Publishing Co. Pte. Ltd., 2007

[103] A. Romagnoni, Renormalizability of N=1/2 Wess-Zumino model in superspace, JHEP **0310**, 016 (2003) [hep-th/0307209].

[104] A.A. Rosly, Geometry of N=1 Yang-Mills theory in curved superspace, J. Phys. A: Math. Gen. **15** (1982) L663-L667.

[105] A.A. Rosly, A. Schwarz, Supersymmetry in a Space with Auxillary Dimensions, Commun. Math. Phys. 105, 645-668, (1986).

[106] M. Rothstein, The axioms of supermanifolds and new structure arising from them, Trans. Amer. Math. Soc. 297 159, 1986.

[107] L.H. Ryder, Quantum Field Theory, 2nd. Ed. , Cambridge University Press, 1996.

[108] T.A. Ryttov, F. Sannino, Chiral Models in Noncommutative N=1/2 Four Dimensional Superspace, [hep-th/0504104].

[109] A. Sako, T. Suzuki, Ring Structure of SUSY * Product and 1/2 SUSY Wess-Zumino Model, Phys.Lett. **B582**, 127-134 (2004) [hep-th/0309076].

[110] A. Sako, T. Suzuki, Recovery of Full N=1 Supersymmetry in Non(anti-)commutative Superspace, JHEP **0411**, 010 (2004) [hep-th/0408226].

[111] J. H. Schwarz and P. Van Niewenhuizen, Speculations Concerning A Fermionic Substructure of Space-Time, Lett. Nuovo Cim. **34**, 21 (1982).

[112] N. Seiberg, Noncommutative superspace, N = 1/2 supersymmetry, field theory and string theory, JHEP **0306**, 010 (2003) [hep-th/0305248].

[113] J.P. Serre, Lie Algebras and Lie Groups, W.A. Benjamin, Inc. 1965.

[114] P. Teofilatto, Enlargeable graded Lie algebras of supersymmetry, J. Math. Phys. **28** (5) (May 1987) 991-996.

[115] S. Terashima, J.T. Yee, Comments on Noncommutative Superspace, JHEP **0312**, 053 (2003) [hep-th/0306237].

[116] J. Wess and J. Bagger, Supersymmetry and Supergravity, Princeton University Press, 1992.

[117] E. Witten, An Interpretation of Classical Yang-Mills Theory, Phys. Lett. B **77** (1978) 394-398.