Name: (please print name as it appears in Canvas here $\rightarrow$) ____ .

MATH 334:	SAM 11: SYSTEMS OF LINEAR ODES [35PTS]

Box or underline your answer. Work to be shown on separate, single-sided, normal sized paper with no fuzzies. Thanks!

Problem 44 (1pts) Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 7 \end{bmatrix}$. Suppose $v_1 = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -7 \\ 2 \end{bmatrix}$. (show work below)

(a.) calculate Av_1

(b.) calculate Av_2

(c.) Show $A[v_1|v_2] = [Av_1|Av_2]$.

Problem 45 (1pts) Show that e^{tA} is a fundamental solution matrix for $\frac{d\vec{r}}{dt} = A\vec{r}$.(show work below)

Problem 46 (3pts) Differentiation of matrices of functions is not hard. Let $X(t) = \begin{bmatrix} e^t & -t \\ 1/t & e^{-t} \end{bmatrix}$.

Note: if $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $M^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Calculate: (show work below)

(a.) $\frac{dX}{dt}$

(b.) $\frac{dX^{-1}}{dt}$

(c.) simplify $\frac{dX}{dt}X^{-1} + X\frac{dX^{-1}}{dt}$.

Problem 47 (2pts) Consider the system of differential equations given by

$$(D^2+1)[z] = y \qquad \& \qquad (D^2-3D+2)[y] = 0$$

where $D = d/dt$. Use reduction of order to rewrite this pair of second order ODEs as a single system of four first order ODEs in matrix normal form $\frac{dx}{dt} = Ax$. Also, use technology to find the eigenvalues of the matrix.

Problem 48 Suppose X and Y are both fundamental solution matrices for $\frac{d\vec{r}}{dt} = A\vec{r}$ where $A \in \mathbb{R}^{n\times n}$. (show work below)

(a.) (1pts) Show there exists a constant invertible matrix B for which $X = BY$.

(b.) (1pts) Find a formula for e^{tA} in terms of X.

Problem 49 (3pts) Linear independence (LI) of vector-valued functions $\{\vec{f}_j : I \subseteq \mathbb{R} \to \mathbb{R}^n \mid j = 1, \dots, k\}$ is defined in the same way as was previously discussed for real-valued functions. In particular, $\{\vec{f}_1, \dots, \vec{f}_k\}$ is LI on $I \subseteq \mathbb{R}$ if $c_1\vec{f}_1(t) + \cdots + c_k\vec{f}_k(t) = 0$ for all $t \in I$ implies $c_1 = 0, \dots, c_k = 0$. We can check LI of n such n-vector-valued functions without any further differentiation; in particular, if there exists $t \in I$ for which $\det[\vec{f}_1(t)|\cdots|\vec{f}_n(t)] \neq 0$ then $\{\vec{f}_1(t), \dots, \vec{f}_n(t)\}$ is LI on I. Determine whether or not each set of vector-valued functions is LI on $\mathbb{R}$. (show work below)

(a.) $\{\, (e^t, e^t),\ (e^t, -e^t)\,\}$

(b.) $\{\, (\cos(t), -\sin(t)),\ (\sin(t), \cos(t))\,\}$,

(c.) $\{\, e^t\vec{u}_1,\ e^t\vec{u}_2\ , e^t\vec{u}_3\,\}$ given $[\vec{u}_1|\vec{u}_2|\vec{u}_3]$ is an invertible matrix

Problem 50 (1pts) Suppose $\frac{dx}{dt} = x + 4y$ and $\frac{dy}{dt} = x + y$. Find the general real solution via the e-vector method.

Problem 51 (1pts) Calculate e^{tA} for $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$. (previous problem relevant)

Problem 52 (1pts) Suppose $\frac{dx}{dt} = 2x + y$ and $\frac{dy}{dt} = 2y$. Find the general real solution via the generalized e-vector method.

Problem 53 (1pts) Suppose $\frac{dx}{dt} = 4x - 3y$ and $\frac{dy}{dt} = 3x + 4y$. Find the general real solution via the e-vector method.

Problem 54 (1pts) Suppose $\frac{dx}{dt} = 5x - 6y - 6z$, $\frac{dy}{dt} = -x + 4y + 2z$ and $\frac{dz}{dt} = 3x - 6y - 4z$. Find the general real solution via the e-vector method.

Problem 55 (1pts) Suppose $\frac{dx}{dt} = 5x - 5y - 5z$, $\frac{dy}{dt} = -x + 4y + 2z$ and $\frac{dz}{dt} = 3x - 5y - 3z$. Find the general real solution via the e-vector method.

Problem 56 (1pts) Suppose $\frac{dx}{dt} = 3x + y$, $\frac{dy}{dt} = 3y + z$ and $\frac{dz}{dt} = 3z$. Find the general real solution via the generalized e-vector method.

Problem 57 (1pts) Suppose A is a 3×3 matrix with nonzero vectors $\vec{u}, \vec{v}, \vec{w}$ such that

$$A\vec{u} = 3\vec{u}, \qquad (A - 3I)\vec{v} = \vec{u}, \qquad A\vec{w} = 0.$$

Write the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ in terms of the given vectors.

Problem 58 (2pts) Suppose $(A - \lambda I)\vec{u}_1 = 0$ and $(A - \lambda I)\vec{u}_2 = \vec{u}_1$ where $\lambda = 6 + 7i$ and $\vec{u}_1 = \vec{a}_1 + i\vec{b}_1$ and $\vec{u_2} = \vec{a}_2 + i\vec{b}_2$ where $\vec{a}_1, \vec{a}_2, \vec{b}_1, \vec{b}_2$ are real vectors.

