§7.4 #41

$$y = (2x+1)^5 (x^4-3)^6$$

$$\ln(y) = 5\ln(2x+1) + 6\ln(x^4-3)$$

Differentiate w.r.t. x,

$$\frac{1}{y}\frac{dy}{dx} = \frac{5}{2x+1}\frac{d}{dx}(2x+1) + \frac{6}{x^4-3}\frac{d}{dx}(x^4-3)$$

$$\frac{dy}{dx} = y\left(\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right)$$

$$= \boxed{(2x+1)^5(x^4-3)^6\left[\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right]}$$

• Notice log-diff. gives simplified answer by default.

§7.4 #43

$$y = \frac{\sin^2(x)\tan^4(x)}{(x^2+1)^2} = \frac{\sin^6(x)}{\cos^4(x)(x^2+1)^2}$$

$$\ln(y) = 6\ln(\sin(x)) - 4\ln(\cos(x)) - 2\ln(x^2+1)$$

Differentiate w.r.t. x,

$$\frac{1}{y}\frac{dy}{dx} = \frac{6}{\sin(x)}\frac{d}{dx}[\sin(x)] - \frac{4}{\cos(x)}\frac{d}{dx}[\cos(x)] - \frac{2}{x^2+1}\frac{d}{dx}[x^2+1]$$

$$= \frac{6\cos(x)}{\sin(x)} - \frac{4\sin(x)}{\cos(x)} - \frac{4x}{x^2+1}$$

Now multiply by y to obtain

$$\boxed{\frac{dy}{dx} = \frac{\sin^2(x)\tan^4(x)}{(x^2+1)^2}\left[\frac{6\cos(x)}{\sin(x)} - \frac{4\sin(x)}{\cos(x)} - \frac{4x}{x^2+1}\right]}$$

$$\boxed{\frac{dy}{dx} = \frac{\sin^2(x)\tan^4(x)}{(x^2+1)^2}\left[6\cot(x) - 4\tan(x) - \frac{4x}{x^2+1}\right]}$$

§7.4 #47

$$y = x^{\sin(x)}$$

$$\ln(y) = \ln(x^{\sin(x)}) = \sin(x)\ln(x)$$

Differentiate,

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[\sin(x)\ln(x)\right]$$

$$= \cos(x)\ln(x) + \sin(x)\frac{d}{dx}\left[\ln(x)\right]$$

$$= \cos(x)\ln(x) + \frac{1}{x}\sin(x)$$

Now multiply by $y = x^{\sin(x)}$,

$$\frac{dy}{dx} = \boxed{\frac{d}{dx}\left(x^{\sin(x)}\right) = x^{\sin(x)}\left[\cos(x)\ln(x) + \frac{\sin(x)}{x}\right]}$$

§7.4 #49 $y = (\cos(x))^x$

$$\ln(y) = \ln(\cos(x)^x) = x\ln(\cos(x))$$

$$\frac{d}{dx}\left[\ln(y)\right] = \frac{d}{dx}\left[x\ln(\cos(x))\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \ln(\cos(x)) + \frac{x}{\cos(x)}\frac{d}{dx}(\cos(x))$$

$$\Rightarrow \boxed{\frac{dy}{dx} = (\cos(x))^x\left[\ln(\cos(x)) - \frac{x\sin(x)}{\cos(x)}\right]}$$

§7.6 # 19) Prove that $\frac{d}{dx}\left(\cot^{-1}(x)\right) = \frac{-1}{1+x^2}$.

To begin let us define $y = \cot^{-1}(x)$ thus $\cot(y) = \cot(\cot^{-1}(x)) = x$.
We differentiate $\cot(y) = x$ implicitly and obtain, (recall $\frac{d}{d\theta}(\cot\theta) = -\csc^2\theta$)

$$-\csc^2(y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\csc^2(y)}$$

We know from earlier work,

$$\cos^2(y) + \sin^2(y) = 1$$

Divide by $\sin^2(y)$ to obtain,

$$\frac{\cos^2(y)}{\sin^2(y)} + \frac{\sin^2(y)}{\sin^2(y)} = \frac{1}{\sin^2(y)}$$

$$\Rightarrow \cot^2(y) + 1 = \csc^2(y)$$

But $\cot(y) = x$ for this problem so $x^2+1 = \csc^2(y)$ and

$$\frac{dy}{dx} = \boxed{\frac{d}{dx}\cot^{-1}(x) = \frac{-1}{x^2+1}}$$

§7.6 # 29) Recall $y = \cos^{-1}(x) \Rightarrow \cos(y) = x \Rightarrow -\sin(y)\frac{dy}{dx} = 1$
thus $\frac{dy}{dx} = \frac{-1}{\sin(y)} = \frac{-1}{\sqrt{1-\cos^2(y)}} = \frac{-1}{\sqrt{1-x^2}} = \frac{d}{dx}\left(\cos^{-1}(x)\right)$ You didn't have to derive that here, but you should be capable of it.

$$\frac{d}{dx}\left[\cos^{-1}(e^{2x})\right] = \frac{d}{dx}\left[\cos^{-1}(u)\right] \quad : \quad u = e^{2x}$$

$$= \frac{-1}{\sqrt{1-u^2}}\frac{d}{dx}\left(e^{2x}\right)$$

$$= \boxed{\frac{-2e^{2x}}{\sqrt{1-e^{4x}}}} \quad : \text{note } \left(e^{2x}\right)^2 = e^{4x}.$$

§7.6 # 31

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left[\tan^{-1}(\cos\theta)\right]$$

$$= \frac{1}{1+\cos^2\theta}\frac{d}{d\theta}\left(\cos\theta\right)$$

$$= \boxed{\frac{-\sin\theta}{1+\cos^2\theta}}.$$

Here I used $\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$. We can prove that by the following argument:

$$w = \tan^{-1}(x)$$

$$\tan(w) = x$$

$$\frac{d}{dx}\left(\tan(w)\right) = \frac{d}{dx}(x)$$

$$\sec^2(w)\frac{dw}{dx} = 1$$

$$\frac{dw}{dx} = \frac{1}{\sec^2(w)}$$

Notice $\sin^2(w) + \cos^2(w) = 1 \Rightarrow \tan^2(w) + 1 = \frac{1}{\cos^2(w)} = \sec^2(w)$

Thus,

$$\frac{dw}{dx} = \frac{1}{1+\tan^2(w)} = \frac{1}{1+x^2}$$

$$\underline{\frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2}}.$$

Again this was not required for this problem but if I were to ask for you to show $\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$ you ought to be ready for it.