Same instructions as Mission 1. Thanks!

Problem 121 Apply the method of Lagrange multipliers to solve the following problem: Let a, b be constants. Maximize xy on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

Problem 122 Apply the method of Lagrange multipliers to solve the following problem: Suppose the base of a rectangular box costs twice as much per square foot as the sides and the top of the box. If the volume of the box must be $12ft^3$ then what dimensions should we build the box to minimize the cost?
[Please state the dimensions of the base and altitude clearly. Include a picture in your solution to explain the meaning of any variables you introduce, thanks!]

Problem 123 A technology company plans to construct a massive data center in a small town. The facility will consume enormous quantities of electricity and water, but the company promises that the project will "bring jobs and economic growth."

To ensure that the project receives prompt approval, the company allocates money to two categories:

$$x = \text{money spent on "economic development"},$$

$$y = \text{money spent on "consulting and community relations."}$$

The company's public-relations department insists that neither category should be interpreted as a bribe to a local official. All the special arrangements made for the data center against the will of the citizens are merely for the greater good and are certainly not due to any sort of kickback deals to local government officials.

The company's total expenditure is modeled by

$$C(x, y) = 2x^2 + y^2.$$

According to the company's consultants, the amount of local political support generated by these expenditures is modeled by

$$4x + 3y = 100.$$

(a.) Use the method of Lagrange multipliers to determine the values of x and y that minimize the company's total expenditure while satisfying the required level of "local political support."

(b.) Determine the minimum value of $C(x, y)$.

Note: The mathematical model is entirely fictional and should not be interpreted as an accusation that any particular technology company engages in bribery, regulatory capture, or environmental exploitation.

Problem 124 Suppose you want to design a soda can to contain volume V of soda. If the can must be a right circular cylinder then what radius and height should you use to minimize the cost of producing the can? *assume the cost is directly proportional to the surface area of the can*

Problem 125 Find any extreme values of xy^2z on the sphere $x^2+y^2+z^2$. *note the sphere is compact and the function* $f(x,y,z) = xy^2z$ *is continuous so this problem will have at least two interesting answers*

Problem 126 A ninja hound is caged in an region with boundary $9x^2+4y^2=36$. Once his energy reaches a level of 5 he can change form into an eagle and escape by flying over the ellipse-shaped fence. Given that his energy level is $f(x,y) = 3+xy$ is there any place(s) in the cage which will allow him the energy to escape?
(please justify your answer by the method of Lagrange Multipliers and multivariate calculus, guessing the answer is only worth 20% credit)

Problem 127 Find all critical points of $f(x,y,z) = x^2y - xy + y^2$ and classify each by the second derivative test either as a local minimum, maximum or saddle:

Problem 128 Consider $f(x,y) = x^3 - 3x - y^2$. Find any critical points for f and use the second derivative test for functions of two variables to judge if any of the critical points yield local extrema.

Problem 129 Consider $f(x,y) = x^3 + y^3 - 3xy$. Find any critical points for f and use the second derivative test for functions of two variables to judge if any of the critical points yield local extrema.

Problem 130 Let $f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy + 39$.

(a.) Find and classify and local extreme values of $f(x,y)$.

(b.) Find absolute maxima and minima for $f(x,y)$ on the triangular region T with vertices $(0,-3), (0,3), (3,0)$.

Problem 131 Find the maximum and minimum values for $f(x,y) = x^4 - 2x^2 + y^2 - 2$ on the closed disk with boundary $x^2+y^2=9$.

Problem 132 Suppose there exist real constants $c_0, c_{i_1 i_2 \dots i_n}$ where such that

$$f(x_1,x_2) = c_0 + \sum_{n=1}^{\infty} \sum_{i_1,\dots,i_n=1}^{2} c_{i_1 i_2 \dots i_n} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

In detail, using $x_1 = x$ and $x_2 = y$,

$$f(x_1,x_2) = c_0 + c_1 x + c_2 y + c_{11}x^2 + c_{12}xy + c_{21}yx + c_{22}y^2 + \cdots .$$

Given the notation above,

(a.) Find formulas for $c_0, c_1, c_2, c_{11}, c_{12}, c_{21}, c_{22}$ in terms of values of f and its partial derivatives evaluated at $(0,0)$.