(a.) Find a pair of complex solutions of $\frac{d\vec{x}}{dt} = A\vec{x}$

(b.) Suppose $A \in \mathbb{R}^{4\times 4}$. Solve $\frac{d\vec{x}}{dt} = A\vec{x}$.

Problem 59 (1pts) Consider A is a 3×3 matrix for which there exist nonzero vectors v_1, v_2, v_3 such that:

$$Av_1 = 10v_1, \quad Av_2 = 10v_2, \quad Av_3 = 10v_3 + v_1$$

derive the general solution for $\frac{d\vec{r}}{dt} = A\vec{r}$ with appropriate arguments based on the matrix exponential.

Problem 60 (1pts) Suppose $A = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}$. Calculate e^{tA} and solve $\frac{d\vec{r}}{dt} = A\vec{r}$ given that $\vec{r}(0) = (1, 2)$.

Problem 61 (1pts) Solve $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}$ where $A = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$ and $\vec{f}(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$

Problem 62 (1pts) Consider the constant coefficient problem $ay'' + by' + cy = 0$ where a, b, c are real constants and $a \neq 0$. Use reduction of order with $x_1 = y$ and $x_2 = y'$ to rewrite the given second order ODE as a system of first order ODEs. Calculate the characteristic equation for your system and comment on how it compares to the usual characteristic equation for the given second order ODE. (show work below)

Problem 63 (4pts) To solve $\frac{d\vec{x}}{dt} = A\vec{x}$ in the case $A = \begin{bmatrix} -3 & 0 & -3 \\ 1 & -2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$ by the following calculations:

(a) find the e-values and corresponding e-vectors $\vec{u_1}, \vec{u_2}, \vec{u_3}$. (you may use technology)

(b) construct $P = [\vec{u_1}|\vec{u_2}|\vec{u_3}]$ and calculate $P^{-1}AP$. (you may use technology)

(c) note the solution of $AP\vec{y} = \frac{d}{dt}[P\vec{y}] = P\frac{d\vec{y}}{dt}$ is easily found since multiplying by P^{-1} yields $P^{-1}AP\vec{y} = P^{-1}P\frac{d\vec{y}}{dt} = I\frac{d\vec{y}}{dt} = \frac{d\vec{y}}{dt}$. Solve $P^{-1}AP\vec{y} = \frac{d\vec{y}}{dt}$. (this should be really easy, just solve 3 first order problems, one at a time)

(d) $AP\vec{y} = \frac{d}{dt}[P\vec{y}]$ means $\vec{x} = P\vec{y}$ solves $\frac{d\vec{x}}{dt} = A\vec{x}$. Solve the original system by multiplying the solution from (c.) by P.

Problem 64 (3pts) If $A^T = A$ then we say A is a symmetric matrix. A rather deep theorem of linear algebra states that a symmetric matrix has real eigenvalues and it is possible to select n-LI eigenvectors $\{\vec{u}_1, \dots, \vec{u}_n\}$ for which $A\vec{u}_j = \lambda_j \vec{u}_j$ and $\vec{u}_i \cdot \vec{u}_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ for all $i, j \in \mathbb{N}_n$. It follows that $P = [\vec{u}_1 | \cdots | \vec{u}_n]$ has $P^T P = I$ which means $P^{-1} = P^T$. This means, if we're studying a system of differential equations $\frac{d\vec{x}}{dt} = A\vec{x}$ with $A^T = A$ we can change coordinates to $\vec{y} = P^T \vec{x}$ and in that new $\vec{y}$-coordinate system the differential equation is simply:

$$\frac{dy_1}{dt} = \lambda_1 y_1, \ \dots, \ \frac{dy_n}{dt} = \lambda_n y_n. \qquad \star.$$

This system is said to be **uncoupled** and it's really the most trivial sort of system you can come across; we can solve each equation in the uncoupled system without knowledge of the remaining variables. Consider $\vec{x} = \langle x, y, z \rangle$ and the differential equation $\frac{d\vec{x}}{dt} = A\vec{x}$ where $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$.

(a.) Find an orthonormal eigenbasis for A

(b.) Find formulas for the eigencoordinates and use them to change coordinates on the given system.

(c.) Solve $\star$ in the context of A. Use the notation $\vec{y} = \langle \bar{x}, \bar{y}, \bar{z} \rangle$, so $y_1 = \bar{x}$ etc..

Problem 65 (1pts) Consider the solution-set of $4xy + 4xz + 4yz = 1$. Rewrite the formula using the barred-coordinates $\bar{x}, \bar{y}, \bar{z}$ from part (b.) of the previous problem. Which Quadric surface is this?

Problem 66 (1pts) If A has n-linearly independent eigenvectors $\vec{u}_1, \dots, \vec{u}_n$ then the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ is given by:

$$\vec{x} = c_1 e^{\lambda_1 t}\vec{u}_1 + \cdots + c_n e^{\lambda_n t}\vec{u}_n.$$

Solve $\frac{d\vec{x}}{dt} = A^k\vec{x}$ where $k \in \mathbb{N}$. (show work below)

Problem 67 (1pts) The Cayley Hamilton Theorem states that a matrix will solve its own characteristic equation. For example, if $P(x) = x^3 - 1$ then $P(A) = A^3 - I = 0$. For this A, calculate e^{tA} in terms of A and the functions $C = \sum_{k=0}^{\infty} \frac{t^{3k+2}}{(3k+2)!}$ and $M = \frac{dC}{dt}$ and $L = \frac{dM}{dt}$. (show work below)