MATH 231	TEST 4

Please work the problems in the white space provided and clearly box your solutions. You are allowed one 3″ × 5″ notecard. Enjoy!.

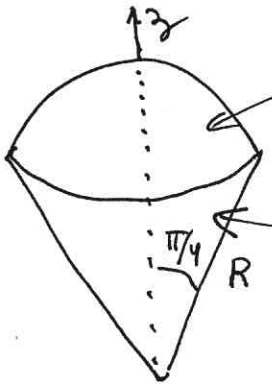
Problem 1 (10pt) Let C be a curve from $(1,2,3)$ to $(5,5,5)$ calculate:

$$\int_C xy^2dx + yx^2dy + dz. = \int_C \nabla\left(\tfrac{1}{2}x^2y^2 + z\right)\cdot d\vec{r}$$

$$= \left(\tfrac{1}{2}x^2y^2 + z\right)\Big|_{(1,2,3)}^{(5,5,5)}$$

$$= \tfrac{1}{2}(25)(25) + 5 - \left(\tfrac{1}{2}(1)(4) + 3\right) = \boxed{312.5 = \frac{625}{2}}$$

Problem 2 (10pt) Find the surface area of the snow-cone-like shape formed by topping off the cone $\phi = \pi/4$ by a portion of a sphere of radius R.



$$dS = R^2\sin\phi\, d\phi\, d\theta$$
$$0 \le \theta \le 2\pi$$
$$0 \le \phi \le \pi/4$$

$$dS = \rho\sin\left(\tfrac{\pi}{4}\right) d\theta\, d\rho$$
$$0 \le \theta \le 2\pi$$
$$0 \le \rho \le R.$$

from our standard example 3. on page 377 of my notes)

$$S = \int_0^{2\pi}\int_0^{\pi/4} R^2\sin\phi\, d\phi\, d\theta + \int_0^R\int_0^{2\pi} \rho\sin\left(\tfrac{\pi}{4}\right) d\theta\, d\rho$$

$$= 2\pi R^2\left(-\cos\phi\Big|_0^{\pi/4}\right) + \frac{2\pi}{\sqrt{2}}\left(\frac{\rho^2}{2}\Big|_0^R\right)$$

$$= 2\pi R^2\left(\frac{-1}{\sqrt{2}} + 1\right) + \frac{2\pi R^2}{2\sqrt{2}}$$

$$= \boxed{\pi R^2\left(2 - \sqrt{2} + \frac{1}{\sqrt{2}}\right)}$$

Problem 3 (10pt) (this problem intends use of divergence theorem) Calculate the flux of $\vec{F}(x,y,z) = \langle x^3,\ -yx^2,\ -zx^2\rangle$ through the positively oriented sphere S_R: $x^2+y^2+z^2=R^2$.

That is, calculate $\iint_{S_R} \vec{F}\cdot d\vec{S}$. $=$

$$= \iiint_{x^2+y^2+z^2\le R^2} (\nabla\cdot\vec{F})\,dV$$

$$= \iiint_{x^2+y^2+z^2\le R^2} (3x^2 - x^2 - x^2)\,dV \qquad \begin{cases} x = \rho\cos\theta\sin\phi \\ dV = \rho^2\sin\phi\, d\phi\, d\theta\, d\rho \end{cases}$$

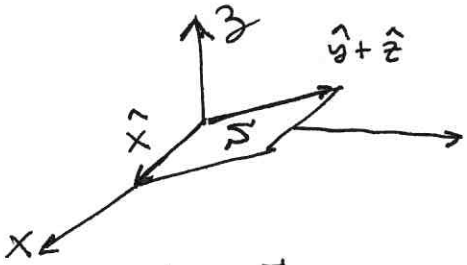
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2\cos^2\theta\sin^2\phi\ \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \left(\int_0^{2\pi}\cos^2\theta\, d\theta\right)\left(\int_0^{\pi}\sin^3\phi\, d\phi\right)\left(\int_0^{R}\rho^4 d\rho\right) = (\pi)\left(\frac{4}{3}\right)\left(\frac{R^5}{5}\right) = \boxed{\frac{4\pi R^5}{15}}$$

