Same instructions as Mission 1. Thanks!

Problem 101 Suppose S is the surface defined by $F(x,y,z) = xyz = 1$. Find the equation of the tangent plane and the parametrization of the normal line through $(1,1,1)$.

Problem 102 Find a parametrization $\vec{X}$ of S from the previous problem which provides a patch in the locality of $(1,1,1)$. Use α, β for your parameters and find the normal-vector field $\vec{N}(\alpha,\beta)$ by computing $\vec{N}(\alpha,\beta) = \frac{\partial \vec{X}}{\partial \alpha} \times \frac{\partial \vec{X}}{\partial \beta}$. Do you obtain the same normal vector at $(1,1,1)$ with this patch?

Problem 103 Let $F(x,y,z) = xy^2z^3 = 67$ define a surface and suppose (a,b,c) is a point on this surface. Find the equation of the tangent plane to the surface at the point (a,b,c).

Problem 104 Consider the graph $z = 3 + x^2 + y^2$. Find the equation of the tangent plane at $(-1,2,8)$.

Problem 105 Let $\vec{r}(s,t) = \langle s^2,\ t,\ st\rangle$ be the parametrization of a surface M. Calculate the normal vector field $\vec{N}(s,t) = \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}$ and find the equation of the tangent plane to M at $(4,-1,2)$.

Problem 106 Label the solution set of $x^2 = y - z^2$ as M.

(a.) present M as a level-surface for some function F. Explicitly state the formula for F. Find the normal vector field on M.

(b.) parametrize M and once more find the normal vector field. This time find $\vec{N}$ explicitly in terms of your chosen parameters.

Problem 107 Suppose $w = x^2 + y - z + \sin(t)$ and $x + y = t$. Calculate the following constrained partial derivatives:

(**a.**) $\left(\dfrac{\partial w}{\partial y}\right)_{x,z}$ (**b.**) $\left(\dfrac{\partial w}{\partial y}\right)_{z,t}$ (**c.**) $\left(\dfrac{\partial w}{\partial z}\right)_{x,y}$ (**d.**) $\left(\dfrac{\partial w}{\partial z}\right)_{y,t}$

Problem 108 Suppose $w = x^2 + yz^3$ where $x + y + e^z = y^2 + z^2$. Calculate $\left(\dfrac{\partial w}{\partial x}\right)_y$.

Problem 109 Suppose $w = xy + yz$ and $x + y^3 + z = 1$. Calculate $\left(\frac{\partial w}{\partial x}\right)_y$ and $\left(\frac{\partial z}{\partial x}\right)_y$.

Problem 110 Suppose $K = \frac{1}{2}mv^2$. Show $\frac{\partial K}{\partial m}\frac{\partial m}{\partial v}\frac{\partial v}{\partial K} = -1$.

Problem 111 For each vector field $\vec{B}$ below, find Ψ such that $\nabla\Psi = \vec{B}$, or explain mathematically why no such Ψ can exist.

(a.) $\vec{B}(x,y,z) = \langle e^{3x} + xy^2z^2,\ y\cos(y^2) + x^2yz^2,\ \cosh^2 z + x^2y^2z\rangle$

(b.) $\vec{B}(x,y,z) = \langle 2x + z,\ x^3,\ y\rangle$.

Problem 112 The task of solving of an exact differential equation and the problem of finding a potential for a conservative vector field on the plane are nearly the same problem. For the differential equation, the solution should be given as a level set $G(x,y) = c$ where the value of c may be specified if a point on the solution curve is given. For the vector field, the solution is an expression for $f(x,y)$ such that the gradient of f reproduces the given vector field. To make the choice of f unique we would need to specify a value for f at some point. In physics, $-f$ is the potential energy for the force $\langle 2xy^2 + \cos(x),\ 2yx^2 + \sinh y \rangle$.

(a.) Solve $(2xy^2 + \cos(x))\,dx + (2yx^2 + \sinh y)\,dy = 0$.

(b.) Find f for which $\nabla f = \langle 2xy^2 + \cos(x),\ 2yx^2 + \sinh y \rangle$.

Problem 113 A differential equation $Mdx + Ndy = 0$ is said to be **exact** if there exists a function G for which $dG = Mdx + Ndy$. In the case the differential equation is exact it has solutions given by $G(x,y) = c$. If the differential equation below is exact then find its solutions, otherwise explain why the given differential equation is not exact.

