Same instructions as Mission 1. Thanks !

Problem 13 Solve, as in find all possible solutions of:

(a.) $e^z = 2i$

(b.) $Log(z^2 - 1) = \frac{i\pi}{2}$

(c.) $e^{2z} + e^z + 1 = 0$

Problem 14 Find all solutions of $x^2 - y^2 = \cos(\pi/3)$ and $2xy = \sin(\pi/3)$.
Hint: go complex, think about z^2

Problem 15 We define (for now) the trigonometric and hyperbolic trigonometric functions in terms of the complex exponential: $\cosh(z) = \frac{1}{2}(e^z + e^{-z})$, $\sinh(z) = \frac{1}{2}(e^z - e^{-z})$, $\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$, $\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$. Observe $\cosh(iz) = \cos z$ and $\sinh(iz) = i\sin z$. Suppose we've shown $e^z e^w = e^{z+w}$.

(a.) Show that $\sin(z + w) = \sin z \cos w + \cos z \sin w$,

(b.) Derive corresponding identity for $\sinh(z + w)$.

Problem 16 Find all solutions of $\sinh(3z + i) = 0$.

Problem 17 Let $A = \{z \in \mathbb{C} \mid 1 \leq |z - 1| \leq 3\}$.

(a.) Show geometrically that A is connected. (draw pictures)

(b.) Explain geometrically why A is not star-centered. (draw pictures)

Problem 18 Prove $S = \{z = x + iy \in \mathbb{C} \mid x < 3\}$ is an open set. Your proof should include arguments showing there exists an open disk around an arbitrary point in S.

Problem 19 Suppose $S \subseteq \mathbb{C}$ and $F : S \to \mathbb{C}$ is a complex function given by $F(z) = w$. We use $z = x+iy$ and $w = u + iv$. In general, if $F(S) = V$ denotes the range of F then $F^{-1} : V \to S$ need not be a function since it may be **multiply-valued**. We must think of $F^{-1}(w)$ as a set of values in general; $F^{-1}(w) = \{z \in S \mid F(z) = w\}$. For example, $F(z) = e^z$ has domain $S = \mathbb{C}$ and range $F(S) = \mathbb{C} - \{0\} = \mathbb{C}^\times$. The inverse relation for the complex exponential is given by solving $e^z = w$ for $z = \log w$. We have $F^{-1}(w) = \log(w)$. We know $\log(w) = \ln|w| + iarg(w)$ is a set of values. To select a particular choice from the many values of an inverse relation is a method to form a **local inverse** of the given function. For example, the **principal logarithm**: $G(w) = Log(w)$ has $G : \mathbb{C}^\times \to \mathbb{R} \times (-\pi, \pi]$. It serves as an inverse for the exponential restricted to $\mathbb{R} \times (-\pi, \pi]$. Let $t < 0$ and observe $G(t + i\varepsilon) \to \ln|t| \pm i\pi$ as $\varepsilon \to 0^\pm$. Thus, G is not continuous along the negative real axis. When we want to insist on a continuous local inverse then it is usually necessary to restrict the domain of the local inverse by deleting a **branch**. The standard **branch-cut** of the logarithm is given by $H(w) = Log(w)$ where $H : \mathbb{C}^- \to \mathbb{R} \times (-\pi, \pi)$ and $\mathbb{C}^- = \{x + iy \mid \text{if } y = 0 \text{ then } x > 0\}$ is the **slit-complex-plane**. Then H is continuous on $\mathbb{C}^-$ as we have removed the points of discontinuity which arose from the angle jump at $\theta = \pm\pi$. We define $Arg_\alpha : \mathbb{C}^\alpha \to (\alpha, \alpha + 2\pi)$ by $Arg_\alpha(z) \in (\alpha, \alpha + 2\pi) \cap arg(z)$ and likewise $Log_\alpha(z) = \ln|z| + iArg_\alpha(z)$ for each $z \in \mathbb{C}^\alpha$. Notice Log_α is a branch-cut of log along the ray $e^{i\alpha}[0, \infty)$. Note $\mathbb{C}^\alpha = \mathbb{C} - e^{i\alpha}[0, \infty)$.

(a.) Find a branch-cut of $\log(z + 1 - i)$ which is continuous everywhere except for the infinite ray $i + [-1, \infty)$,

(b.) Find a branch-cut of $\log(iz)$ which is continuous everywhere except for the positive real axis,

(c.) Which $\alpha \in [0, 2\pi)$ give a branch-cut $Log_\alpha(z - i)$ of $\log(z - i)$ which is continuous on horizontal strip $|y - 3| < 1$

Problem 20 Show $f(z) = \frac{1}{2i}\text{Log}\left(\frac{1+iz}{1-iz}\right)$ serves as a local inverse to the complex tangent function. In particular, show $f : R \to T$ is a bijection where $T = \{w \in \mathbb{C} \mid -\pi/2 < \text{Re}(w) < \pi/2\}$ and $R = \{z \in \mathbb{C} \mid z \notin [i, i\infty] \cup [-i\infty, -i]\}$ which is a notation to say R is the complex plane with all but $(-i, i)$ on the imaginary axis deleted.

Problem 21 Find a branch $g(w)$ of $w^{1/3}$ which serves as the inverse function of $f(z) = z^3$ with $\text{dom}(f) = \{z \in \mathbb{C}^\times \mid \text{Arg}(z) \in [0, 2\pi/3)\}$. Relate $g(w)$ to the principal branch $\sqrt[3]{w}$. Recall, the principal branch is defined by $\sqrt[3]{w} = \sqrt[3]{|w|}\exp\left(\frac{i\text{Arg}(w)}{3}\right)$ for each $w \in \mathbb{C}^-$ and serves as the inverse function to $h(z) = z^3$ with $\text{dom}(h) = \{z \in \mathbb{C}^\times \mid \text{Arg}(z) \in (-\pi/3, \pi/3)\}$. Draw a few pictures to organize your thoughts.

Problem 22 A supplementary video was posted to give the proof that a map on the plane which is continuously differentiable at p is likewise Frechet differentiable at p. In the proof we construct the differential using the given partial derivatives. What theorem of first semester calculus was crucial to proving the differential did properly approximate the change in the map at p ?

Problem 23 Notice $f(z) = u(z) + iv(z)$ naturally corresponds to $F(x, y) = (u(x, y), v(x, y))$. With this correspondence in mind, calculate the Jacobian matrix for:

(a.) $f(x + iy) = (x + iy)^3$

(b.) $f(x + iy) = \sin(x + iy)$

(c.) $f(x + iy) = (x - iy)^2$

(d.) $f(x + iy) = Log(x + iy)$

Problem 24 We define the matrix corresponding to a complex number by: $M(a + ib) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

(a.) of the functions given in the previous problem, which one had a Jacobian matrix which was not the matrix corresponding to a complex number ?

(b.) of the functions given in the previous problem whose Jacobian matrix was a matrix corresponding to a complex number, how did that number relate to the partial derivative(s) of f ?