$*$

$$\int_0^{\pi}\sin^3\phi\, d\phi = \int_0^{\pi}(1-\cos^2\phi)\sin\phi\, d\phi$$
$$= -\int_1^{-1}(1-u^2)\,du$$
$$= \left(u - \tfrac{1}{3}u^3\right)\Big|_{-1}^{1}$$
$$= 2 - \tfrac{1}{3}(2) = \tfrac{6-2}{3} = \tfrac{4}{3}. \quad *$$

Problem 4 (10pt) Let Q be the rectangle formed by $(0,0,0)$, $(1,0,0)$, $(0,1,1)$ and $(1,1,1)$ oriented CCW as viewed from above (from the positive z-axis). Use Stokes' Theorem to calculate $\int_Q x^2dx + y^2dy - xdz$.



normal : $\hat{x}\times(\hat{y}\times\hat{z}) = -\hat{y}+\hat{z} = \langle 0,-1,1\rangle$

Through origin $\Rightarrow$ $-y+z=0 \rightsquigarrow z=y$.

all 4 ptr. satisfy this eq^n.

S : $\vec{r}(x,y) = \langle x, y, y\rangle$

$$\vec{N} = \frac{\partial\vec{r}}{\partial x}\times\frac{\partial\vec{r}}{\partial y} = \langle 1,0,0\rangle\times\langle 0,1,1\rangle = \langle 0,-1,1\rangle$$

no surprise!

$$d\vec{S} = \langle 0,-1,1\rangle\, dx\,dy \quad : \quad \underline{0\le x\le 1,\ 0\le y\le 1}.$$

Next, the curl,

$$\nabla\times\vec{F} = \det\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ x^2 & y^2 & -x \end{bmatrix} = \langle 0,1,0\rangle$$

Hence,

$$\int_Q \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot d\vec{S} = \int_0^1\int_0^1 -\,dx\,dy = \boxed{-1.}$$

Problem 5 (10pt) Calculate the surface integral $\iint_S \vec{F} \cdot d\vec{S}$ for S where S is the subset of the plane $x+y+z=1$ bounded by the cylinder $x^2+y^2=1$ and $\vec{F}(x,y,z) = \langle 1,\ x^3,\ 0 \rangle$



$d\vec{S} = \langle 1,1,1 \rangle dx\,dy$

for $x^2+y^2 \le 1$.

$\vec{F} \cdot d\vec{S} = \langle 1, x^3, 0 \rangle \cdot \langle 1,1,1 \rangle dx\,dy = x^3\,dx\,dy$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{x^2+y^2\le 1} x^3\,dx\,dy = \int_0^{2\pi}\int_0^1 r^3\cos^3\theta\, r\,dr\,d\theta$$

$$= \int_0^1 r^4\,dr \int_0^{2\pi} (1-\sin^2\theta)\cos\theta\,d\theta$$

$$= \left(\tfrac{1}{5}\right)\int_0^0 (1-u^2)\,du$$

$$= \boxed{0.}$$

Problem 6 (10pt) Suppose $\vec{F}(x,y,z) = \langle e^y - x,\ z^3,\ y^2+z \rangle$

(a.) calculate $\nabla \times \vec{F}$

$$\nabla \times \vec{F} = \langle \partial_y(y^2+z) - \partial_z(z^3),\ \partial_z(e^y-x) - \partial_x(y^2+z),\ \partial_x(z^3) - \partial_y(e^y-x) \rangle$$

$$= \langle 2y - 3z^2,\ 0,\ -e^y \rangle \neq 0 \quad \therefore \vec{F} \neq \nabla\Phi \Rightarrow \text{have to calculate below}$$

(b.) let C be the line segment from $(0,0,0)$ to $(1,2,3)$ for $0 \le t \le 1$. Calculate $\int_C \vec{F} \cdot d\vec{r}$.