(b.) Show $f(x_1,x_2) = f(0) + (\nabla f)(0) \bullet \langle x, y\rangle + \frac{1}{2} f_{xx}(0)x^2 + f_{xy}(0)xy + \frac{1}{2} f_{yy}(0)y^2 + \cdots$.

(c.) What is the formula for $c_{i_1 i_2 \dots i_n}$ in terms of f and its derivatives at $(0,0)$?

Problem 133 Let $f(x,y,z) = 13 + x - y + 24x^2 + 14xy + z^2 + \cdots$ be the multivariate Taylor expansion of f based at $(0,0,0)$.

(a.) Find $f(0,0,0)$,

(b.) Find $(\nabla f)(0,0,0)$,

(c.) Find the value of all six distinct second partial derivatives at $(0,0,0)$.

Problem 134 Let $f(x,y) = 23 + 4(x+1)^2 + 10(x+1)(y-3) - 18(y-3)^2 + \cdots$ describe the power series expansion about $(-1,3)$ for f.

(a.) Is $(-1,3)$ a critical point of f ?

(b.) Is $f(-1,3)$ a local minimum, maximum or neither ? What is the value of $f(-1,3)$?

Problem 135 Suppose $f(x,y) = \frac{1}{1-x^2-y^2}$.

(a.) Find the multivariate power series based at $(0,0)$ up to order 4 by an appropriate application of the geometric series result

(b.) Find the multivariate power series based at $(2,0)$ up to order 2.

Problem 136 Find the power series expansions up to order four centered about the origin for:

(a.) $\sin(x+y)$,

(b.) $(x+y)e^{-2x}$.

Problem 137 Power series centered about $(0,0,0)$ for a function are given below. In each part, decide if $(0,0,0)$ is a critical point and then if it is a critical point determine if $f(0,0,0)$ is a local maximum, local minimum or a saddle point. In the case it is a saddle point, please show your work.

(a.) $f(x,y,z) = 16 - x^2 - y^2 - z^2 + x^2y + xy^2 + \cdots$,

(b.) $f(x,y,z) = 3 - x + y + x^2 + y^2 + \cdots$,

(c.) $f(x,y,z) = 13 + x^4 + y^4 + z^4 + \cdots$,

(d.) $f(x,y,z) = 6 + x^2 - 6xy - y^2 - z^2 + \cdots$,

Remark: the problems which follow are not directly about optimization. Instead, we return to the general study of differentiable functions and their properties:

Problem 138 Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be differentiable functions of $x_1, \dots, x_n$. Show that

$$\nabla(fg) = (\nabla f)g + f(\nabla g).$$

Problem 139 Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable.

(a.) We say $U \subseteq \mathbb{R}^n$ is **convex** if and only if any pair of points in U can be connected by line-segment within U. Show that if $\nabla f = 0$ on a convex set $U \subseteq \mathbb{R}^n$ then $f(\vec{x}) = c$ for each $\vec{x} \in U$. (please use a theorem from Calculus I to complete your argument)

(b.) Show that if $\nabla f = \nabla g$ on a convex set $U \subseteq \mathbb{R}^n$ then $f(\vec{x}) = g(\vec{x}) + c$ for each $\vec{x} \in U$.

Problem 140 Prove the mean-value theorem for functions $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$. In particular, show that if f is differentiable at each point of the line-segment connecting P and Q then there exists a point C on the line-segment $\overline{PQ}$ such that $\nabla f(C) \bullet (Q - P) = f(Q) - f(P)$.
Hint: parametrize the line-segment and construct a function on $\mathbb{R}$ to which you can apply the ordinary mean value theorem, use the multivariate chain-rule and win.