Name: (please print name as it appears in Canvas here $\rightarrow$) .

MATH 334:	SAM 7: THEORY FOR LINEAR ODES [20PTS]

Box or underline your answer, thanks! Work should be given on separate, single-sided, normal sized paper with no fuzzies. Problems should be in order and if any problems are skipped there should be white space and the word SKIP to help the grader. Thanks!

Problem 1 Let $\omega > 0$. Show $\{\cos(\omega t), \sin(\omega t), t\cos(\omega t), t\sin(\omega t)\}$ is a linearly independent set of functions on $\mathbb{R}$.

Problem 2 Let $a \in \mathbb{R}$. Show $\{e^{at}, te^{at}, \dots, t^{k-1}e^{at}\}$ is a linearly independent set of functions on $\mathbb{R}$.

Problem 3 Let $R_1, R_2, R_3 \in \mathbb{R}$ be distinct values. Show $\{x^{R_1}, x^{R_2}, x^{R_3}\}$ is a linearly independent set of functions on $(0, \infty)$.

Problem 4 Let $y_1(x) = x^3$ and $y_2(x) = |x|^3$. Show that $W(y_1, y_2)(x) = 0$ for all $x \in \mathbb{R}$. However, explain why $\{y_1, y_2\}$ is linearly independent on $\mathbb{R}$. Does there exist a linear ODE for which $\{y_1, y_2\}$ forms the fundamental solution set? Discuss.

Problem 5 Let $T = x\frac{d}{dx}$ and $S = x^2\frac{d}{dx} + \frac{d}{dx}$. Calculate $[T, S] = TS - ST$. Your work should be based on studying how these operators act on some test function y.

Problem 6 Show that $(xD)^2 \neq x^2D^2$. Determine how these are related using calculus on a test function y. Here $D = d/dx$ and we suppose y is a differentiable function of x.

Problem 7 Show that $\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}$ for $\lambda = \alpha + i\beta$. Note $\frac{d}{dt}(u+iv) = \frac{du}{dt} + i\frac{dv}{dt}$ defines real derivatives of a complex valued function where u, v are both real functions of t. Furthermore,

$$e^{(\alpha+i\beta)t} = e^{\alpha t}\left(\cos\beta t + i\sin\beta t\right)$$

defines the exponential of a complex quantity.

Problem 8 Given $x > 0$, define $x^{a+ib} = x^a\left(\cos(b\ln(x)) + i\sin(b\ln(x))\right)$. Let $R = a + ib$ and show

$$\frac{d}{dx}\left(x^R\right) = Rx^{R-1}.$$

Problem 9 Derive Abel's formula for $y'' + p(x)y' + q(x)y = 0$ given that $p(x)$ and $q(x)$ are continuous on $\mathbb{R}$.

Problem 10 Suppose that y_1 is a nontrivial solution of $y'' + p(x)y' + q(x)y = 0$. We seek a method to derive a second LI solution. Let y_2 be such a solution and **show that it must satisfy** $\frac{d}{dx}\left[\frac{y_2}{y_1}\right] = \frac{W(y_1,y_2)}{y_1^2}$. Now, use Abel's formula to find a nice formula for y_2.