(a.) $x^3\,dx + \cos(y)\,dy = 0$

(b.) $ydx + (1 + xy)dy = 0$

(c.) $\left(\frac{-y}{x^2+y^2}\right)dx + \left(\frac{x}{x^2+y^2}\right)dx = 0$

Problem 114 Let $x = r\cos\theta$ and $y = r\sin\theta$. The kinetic energy of a mass m is given by $K = \frac{1}{2}mv^2$. Derive the formula for kinetic energy in polar coordinates using the dot-notations $\frac{dr}{dt} = \dot{r}$ and $\frac{d\theta}{dt} = \dot{\theta}$ to formulate your answer.

Problem 115 Let $f(x,y) = \tan^{-1}(y/x)$.

(a.) Calculate ∇f and simplify answer in Cartesian coordinates,

(b.) Calculate ∇f and simplify answer in polar coordinates,

Problem 116 We defined spherical coordinates by $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$ and $z = \rho\cos\phi$. Derive the formula for $\widehat{\phi}$ (the unit-vector field in the direction of increasing ϕ).

Problem 117 We derived formulas for the gradient in the spherical coordinate system and frame. Answer the following questions in light of those formulas:

(a.) Let $f(x,y,z) = (x^2 + y^2 + z^2)\tan^{-1}(y/x) + \cos^{-1}(z/\sqrt{x^2 + y^2 + z^2})$. Calculate ∇f.

(b.) Suppose $\vec{F} = \rho^2\,\widehat{\rho} + \frac{1}{\rho}\sin^3(\phi)\,\widehat{\phi}$ find f such that $\nabla f = \vec{F}$.

Problem 118 Let $\alpha = ax + by$ and $\beta = cx + dy$. Find the formula for ∇f in terms of derivatives with respect to α, β and the unit-vector fields in the dierction of increasing α and β which we denote $\hat{\alpha}$ and $\hat{\beta}$. (*that is, find the formula like* $\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta}$ *for the* α, β *coordinate system*)

Problem 119 *A curve C in the plane may be described implicitly or parametrically. Suppose C is given* **implicitly** *as the solution set of $G(x,y) = k$. In other words, $C = G^{-1}\{k\}$ where G is a* **level function** *for C. A path $t \mapsto \vec{r}(t) = (x(t), y(t))$ is a* **parametrization** *of C for $t \in I \subseteq \mathbb{R}$ if $G(\vec{r}(t)) = k$ for each $t \in I$. In other words, $\vec{r}(I) = C$ and $\vec{r}$ is a* **parametrization** *of C. Suppose both G and $\vec{r}$ are smooth in what follows so all the rules of calculus apply. Suppose $dG = M dx + N dy$.*

(a.) *Use the chain rule to determine how $\langle M, N\rangle$ and $\frac{d\vec{r}}{dt}$ relate at $p = \vec{r}(t_0) \in C$.*

(b.) *Explain why C is a solution of $\frac{dy}{dx} = \frac{N}{M}$.*

Problem 120 *Let $\vec{F} = \langle M, N\rangle$ be a vector field. A path $t \mapsto \vec{r}(t) = (x(t), y(t))$ is an* **integral curve** *or* **stream line** *of $\vec{F}$ if $\vec{F}(\vec{r}(t)) = \frac{d\vec{r}}{dt}$. Explicitly,*

$$\langle M(\vec{r}(t)),\ N(\vec{r}(t))\rangle = \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle \quad \Leftrightarrow \quad M(\vec{r}(t)) = \frac{dx}{dt} \ \&\ N(\vec{r}(t)) = \frac{dy}{dt}.$$

Thus, technically, to find the integral curves (parametrically) to a given vector field we have to solve a pair of first order differential equations. These could be rather challenging in general. That said, there is a nice trick in the case that the vector field depends only on coordinates of space. In this static context, we can make the following formal calculus steps: for y and x on the integral curve,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{N}{M}.$$

Thus, to find integral curves of $\langle M, N\rangle$ we need only solve $\frac{dy}{dx} = \frac{N}{M}$. But, we learned how to solve such differential equations in Calculus II. With this background in mind, let us consider the following tasks:

(a.) *Let $\vec{F} = \langle -y,\ x\rangle$. Show $\vec{r}(t) = \langle R\cos t,\ R\sin t\rangle$ is an integral curve of $\vec{F}$.*

(b.) *Solve the differential equation $\frac{dy}{dx} = \frac{N}{M}$ for the vector field given in (a.). Verify your Carteisan coordinate based solution has the path given in (a.) as a parametrization.*