MATH 231	QUIZ 4

Please work the problems in the white space provided and clearly box your solutions. You are allowed one 3″ × 5″ notecard. Enjoy!.

Problem 1 (10pt) Suppose C_1 is a curve parametrized by $x = t, y = t^2$ and $z = t^3$ for $0 \le t \le 1$. Calculate the line integral $\int_{C_1} z\,dx + y^2\,dy + x^3\,dz$.

$$\int_{C_1} z\,dx + y^2\,dy + x^3\,dz = \int_0^1 t^3\,dt + (t^2)^2\,2t\,dt + t^3 \cdot 3t^2\,dt$$

$$= \int_0^1 \left[t^3 + 2t^5 + 3t^5 \right] dt$$

$$= \frac{1}{4} + \frac{2}{6} + \frac{3}{6}$$

$$= \frac{3 + 4 + 6}{12}$$

$$= \boxed{13/12}.$$

Problem 2 (5pt) Suppose C_2 is another path from $(0,0,0)$ to $(1,1,1)$. What can we say about $\int_{C_2} z\,dx + y^2\,dy + x^3\,dz$. Will the value be the same as in the previous problem? Describe the possibilities.

Notice $\nabla \times \langle z, y^2, x^3 \rangle = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ z & y^2 & x^3 \end{bmatrix}$

$$= \langle 0, 1 - 3x^2, 0 \rangle$$

Hence $\vec{F} = \langle z, y^2, x^3 \rangle$ is <u>not</u> conservative

Hence $\vec{F}$ is <u>not</u> path-independent. It follows we cannot know the value of $\int_{C_2} \vec{F} \cdot d\vec{r}$ w/o further data.

Problem 3 (5pt) Is $\vec{F}(x, y, z) = \langle e^x,\ ye^{y^2},\ 3z^2 \rangle$ a conservative vector field on $\mathbb{R}^3$? Explain why it is, or why it is not.

Observe $\Phi(x,y,z) = e^x + \frac{1}{2}e^{y^2} + z^3$

has $\nabla \Phi = \langle e^x, ye^{y^2}, 3z^2 \rangle = \vec{F}$

$\therefore$ $\vec{F}$ is conservative on $\mathbb{R}^3$.

Problem 4 (16pt) Suppose $\vec{F}(x,y,z) = \langle 2xy^2,\ 2x^2y + 2y + 2z,\ y^2 \rangle$

(a.) calculate $\nabla \cdot \vec{F}$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2xy^2) + \frac{\partial}{\partial y}(2x^2y + 2y + 2z) + \frac{\partial}{\partial z}[y^2]$$

$$= \boxed{2y^2 + 2x^2 + 2z + 2}$$

(b.) calculate $\nabla \times \vec{F}$

$$\nabla \times \vec{F} = \langle \partial_y(y^2) - \partial_z(2x^2y + 2y + 2z),\ \partial_z(2xy^2) - \partial_x(y^2),\ \partial_x(2x^2y + 2y + 2z) - \partial_y(2xy^2) \rangle$$

$$= \langle 2y - 2y,\ 0,\ 4xy - 4xy \rangle$$

$$= \boxed{\langle 0, 0, 0 \rangle} \Rightarrow \vec{F} \text{ is conservative as } \mathbb{R}^3 \text{ is simply connected domain}$$

(c.) if possible find f such that $\vec{F} = \nabla f$.

$$\frac{\partial f}{\partial x} = 2xy^2 \Rightarrow f = x^2y^2 + C_1(y,z)$$

$$\frac{\partial f}{\partial y} = 2x^2y + 2y + 2yz = 2x^2y + \frac{\partial C_1}{\partial y} \Rightarrow C_1 = y^2 + y^2z + C_2(z).$$

$$\frac{\partial f}{\partial z} = y^2 = \frac{\partial}{\partial z}[x^2y^2 + y^2 + y^2z + C_2] = y^2 + \frac{\partial C_2}{\partial z} \quad \therefore\ C_2 = \text{const.}$$

$$\boxed{f(x,y,z) = x^2y^2 + y^2 + y^2z}$$

(d.) let C be the curve parametrized by $x = 3+2t, y = 2t^2$ and $z = 4t^3$ for $0 \le t \le 1$. Calculate $\int_C \vec{F} \cdot d\vec{r}$

$$\int_c \vec{F} \cdot d\vec{r} = \int_c \nabla f \cdot d\vec{r} = f(5,2,4) - f(3,0,0)$$

$$= [(25)(4) + 4 + 4(4)] - [0 + 0 + 0]$$

$$= 100 + 4 + 16$$

$$= \boxed{120.}$$

Problem 5 (6pt)Let C be a circle of radius R centered at $(0,0)$. Let $\vec{F}(x,y) = \langle x^3, y^3 \rangle$. Calculate the flux of $\vec{F}$ through C.

$$\Phi_C = \int_C (\vec{F}\cdot\hat{n})\,ds = \iint_{x^2+y^2\le R^2} (\nabla\cdot\vec{F})\,dA \qquad (\text{div. form of Green's th}^{\text{m}})$$