$x = t \quad dx = dt$
$y = 2t \quad dy = 2\,dt$
$z = 3t \quad dz = 3\,dt$

$$\int_C (e^y - x)\,dx + z^3\,dy + (y^2+z)\,dz = \int_0^1 \left(e^{2t} - t + (3t)^3(2) + \left[(2t)^2 + 3t\right](3) \right) dt$$

$$= \int_0^1 \left(e^{2t} + 8t + 54t^3 + 12t^2 \right) dt$$

$$= \frac{1}{2}\left(e^2 - 1\right) + 4 + \frac{54}{4} + \frac{12}{3} = \boxed{21 - e^2/2}$$

$\Rightarrow \boxed{e^2 + 7 + \frac{27}{2}} = e^2 + \frac{41}{2}$ etc...

Problem 7 (5pt) Suppose that $\vec{F} = \langle P, Q \rangle$ has $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ at all points except the at the centers of the CCW oriented circles C_1, C_2.It is given that $\int_{C_1} \vec{F} \cdot d\vec{r} = 1$ and $\int_{C_2} \vec{F} \cdot d\vec{r} = 2$. Find $\int_C P dx + Q dy$ where C is pictured below:

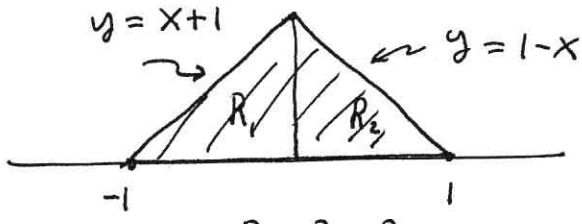


$$\int_C \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \quad (0)$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} = \boxed{2}$$

Problem 8 (10pt) Let C be the CCW triangle with vertices $(-1,0)$, $(1,0)$ and $(0,1)$. Calculate

$$\int_C (xe^x - y^3)dx + (y^3 \cos(y) + x)dy$$



$R = R_1 \cup R_2$

$$\int_{\partial R = C} (xe^x - y^3)\,dx + (y^3 \cos(y) + x)\,dy = \iint_R \left(\frac{\partial}{\partial x}\left[y^3 \cos y + x \right] - \frac{\partial}{\partial y}\left[xe^x - y^3 \right] \right) dA \quad (0, 0)$$

$$= \iint_R (1 + 3y^2)\,dA$$

$$= \frac{1}{2}(2)(1) + 3\iint_{R_1} y^2\,dA + 3\iint_{R_2} y^2\,dA$$

$$= 1 + 6\iint_{R_2} y^2\,dA \quad \text{by symmetry.}$$

$$= 1 + 6\int_0^1 \int_0^{1-x} y^2\,dy\,dx$$

$$= 1 + 6\int_0^1 \frac{1}{3}(1-x)^3\,dx$$

$$= 1 + \left[\frac{-6}{12}(1-x)^4 \right]\Big|_0^1$$

$$= 1 - \frac{1}{2}(0 - 1)$$

$$= \boxed{3/2}$$

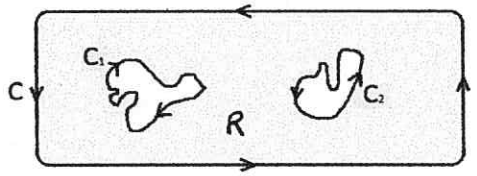
Problem 9 (5pt) Show that $\nabla \times \nabla f = 0$.

$$\nabla \times \langle \partial_x f, \partial_y f, \partial_z f \rangle =$$

$$= \langle \partial_y \partial_z f - \partial_z \partial_y f \,,\, \partial_z \partial_x f - \partial_x \partial_z f \,,\, \partial_x \partial_y f - \partial_y \partial_x f \rangle$$

$$= \langle 0, 0, 0 \rangle$$ as partial derivatives commute (we assume f is twice continuously diff.)

