Same instructions as Mission 1. Thanks !

Problem 85 The Riemann zeta function is defined by $\zeta(z) = \displaystyle\sum_{j=1}^{\infty} \frac{1}{j^z}$ for $Re(z) > 1$ where $j^z = exp(zLog(j))$. Show $\zeta(z)$ is analytic on $Re(z) > 1$. *Note: this problem could have been part of Mission 5*

Problem 86 Let $n \in \mathbb{N}$. Show that if $P(z) = (z - z_1)^{m_1}(z - z_2)^{m_2} \cdots (z - z_n)^{m_n}$ then

$$\frac{P'(z)}{P(z)} = \frac{m_1}{z - z_1} + \frac{m_2}{z - z_2} + \cdots + \frac{m_n}{z - z_n}.$$

Problem 87 Suppose $P(z)$ is a polynomial with no zeros along the simple closed contour Γ (this means $\Gamma = \partial D$ where D is a simply connected subset of $\mathbb{C}$) then show that the number of zeros inside Γ, counting multiplicities, is given by:

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{P'(z)}{P(z)} dz.$$

Problem 88 Show that $e^{-z} = 4 - z$ has only one solution in the right-half plane. Or, show the claim is false.

Problem 89 Let $m, n \in \mathbb{N}$. Show the polynomial $p(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^m}{m!} + 3z^n$ has exactly n zeros in the unit-disk.

Problem 90 Determine the number of zeros for the polynomial $p(z) = z^6 + 4z^2 + 13z + 2$ in the annulus $1 < |z| < 2$.

Problem 91 Use methods of contour integration and/or residue theory to calculate the integral that follows:

$$\int_0^{2\pi} e^{2\cos(\theta)}\, d\theta.$$

Answer can be left in a closed form infinite series that has a factorial.

Problem 92 Use methods of contour integration and/or residue theory to calculate the integral that follows:

$$\int_0^{\infty} \frac{\sin^3(x)}{x^3}\, dx$$

Leave your answer in terms of $3, \pi$ and 8.

Problem 93 Use methods of contour integration and/or residue theory to calculate the integral that follows (assume $b > 0$):

$$\int_0^{\infty} \frac{\ln(x)}{x^2 + b^2}\, dx$$

Leave your answer in terms of $2, \pi$ and b and $\ln(b)$.

Problem 94 Problem above worth double.

Problem 95 Problem below borrowed from the excellent text by Saff and Snider.

Let $f(z)$ be a rational function of the form $P(z)/Q(z)$, where $\deg Q \geq 2 + \deg P$. Assume that no poles of $f(z)$ occur at the integer points $z = 0, \pm 1, \pm 2, \ldots$. Complete each of the following steps to establish the summation formula

$$\lim_{N\to+\infty} \sum_{k=-N}^{N} f(k) = -\{\text{sum of the residues of } \pi f(z)\cot(\pi z) \text{ at the poles of } f(z)\}. \tag{10}$$

(a) Show that for the function $g(z) := \pi f(z) \cot(\pi z)$, we have

$$\operatorname{Res}(g; k) = f(k), \quad k = 0, \pm 1, \pm 2, \ldots.$$

(b) Let Γ_N be the boundary of the square with vertices at $(N + \frac{1}{2})(1 + i)$, $(N + \frac{1}{2})(-1 + i)$, $(N + \frac{1}{2})(-1 - i)$, $(N + \frac{1}{2})(1 - i)$, taken in that order, where N is a positive integer. Show that there is a constant M independent of N such that $|\pi \cot(\pi z)| \leq M$ for all z on Γ_N.

(c) Prove that

$$\lim_{N\to+\infty} \int_{\Gamma_N} \pi f(z) \cot(\pi z)\, dz = 0,$$

where Γ_N is defined previously.

(d) Use the residue theorem and parts (a) and (c) to derive (10).

Problem 96 Problem below borrowed from the excellent text by Saff and Snider.

Using the summation formula in Prob. 95 verify that

(a) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{k^2+1} = \pi \coth(\pi)$ [HINT: Take $f(z) = 1/(z^2 + 1)$.]

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ [HINT: The formula in Prob. 14 needs to be modified to compensate for the pole of $f(z) = 1/z^2$ at $z = 0$.]