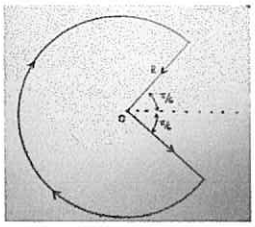
$$= \iint_{x^2+y^2\le R^2} (3x^2+3y^2)\,dA$$

$$= \int_0^{2\pi}\int_0^R 3r^3\,dr\,d\theta$$

$$= 6\pi\,\frac{r^4}{4}\Big|_0^R = \boxed{\frac{3\pi R^4}{2}}.$$

Problem 6 (10pt) Let C be the curve illustrated below which is formed by part of a circle of radius R centered at the origin and two line-segments at $\theta = \pm\pi/6$. Calculate

$$\int_C x^2y\,dx - y^2x\,dy.$$



$$\left.\begin{array}{l} 0 \le r \le R \\ \frac{\pi}{6} \le \theta \le 2\pi - \frac{\pi}{6} = \frac{11\pi}{6} \end{array}\right\} S$$

C is CW-oriented.

$$\int_C x^2y\,dx - y^2x\,dx = -\iint_S \left(\frac{\partial}{\partial x}(-y^2x) - \frac{\partial}{\partial y}(x^2y)\right) dA$$

$$= \iint_S (y^2+x^2)\,dA$$

$$= \int_{\pi/6}^{11\pi/6}\int_0^R r^3\,dr\,d\theta$$

$$= \theta\Big|_{\pi/6}^{11\pi/6}\,\frac{r^4}{4}\Big|_0^R$$

$$= \left(\frac{11\pi}{6} - \frac{\pi}{6}\right)\left(\frac{R^4}{4}\right) = \frac{10\pi R^4}{4(6)} = \boxed{\frac{5\pi R^4}{12}}$$

Problem 7 (10pt) Suppose $\vec{F}(x,y,z) = \langle z,\ \cosh^2(x+y+z),\ x^3\rangle$ and suppose S is the surface parametrized by

$$\vec{r}(u,v) = \langle u,\ 3+v,\ 2-u^2\rangle$$

for $(u,v) \in [0,1]\times[0,1]$. Calculate the flux of $\vec{F}$ through S.

$$\vec{N} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \langle 1,\ 0,\ -2u\rangle \times \langle 0,\ 1,\ 0\rangle$$

$$= (\hat{x} - 2u\hat{z}) \times \hat{y}$$

$$= \langle 2u,\ 0,\ 1\rangle \quad \therefore \quad \underline{d\vec{S} = \langle 2u, 0, 1\rangle\, du\,dv}.$$

$$\vec{F}(\vec{r}(u,v)) = \langle 2-u^2,\ \cosh^2(u+3+v+2-u^2),\ u^3\rangle$$

$x = u$
$y = 3+v$
$z = 2-u^2$

$$\vec{F}(\vec{r}(u,v)) \cdot d\vec{S} = (2u - 2u^3 + u^3)\,du\,dv$$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^1\int_0^1 (2u - u^3)\,dv$$

$$= \int_0^1 dv \int_0^1 (2u - u^3)\,du$$

$$= (1)\left(\frac{2}{2} - \frac{1}{4}\right)$$

$$= \boxed{\frac{3}{4}}.$$

Problem 8 (11pt) Find the flux of $\vec{F}(x,y,z) = \langle xy^2, y^3, zy^2\rangle$ through the surface of the positively oriented cube $E = [0,1]^3$. That is calculate $\int_{\partial E} \vec{F}\cdot d\vec{S}$.

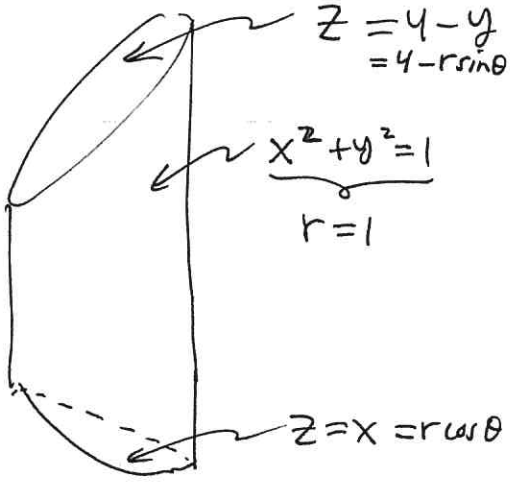
$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(zy^2) = y^2 + 3y^2 + y^2 = 5y^2$$

$$\iint_{\partial E} \vec{F}\cdot d\vec{S} = \iiint_E (\nabla\cdot\vec{F})\,dV \quad \therefore \text{ Gauss' Th}^{m}$$

$$= \int_0^1\int_0^1\int_0^1 5y^2\,dy\,dz\,dx$$

$$= \boxed{\frac{5}{3}}.$$

Problem 9 (15pt) Find the flux of the vector field $\vec{F} = \langle x, y, z \rangle$ through the boundary of the simple solid E which is bounded by $x^2 + y^2 = 1$ and $z = x$ and $z = 4 - y$.