Problem 10 (5pt) Suppose we are given $\int_{C_1} \vec{F} \cdot d\vec{r} = 4$ and $\int_{C_2} \vec{F} \cdot d\vec{r} = 11$. Calculate $\int_C \vec{F} \cdot d\vec{r}$ given that $\vec{F}$ is conservative (locally) on the domain between C and the curves C_1, C_2. Is $\vec{F}$ conservative on $\mathbb{R}^2$? Explain.



$\vec{F}$ conservative locally $\Rightarrow \vec{F} = \nabla f$ on simply connected subsets $\Rightarrow \nabla \times \vec{F} = \nabla \times \nabla f = 0$

Hence Green's Th^{m} applies to give

$$\int_C \vec{F} \cdot d\vec{r} + \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = \iint_R (\nabla \times \vec{F})_3 \, dA = 0$$

$$\int_C \vec{F} \cdot d\vec{r} = -\int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = -4 + 11 = \boxed{7.}$$

No, $\vec{F}$ is not conservative on $\mathbb{R}^2$ as $\int_{C_1} \vec{F} \cdot d\vec{r} = 4 \neq 0$. If $\vec{F}$ was conservative on $\mathbb{R}^2$ then all loops in $\mathbb{R}^2$ must integrate to zero for $\vec{F}$.

Problem 11 (5pt) Show that if f solves Laplaces equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ on a simply connected region R then the flux of ∇f through the boundary ∂R is zero.

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \nabla \cdot \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \nabla \cdot \nabla f$$

$$\int_{\partial R} (\nabla f) \cdot \hat{n} \, ds = \iint_R \nabla \cdot (\nabla f) \, dA$$ by divergence form of Green's Th^{m} which applies to simply connected R.

$$= \iint_R (0) \, dA$$

$$= \boxed{0.}$$

Problem 12 (10pt) Set-up, but do not evaluate, the surface-integral of $\vec{F}(x,y,z) = \langle 1, 2, x+y+z\rangle$ on the positively oriented ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. (I want you to state the parameterization you would use and work out the normal vector field)

$$\vec{r}(\phi,\theta) = \langle a\cos\theta\sin\phi,\ b\sin\theta\sin\phi,\ c\cos\phi\rangle$$

$$\frac{\partial\vec{r}}{\partial\phi} = \langle a\cos\theta\cos\phi,\ b\sin\theta\cos\phi,\ -c\sin\phi\rangle$$

$$\frac{\partial\vec{r}}{\partial\theta} = \langle -a\sin\theta\sin\phi,\ b\cos\theta\sin\phi,\ 0\rangle = \langle -a\sin\theta,\ b\cos\theta,\ 0\rangle\sin\phi$$

$$\vec{N}(\phi,\theta) = \frac{\partial\vec{r}}{\partial\phi}\times\frac{\partial\vec{r}}{\partial\theta} = \sin\phi\ \det\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ a\cos\theta\cos\phi & b\sin\theta\cos\phi & -c\sin\phi \\ -a\sin\theta & b\cos\theta & 0 \end{bmatrix}$$

$$= \sin\phi\ \langle bc\sin\phi\cos\theta,\ ac\sin\phi\sin\theta,\ ab\cos\phi\rangle$$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi} \langle 1, 2, a\cos\theta\sin\phi + b\sin\theta\sin\phi + c\cos\phi\rangle \cdot \langle bc\sin^2\phi\cos\theta,\ ac\sin^2\phi\sin\theta,\ ab\sin\phi\cos\phi\rangle\ d\phi\,d\theta$$

Problem 13 (10pt) Continuing the previous problem, use the divergence theorem to calculate the flux of $\vec{F}$ through the ellipsoid.

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_{x^2/a^2+y^2/b^2+z^2/c^2\le 1} \left[\frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2) + \frac{\partial}{\partial z}(x+y+z)\right] dV$$

$$= \iiint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1} dV$$

$$x = au,\quad y = bv,\quad z = cw \qquad \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = abc$$

$$= \iiint_{u^2+v^2+w^2\le 1} abc\ du\,dv\,dw$$

$$= \frac{4\pi abc}{3}$$