Math 331: Harmonic Functions, Complex Integral	Mission 4 [20pts]

Same instructions as Mission 1. Thanks !

Problem 37 Prove $\cos^2 z + \sin^2 z = 1$ for all $z \in \mathbb{C}$. Also, show $|\cos z|$ and $|\sin z|$ are unbounded on $\mathbb{C}$. How is this possible ?

Problem 38 Suppose $f = u + iv$ is holomorphic on some domain D.

(a.) Show $\|\nabla u\| = \|\nabla v\| = |f'(z)|$ on D,

(b.) Show ∇u and ∇v are related by a rotation of 90^o,

(c.) Show $u = 67$ on D implies f is constant.

(d.) Show u, v are harmonic on D.

Problem 39 Let $u = e^{x^2-y^2}\cos(2xy)$. Find the harmonic conjugate of u on $\mathbb{C}$.

Problem 40 Let $u = Arg_\alpha(x+iy)$ for $x+iy \in \mathbb{C}^\alpha$. Find the harmonic conjugate of u on $\mathbb{C}^\alpha$.

Problem 41 Let $u(re^{i\theta}) = \theta\ln(r)$. Find the harmonic conjugate v via the polar Cauchy Riemann equations. Prove u, v are harmonic by deriving the formula for $u + iv$ in terms of z (you should obtain a complex relation whose branches are clearly holomorphic)

Problem 42 Let γ be the CCW-oriented circle $|z| = R$. Calculate $\displaystyle\int_\gamma \bar{z}\,dz$.

Problem 43 Let C be given by $z = e^{i\theta}$ where $-\pi \leq \theta \leq \pi$. Show for any $a \in \mathbb{R}$ that

$$\int_C \frac{e^{az}}{z}\,dz = 2\pi i.$$

Then show the real integral $\int_0^\pi e^{a\cos\theta} cos(a\sin\theta)\,d\theta = \pi$.

Problem 44 I mentioned we can define $\text{Log}(z) = \displaystyle\int_1^z \frac{dz}{z}$ for any $z \in \mathbb{C}^-$. We can appreciate the need for limiting z to the slit complex plane from the calculations which follow. I'll focus our attention to $z = 2i$ to keep it relatively simple.

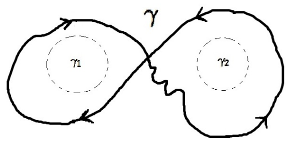
(a.) Let C_1 be the CCW arc on the unit-circle from $z = 1$ to $z = i$ and $C_2 = [i, 2i]$. If $C = C_1 \cup C_2$ the calculate $\displaystyle\int_C \frac{dz}{z}$

(b.) Show the integral calculated in part (a.) agrees with $\text{Log}(2i)$ as we before defined it.

(c.) Let C_3 be the CW arc on the unit-circle from $z = 1$ to $z = i$ and $C_2 = [i, 2i]$. If $C' = C_3 \cup C_2$ then calculate $\displaystyle\int_{C'} \frac{dz}{z}$

(d.) Give a particular choice of α for which $\text{Log}_\alpha(2i)$ agrees with part (c.)

Problem 45 Let γ_1 and γ_2 be CCW-oriented circles as illustrated by the dotted circles in the diagram. Let γ be as indicated. Given $\int_{\gamma_1} f(z)dz = -21$ and $\int_{\gamma_2} f(z)dz = 21$, calculate the value of $\int_{\gamma} f(z)dz$. You are given that $f(z)dz$ is a closed form outside dotted circles.



Problem 46 Let C be the CCW-oriented square which forms the positively oriented boundary of the solid square $[-2,2] \times [-2,2]$. Use an appropriate theorem to calculate:

(a.) $\displaystyle\int_C \frac{e^{-z}dz}{z - \pi i/2}$

(b.) $\displaystyle\int_C \frac{\cos z\, dz}{z^2+8}$

(c.) $\displaystyle\int_C \frac{z\, dz}{2z+1}$

(d.) $\displaystyle\int_C \frac{\tan(z/2)\, dz}{(z-x_o)^2}$ where $-2 < x_o < 2$

(e.) $\displaystyle\int_C \frac{\cosh z\, dz}{z^4}$

Problem 47 Let $a \in \mathbb{R}$ with $a \neq 0$. Also let C_R be the CCW-oriented circle of radius R centered at the origin. Show $\displaystyle\int_{C_R} (z-z_o)^{a-1}\, dz = \frac{2iR^a}{a}\sin(a)$. You should choose a branch of the power function and understand the integration in a limiting sense. Probably a good choice would be to parametrize the circle to begin and end at the negative real axis and allow the endpoints to squeeze down to the point $z = -R$. If you make that choice then the power function can be formulated with the principle logarithm *Log*.

Problem 48 Let $\gamma(t) = 2\sqrt{3}e^{it}$ for $\pi/2 \leq t \leq 3\pi/2$. Calculate $\displaystyle\int_\gamma \frac{dz}{z+2}$. A wandering math ninja stumble across your path an mutters $\tan(\pi/3) = \sqrt{3}$.