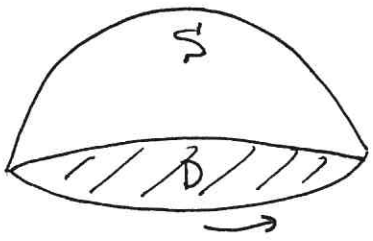
$$\nabla \cdot \vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

$$\iint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_E (\nabla \cdot \vec{F})\, dV = 3 \underbrace{\iiint_E dV}_{\text{Vol.}} \quad (\star)$$

$$\begin{aligned} \text{Vol} &= \int_0^{2\pi} \int_0^1 \int_{r\cos\theta}^{4 - r\sin\theta} r \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 \left(4r - r^2 \sin\theta - r^2 \cos\theta \right) dr \, d\theta \\ &= 8\pi \left. \frac{r^2}{2} \right|_0^1 \\ &= 4\pi, \text{ by } \star \end{aligned}$$

$$\therefore \quad \boxed{\Phi_{\vec{F}} = 12\pi}$$

Problem 10 (10pt) Calculate $\int_S \left(\nabla \times \vec{F} \right) \cdot d\vec{S}$ for $\vec{F}(x, y, z) = \langle zx, zy, z^2 \rangle$ over the positively oriented hemisphere $\rho = R$ with $z \geq 0$. Hint: use Stokes' theorem to trade the hemisphere for an easier surface.



$\partial S = \partial D$

moreover, $\vec{F}|_{\partial S} = \vec{F}|_{\partial D} = 0.$

Oh, not as interesting as I had hoped ☹

$$\int_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r} = \underbrace{\int_{\partial D} \vec{F} \cdot d\vec{r}}_{\text{not needed}} = \boxed{0}$$

can see $F|_{\partial S} = 0$ just the same.

$$\nabla\cdot(\nabla\times\vec{G}) = \sum_{k=1}^{3}\partial_k(\nabla\times G)_k = \sum_k \partial_k\Big(\sum_{ij}\epsilon_{ijk}\,\partial_i G_j\Big) = \sum_{i,j,k}\epsilon_{ijk}\,\partial_k\partial_i G_j = 0$$

(antisymm. in ik; symmetric in (ik))

anytime we contract a pair of sym. and antisym. indices the sum vanishes.

Problem 11 (6pt) Show that $\nabla\cdot(\nabla\times\vec{G}) = 0$.

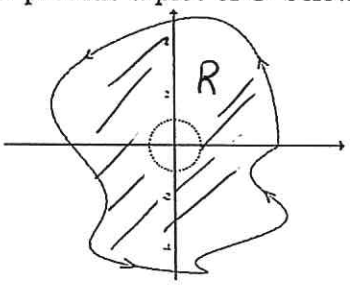
$$\nabla\cdot(\nabla\times\vec{G}) = \partial_x(\partial_y G_z - \partial_z G_y) + \partial_y(\partial_z G_x - \partial_x G_z) + \partial_z(\partial_x G_y - \partial_y G_x)$$

$$= \underbrace{\partial_x\partial_y G_z - \partial_y\partial_x G_z}_{\text{cancel}} \underbrace{- \partial_x\partial_z G_y + \partial_z\partial_x G_y}_{\text{cancel}} \underbrace{+ \partial_y\partial_z G_x - \partial_z\partial_y G_x}_{\text{cancel}}$$

partial derivatives commute.

$$= 0.$$

Problem 12 (6pt) Consider $\vec{F}(x,y) = \frac{1}{\sqrt{x^2+y^2}}\langle -y, x\rangle$. Calculate the line-integral of $\vec{F}$ around the curve C parametrized by $x = 4\cos(t) + \sin(2t^2/\pi)$ and $y = 5\sin(t)$ for $0 \le t \le 2\pi$. I provide a plot of C below (the dotted curve is the unit-circle)



$$\vec{F}(x,y) = \frac{1}{x^2+y^2}\langle -y, x\rangle$$

$$\frac{\partial}{\partial y}\left[\frac{-y}{x^2+y^2}\right] = \frac{-1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2}\right] = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$\Big\}\ (\nabla\times\vec{F})_z = 0.$

Apply Green's Thm to R pictured, (with hole we subtract the inside ccw)

$$-\int_{x^2+y^2=1}\vec{F}\cdot d\vec{r} + \int_C \vec{F}\cdot d\vec{r} = \iint_R (\nabla\times\vec{F})_z\,dA = 0$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{x^2+y^2=1}\frac{-y\,dx + x\,dy}{x^2+y^2}$$

$x = \cos\theta \quad dx = -\sin\theta\,d\theta$
$y = \sin\theta \quad dy = \cos\theta\,d\theta$

$$= \int_0^{2\pi}\frac{(\sin^2\theta + \cos^2\theta)\,d\theta}{\cos^2\theta + \sin^2\theta}$$

$$= \boxed{2\pi}$$